Int Jr. of Mathematics Sciences &Applications
Vol.3, No.1, January-June 2013

ISSN No: 2230-9888
www.journalshub.com

SUFFICIENT CONDITIONS FOR CLOSEDNESS OF MAPS

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ABSTRACT

A function f:X→Y is said to have closed graph if graph of f i.e. the set (x,f(x)) is a closed subset of the product space X × Y. It is well established fact that a function with closed graph is KC as well as Inversely KC. Moreover, a function with closed graph is closed if X is compact .In the present paper, some conditions are investigated under which an inversely KC function or KC function becomes closed.

KEY WORDS: Closed graph, KC, Inversely KC, closed, Frechet space.
2000 AMS SUBJECT CLASSIFICATION CODES. 54C10.

INTRODUCTION

By a space, we shall mean a topological space. No separation axioms are assumed and no function is assumed to be continuous or onto unless mentioned explicitly; cl(A) will denote the closure of the subset A in the space X. If A is a subset of X, we say that X is T1 at A if each point of A is closed in X. X is said to be Frechet space (or closure sequential in the terminology of Wilansky [1]) if for each subset A of X, $x \in cl(A)$ implies there exists a sequence {xn} in A converging to x.X is said to be a k-space if O is open (equivalently :closed) in X whenever $O \cap K$ is open (closed) in K for every compact subset K of X. Every space which is either locally compact or Frechet is a k-space.

A function f:X→Y is said to be compact preserving (compact) if image (inverse image) of each compact set is compact. f:X→Y will be called KC [1] (inversely KC) if image (inverse image) of every compact set is closed. For study of closed functions , see references [4],[5],[6] and [7].

In 1968,Fuller [2] has proved the following

THEOREM : Let f:X→Y have closed graph. Then f is closed if f is inversely subcontinuous.

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THEOREM : Let $f:X\to Y$ have closed graph and be compact where Y is a k-space.Then f is closed.

In the present paper, the condition of closed graph on the function f is weakened by assuming the function KC or inversely KC and the following results are obtained. Theorems 1 and 2 (Theorems 3 and 4) give conditions under which a KC function (inversely KC function) becomes closed.

THEOREM 1. Let $f:X\to Y$ be KC with closed fibers where X is locally compact, regular, countable compact and Y is Frechet .Then f is closed.

PROOF. Let F be a closed subset of X and let $y\in \text{cl}f(F)-f(F)$. Since Y is a Frechet space, there exists a sequence $\{x_n\}$ of points in F such that $f(x_n)\to y$. Now F being a closed subset of countable compact space X is countable compact,the set $\{x_n:n\in N\}$ has a cluster point x in F, $y\neq f(x)$. Since f has closed fibers, $f^{-1}(y)$ is a closed set and $x\notin f^{-1}(y)$.Then X is locally compact, regular implies there exists an open set W containing x such that clW is compact and $\text{cl}W\cap f^{-1}(y)=\phi$. Since f is KC, f(clW) is closed which is a contradiction as $y\in \text{cl}\ f(\text{cl}W)-f(\text{cl}W)$. Hence f must be closed.

THEOREM 2. Let $f:X\to Y$ be KC and compact where Y is a k-space. Then f is closed.

PROOF. Let F be a closed subset of X. To prove f(F) is a closed subset of Y, we prove $K\cap f(F)$ is a closed subset of K for every compact subset K of Y, since Y is a k-space. So let K be a compact subset of Y .Then $f^{-1}(K)$ is a compact subset of X, since f is compact. Now $F\cap f^{-1}(K)$ being a closed subset of $f^{-1}(K)$ is compact and therefore $f(F\cap f^{-1}(K))=K\cap f(F)$ is closed in K as f is KC. This completes the proof.

THEOREM 3. Let $f:X\to Y$ be Inversely KC, compact where Y is a Frechet and T_1 is at f(X). Then f is closed.

