International Journal of Mathematical Sciences and Applications
Vol. 1 No. 2 (May, 2011)
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www.ijmsa.yolasite.com

THE STUDY OF MATHEMATICAL MODELLING OF HUMAN BLOOD CIRCULATORY SYSTEM

H. M. Pandey, D. S. Negi & M. S. Bisht
Department of Mathematics,
H. N. B. Garhwal University,
Campus - Srinagar (Garhwal),
Uttarakhand, India- 246174.
E-mail : harshmanipandey@gmail.com

Abstract. A Mathematical model for the flow of blood in narrow tubes is described. The model consists of a central core of suspended erythrocytes and a cell-free layer surrounding the core. It is assumed that the viscosity in the cell-free layer differs from that of plasma, as a result of additional dissipation of energy near the wall caused by the red blood cell motion near the cell-free layer. A non-Newtonian shear-thinning constitutive relation is proposed to study the flow of blood.

Key words. Mathematical model, energy dissipation, relative viscosity, non-Newtonian shear thinning.

AMS Subject classification (2000) 76S05, 76S99

1. Introduction

Quantitative understanding of the blood flow through arterioles and venules is necessary for assessing the hemodynamic resistance and its regulation in the microcirculation as well as for analyzing mass transport processes. When blood flows through tubes, the two-phase nature of blood as a suspension becomes important as the diameter of the red blood cell (RBC) becomes comparable to the tube diameter. Chien et al. [3] reviewed the studies on blood flow in narrow tubes. Pries at al. [4] reviewed biophysical aspects of microvascular blood flow in vivo as well as in vitro.

Nair et al. [8] used a two-phase model for the blood in modelling transport of oxygen in the arterioles. They considered a cell-rich core surrounded by a cell-free plasma layer. In the cell-rich core, the radial hematocrit distribution was expressed as a power law profile with maximum at the centre of the tube. A parabolic velocity profile was taken for the plasma in the cell-free layer. Different velocities were used for the plasma and RBCs in the cell-rich core. The parabolic profiles were slightly blunted and different by a constant called the slip.

Gupta et al. [13] divied the outer layer into a cell-free plasma layer and cell-depleted layer. In both these studies, the velocity profile in the core was assumed to

follow a power law. Damiano [5] has presented a semi-empirical model for the blood flow in glycocalyx-lined microvessels greater than 20 μm in diameter. The model assumes a steady axisymmetric flow of a viscous fluid having a smoothly varying cross-sectional viscosity distribution throughout the tube. The viscosity is assumed to decrease monotonically from a maximum in the cell-rich core corresponding to a value for whole blood obtained from rotational viscometry studies to the value near the tube wall corresponding to blood plasma. Also, thickness of the cell-depleted region was assumed a constant and the variation in the core hematocrit with the discharge hematocrit was not considered.

It has been well established in a number of studies that blood possesses significant nonlinear rheological properties [6,7,23,25,26,27,28,31]. Considerable attention has been given to the study of blood flow in the cardiovascular system. The majority of such hemorheological studies in straight tubes have assumed a Newtonian model for blood. The validity of this assumption under certain circumstances however, is somewhat dubious [7,11,14]. Although a Newtonian model appears to be reasonable at high shear rates [7], such as might arise near the tube wall, a significant fraction of the tube cross section might be at a sufficiently low shear rate to show manifestations of non-Newtonian behaviour [1,27]. Since, under pulsatile blood flow conditions, shear rates vary substantially both spatially and temporally, it might be important to account for variations in blood viscosity over the tube cross section [15].

2. Mathematical model

Consider a two layer model (Fig. 1) for the blood flow within a cylindrical tube or vessel of radius R consisting of a central core of radius r_h and effective viscosity μ_c, which contains an erythrocyte suspension of uniform hematocrit H_c and a cell-free layer outside the core containing the plasma with an effective viscosity μ_0. The viscosity μ_0, in general, will be estimated to be higher than the true plasma viscosity, μ_{pl}. The shear rates are assumed high enough so that the fluid in each region can be regarded as Newtonian. Thus we consider flow of two immiscible Newtonian fluids, consisting of a cylindrical core surrounded by a less viscous annular cell-free layer, with viscosity being discontinuous at the interface between the two regions.

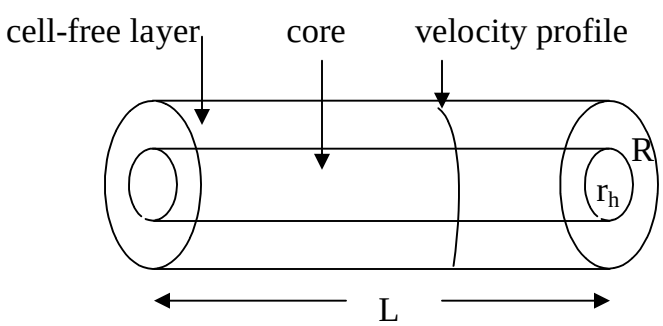

Fig. 1. Schematic diagram of the model.