PROOF. Let F be a closed subset of X and let $y\in \text{cl}f(F)-f(F)$. Since Y is a Frechet space, there exists a sequence $\{x_n\}$ of points in F such that $f(x_n)\to y$. Now $K=\{f(x_n):n\in N\}\cup\{y\}$is compact, f is Inversely KC and compact implies $f^{-1}(K)$ is closed and compact set and therefore the set $\{x_n:n\in N\}$ has a cluster point x in the set $F\cap f^{-1}(K)$.Since $y\neq f(x)$ and Y is T_1 at f(X), $V=Y-\{f(x)\}$ is an open set containing y. Then $f(x_n)\to y$ implies there exists an integer n_o such that $f(x_n)\in V$ for all $n\geq n_o$. Let $H=\{f(x_n): n\geq n_o\}\cup\{y\}$.Then H is compact and f is inversely KC implies $f^{-1}(H)$ is closed which is a contradiction as $x\in \text{cl}\ f^{-1}(H) - f^{-1}(H)$. Hence f must be closed.

THEOREM 4. Let $f: X\to Y$ be Inversely KC, where X is countable compact and Y is Frechet and T_1 is at f(X).Then f is closed.

PROOF. Let F be a closed subset of X and let $y\in \text{cl}f(F)-f(F)$. Since Y is a Frechet space, there exists a sequence $\{x_n\}$ of points in F such that $f(x_n)\to y$. Now F being a closed subset of countable compact space X is countable compact, the set $\{x_n: n\in N\}$ has a cluster point x in F,$y\neq f(x)$. Since Y is T_1 at f(X), the set $V=Y-\{f(x)\}$ is an open set containing y.

Then $f(x_n) \to y$ implies there exists an integer n_o such that $f(x_n) \in V$ for all $n \geq n_o$. Let $K=\{f(x_n): n \geq n_o\} \bigcup \{y\}$.Then K is compact and f is inversely KC implies $f^{-1}(K)$ is closed which is a contradiction as $x \in cl\, f^{-1}(K) - f^{-1}(K)$. Hence f must be closed.

The following example shows that none of the condition on the domain and range space can be weakened.

EXAMPLE: Let X=N, the set of positive integers, with a base for a topology on X the family of all sets of the form (2n-1,2n) $n \in N$ and Y= $(0, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n})$ as a subspace of the real line. The function f: $X \to Y$ defined by $f(2n-1)= \frac{1}{n-1}$ =f(2n) for $n \geq 2$ and f(1)=0=f(2) is Inversely KC as well KC but is not closed although X is B-W compact and Y is Frechet , T_2.

REFERENCES

[1] Wilansky, A. Topology for analysis Xerox College Publishing Lexington, Massachusetts, Toronto,1970.

[2]Fuller, R.V. Relations among continuous and various non continuous functions, Pacific J. Math. 25, (1968)495-509.

[3] Piotrowski,Z & Szymanski,A. A Closed Graph Theorem:Topological Approach, Rendiconti Del Circolo Matematico Di Palermo,serie II, Tomo XXXVII(1988) 88-99.

[4] Goel Asha,& Garg,G.L. Conditions Implying Closedness (openness) and Perfectness of Maps, Acta Mathematica Hungarica ,Vol.78(4)(1998) 327-332.

[5] Goel Asha,& Garg,G.L.On Maps:Continuous,Closed, Perfect, and with Closed Graph, International Journal of Mathematics & Maths. Sciences, Florida, USA, Vol. 20, No. 2(1997) 405-408.

[6] Goel Asha,& Garg,G.L.Convergence Conditions and Closed Maps, Soochow Journal of Mathematics,vol 33,No 2(2007)257-261.

[7] Noorie, N.S. & Bala, Rajni Some characterizations of open,closed and continuous mappings,International journal of mathematics and mathematical sciences,Vol.2008,Article ID 527106.

[8] Thron, W.J. Topological structures,Holt, Rinehart and Winston.1966.