2.1. Conservation equations and boundary conditions

The equations of motion for incompressible steady fully developed flow in a tube reduces to

$$-\frac{\partial p}{\partial z}+\frac{\mu_c}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_c}{\partial r}\right) = 0, \qquad 0 \le r \angle r_h \qquad 2.1(a)$$

for the central core with red blood cells and

$$-\frac{\partial p}{\partial z}+\frac{\mu_0}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_0}{\partial r}\right) = 0, \qquad r_h \angle r \angle R \qquad 2.1(b)$$

for the cell-free layer, where μ_c and μ_0 are respectively the velocities in the core and the plasma layer in the axial direction, p is the hydraulic pressure and r and z represent the axial directions in the tube.

The flow is subject to the following boundary conditions

(a) Due to symmetry, the velocity gradient vanishes along the axis of the tube

$$\frac{\partial u_c}{\partial r} = 0 \qquad \text{at} \quad r = 0. \qquad 2.1(c)$$

(b) No slip condition is assumed at the wall

$$u_0 = 0 \qquad \text{at} \quad r = R. \qquad 2.1(d)$$

(c) The velocity and shear stress are continuous at the interface of plasma and the core

(i) $$u_c\Big|_{r = r_h} = u_0\Big|_{r = r_h}, \qquad 2.1(e)$$

(ii) $$\mu_c \frac{\partial u_c}{\partial r}\Big|_{r = r_h} = \mu_0 \frac{\partial u_0}{\partial r}\Big|_{r = r_h}. \qquad 2.1(f)$$

The solution to Equations 2.1(a) and 2.1(b), subject to the boundary conditions 2.1(c) -2.1(f), is given by

$$u_c(\xi) = \frac{PR^2}{4\mu_0}\left[1-\lambda^2+\frac{\mu_0}{\mu_c}(\lambda^2-\xi^2)\right], \qquad 0 \le \xi \le \lambda, \tag{2.1(g)}$$

$$u_0(\xi) = \frac{PR^2}{4\mu_0}\left[1-\xi^2\right], \qquad \lambda \le \xi \le 1, \tag{2.1(h)}$$

where

$$\xi = \frac{r}{R}, \qquad \lambda = \frac{r_h}{R} \quad \text{and} \quad P = \frac{\Delta p}{L},$$

in which Δp represents the pressure drop along the length L of the tube.

Velocity in the core can be expressed as

$$u_c(\xi) = u_{max}(1-B\xi^2), \tag{2.1(i)}$$

where

$$u_{max} = \frac{PR^2}{4\mu_0}\left\{1-\lambda^2\left(1-\frac{\mu_0}{\mu_c}\right)\right\}, \qquad B = \frac{\mu_0/\mu_c}{1-\lambda^2(1-\mu_0/\mu_c)} \quad . \tag{2.1(j)}$$

The parameter B is the bluntness of the velocity profile and represents the deviation from parabolic flow. When $\mu_c \to \mu_0$, $B \to 1$ and the velocity profile becomes parabolic throughout the entire cross section of the tube.

The volumetric flow rate of the blood is given by

$$Q = 2\pi R^2\int_0^\lambda u_c(\xi)\xi d\xi + 2\pi R^2\int_\lambda^1 \mu_0(\xi)\xi d\xi \quad . \tag{2.1(k)}$$

Overall mass balance of the cells in the tube is defined by

$$QH_D = 2\pi R^2\int_0^1 \xi u(\xi)h(\xi)\, d\xi, \tag{2.1(l)}$$

in which H_D is the discharge hematocrit and

$$h(\xi) = \begin{cases} H_c, & 0 \le \xi \le \lambda, \\ 0, & \lambda \angle \xi \le 1. \end{cases} \tag{2.1(m)}$$

Evaluating the integrals in 2.1(k) and 2.1(l) with the velocity from equation 2.1(g), we get

$$Q = \frac{\pi P R^4}{8\mu_0}\left[\frac{\mu_0}{\mu_c}\lambda^4 + 1 - \lambda^4\right], \qquad 2.1(n)$$

$$QH_D = \frac{\pi P R^4 H_c}{8\mu_0}\left[\frac{\mu_0}{\mu_c}\lambda^4 + 2\lambda^2(1-\lambda^2)\right] \quad . \qquad 2.1(o)$$

Equation 2.1(n) can be written as

$$Q = \frac{\pi P R^4}{8\mu_{app}} \quad , \qquad 2.1(p)$$

where μ_{app} is the apparent total tube flow viscosity given by

$$\mu_{app} = \mu_0\left[\frac{\mu_0}{\mu_c}\lambda^4 + 1 - \lambda^4\right]^{-1} \quad . \qquad 2.1(q)$$

Notice that as $\lambda \to 1$, $\mu_{app} \to \mu_c$, which corresponds to the bulk viscosity of the blood in a large tube at high shear rates.

The tube hematocrit H_T is defined by

$$H_T = 2\int_0^1 h(\xi)\xi \, d\xi \quad , \qquad 2.1(r)$$

which yields

$$H_T = \lambda^2 H_c \ . \qquad 2.1(s)$$

Eliminating Q from equations 2.1(n) and 2.1(o), we get

$$\frac{H_c}{H_D} = \frac{(\mu_0/\mu_c)\lambda^4 + 1 - \lambda^4}{(\mu_0/\mu_c)\lambda^4 + 1 - \lambda^4 - (1-\lambda^2)^2} \quad . \qquad 2.1(t)$$

From 2.1(t), the ratio $H_c / H_D \geq 1$ if and only if $\lambda \leq 1$. This shows that the core hematocrit must be large than the discharge hematocrit to ensure that the core diameter is less than the tube diameter.

The maximum velocity u_{max} can be expressed in terms of apparent viscosity and average velocity of the blood ($\overline{U}$) as

$$\frac{u_{max}}{\overline{U}} = \frac{2\mu_{app}}{\mu_0}\left[1 - \lambda^2 + \frac{\mu_0}{\mu_c}\lambda^2\right] \quad . \qquad 2.1(u)$$

The shear stress at the wall is obtained as

$$\tau_w = \frac{PR}{2} = \frac{4\mu_{app}\overline{U}}{R} \ , \qquad 2.1(v)$$

i.e., the expression for the shear stress at the wall in this flow model is the same as that in homogeneous flow if the bulk viscosity is replaced with the apparent viscosity. Equations 2.1(q), 2.1(s) and 2.1(t) allow us to express μ_{app}, H_D and H_C in terms of λ, H_D, μ_0 and μ_c. In general, the tube diameter, D and discharge hematocrit, H_D, may be regarded as independent variables, while μ_{app} and H_T are available from experimental data in terms D and H_D.

3. Non-Newtonian model

By virtue of the complex nature of its particular suspension, blood has defined characterization by a comprehensive constitutive theory. However, it can be confidently stated that, in relatively large tubes, it can well be regarded as a continuum material.

3.1. Momentum and constitutive equations

Employing a continuum approximation, we model blood as a constant density incompressible fluid. The momentum equation in direct notation is given by

$$\rho \frac{Dv}{Dt} = \nabla . \sigma \ , \qquad 3.1(a)$$

where σ is the Cauchy stress tensor and v is the divergent-free velocity field. Below, we develop a constitutive equation for a shear-thinning, frame-invariant, nonlinearly viscous fluid, which we use in conjunction with the mass and momentum conservation equations to obtain a closed model of the system.

Under general flow conditions, the Cauchy stress tensor, σ, for a fluid is the sum of the isotropic part due to the pressure, p and the deviatoric stress tensor, τ, according to

$$\sigma = \text{-pI} + \tau \ . \qquad 3.1(b)$$

Based on the viscometric data of Chien et al. [7], we assume the viscosity, μ, is only a function of the rate-of-deformation tensor, D and propose the constitutive relationship given by

$$\tau = 2\mu(\mathrm{D})\mathrm{D} \ . \qquad 3.1(c)$$

Note that since D is frame invariant, the latter expression satisfies the principle of material frame indifference [16].

Combining the constitutive equation with the momentum equation, we obtain the material-specific momentum equation given by

$$\rho \frac{\mathrm{Dv}}{\mathrm{Dt}} = -\nabla \mathrm{p} + 2\mu(\mathrm{D})\nabla.\mathrm{D} + 2\mathrm{D}(\nabla\mu(\mathrm{D})) \ . \qquad 3.1(d)$$

$$\rho \frac{\mathrm{Dv_i}}{\mathrm{Dt}} = -\frac{\partial \mathrm{P}}{\partial \mathrm{x_i}} + 2\mu(\mathrm{D})\frac{\partial \mathrm{D_{ij}}}{\partial \mathrm{x_j}} + 2\mathrm{D_{im}}\frac{\partial\mu(\mathrm{D})}{\partial \mathrm{D_{mn}}}\frac{\partial \mathrm{D_{nk}}}{\partial \mathrm{x_k}} \ . \qquad 3.1(e)$$

Introducing the Laplacian of the velocity field, where, for an incompressible fluid,

$$\nabla^2 \mathrm{v} = \text{-}\nabla \times \nabla \times \mathrm{v} = 2\nabla.\mathrm{D} \ . \qquad 3.1(f)$$

We can write 3.1(d) as

$$\rho \frac{\mathrm{Dv}}{\mathrm{Dt}} = -\nabla.(\mathrm{pI}) + \mu(\mathrm{D})\nabla^2\mathrm{v} + \mathrm{D}\frac{\partial\mu(\mathrm{D})}{\partial \mathrm{D}}\nabla^2\mathrm{v} \ . \qquad 3.1(g)$$

Notice that in the high shear-rate limit, where μ is independent of D, 3.1(g) reduces to the Navier-Stokes equation for an incompressible fluid with constant viscosity.

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