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Marichev-Saigo-Maeda operational relationships of Multivariable Aleph-functions

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ABSTRACT
In this document, we introduced product of multivariable Aleph-functions, general class of polynomials and M-series by using Marichev-Saigo-Maeda fractional integral operator and we find out some image formulae by employing product of multivariable Aleph-functions, general class of polynomials and M-series and generalized fractional integral operator due to Marichev-Saigo-Maeda involving Appell's function $F_3(.)$.

Keywords :Multivariable Aleph-function, general class of polynomials, Marichev-Saigo-Maeda fractional integral operator, Pathway integral operator M-serie.

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1.Introduction

Let $\mu,\mu',\upsilon,\upsilon',\eta\in\mathbb{C}(Re(\eta)>0)$ and $x>0$, then the generalized fractional integral operator is defined as :

$$\left(I_{0,x}^{\mu,\mu',\upsilon,\upsilon',\eta}\right)f(x)=\frac{x^{-\mu}}{\Gamma(\eta)}\int_0^x(x-t)^{\eta-1}t^{-\mu'}F_3\left(\mu,\mu',\upsilon,\upsilon';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)\mathrm{d}t \tag{1.1}$$

$$\left(I_{x,\infty}^{\mu,\mu',\upsilon,\upsilon',\eta}\right)f(x)=\frac{x^{-\mu'}}{\Gamma(\eta)}\int_x^\infty(t-x)^{\eta-1}t^{-\mu'}F_3\left(\mu,\mu',\upsilon,\upsilon';\eta;1-\frac{x}{t},1-\frac{t}{x}\right)f(t)\mathrm{d}t \tag{1.2}$$

where $F_3(.)$ is Appell's function (also see Erdelyi [1]) defined by :

$$F_3\left(\mu,\mu',\upsilon,\upsilon';\eta;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\mu)_m(\mu')_n(\upsilon)_m(\upsilon')_n}{(\eta)_{m+n}}\frac{x^my^n}{m!n!}\ ;\ \left(max\{|x|,|y|\}<1\right) \tag{1.3}$$

The generalized fractional integral operator defined through equations (1.1) and(1.2) has been introduced by Marichev [2] and later extended and studied by Saigo and Maeda [11]. These operators are known as the Marichev-Saigo-Maeda operators and properties of these operators were studied by Saigo and Maeda [4]. These fractional integral operators have many interesting applications in various subfields of mathematical analysis.

In this document, we use the following results due to Saigo and Maeda [4]

$$\left(I_{0,x}^{\mu,\mu',\upsilon,\upsilon',\eta}t^{\rho-1}\right)(x)=\frac{\Gamma(\rho)\Gamma(\rho+\eta-\mu-\mu'-\upsilon)\Gamma(\rho+\upsilon'-\mu')}{\Gamma(\rho+\upsilon')\Gamma(\rho+\eta-\mu-\mu')\Gamma(\rho+\eta-\mu'-\upsilon)}x^{\rho+\eta-\mu-\mu'-1} \tag{1.4}$$

where $Re(\eta)>0,Re(\rho)>max\left\{0,Re(\mu+\mu'+\upsilon-\eta),Re(\mu'-\upsilon')\right\}$

$$\left(I_{x,\infty}^{\mu,\mu',\upsilon,\upsilon',\eta}t^{\rho-1}\right)(x)=\frac{\Gamma(1-\rho-\upsilon)\Gamma(1-\rho-\eta+\mu+\mu')\Gamma(1-\rho-\eta+\mu+\upsilon')}{\Gamma(1-\rho)\Gamma(1-\rho-\eta+\mu+\mu'+\upsilon')\Gamma(1-\rho+\eta+\mu-\upsilon)}x^{\rho+\eta-\mu-\mu'-1} \tag{1.5}$$

where $Re(\eta)>0,0<Re(\rho)<max\left\{Re(-\upsilon),Re(\mu+\mu'-\eta),Re(\mu+\upsilon'-\eta)\right\}$

The generalized polynomials of multivariables defined by Srivastava [9], is given in the following manner :

$$S_{N_1,\cdots,N_u}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_u}[y_1,\cdots,y_u]=\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_u=0}^{[N_u/\mathfrak{M}_u]}\frac{(-N_1)_{\mathfrak{M}_1K_1}}{K_1!}\cdots\frac{(-N_u)_{\mathfrak{M}_uK_u}}{K_u!}$$

$$A[N_1,K_1;\cdots;N_u,K_u]y_1^{K_1}\cdots y_u^{K_u} \tag{1.6}$$

Where $\mathfrak{M}_1, \cdots, \mathfrak{M}_u$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \cdots; N_u, K_u]$ are arbitrary constants, real or complex.

Srivastava and Garg introduced and defined a general class of multivariable polynomials [11] as follows

$$S_E^{F_1,\cdots,F_\mathfrak{v}}[z_1,\cdots,z_\mathfrak{v}]=\sum_{L_1,\cdots,L_\mathfrak{v}=0}^{F_1L_1+\cdots F_\mathfrak{v}L_\mathfrak{v}\leqslant E}(-E)_{F_1L_1+\cdots+F_\mathfrak{v}L_\mathfrak{v}}B(E;L_1,\cdots,L_\mathfrak{v})\frac{z_1^{L_1}\cdots z_\mathfrak{v}^{L_\mathfrak{v}}}{L_1!\cdots L_\mathfrak{v}!} \tag{1.7}$$

The M-serie is defined, see Sharma [8].

$${}_{p'}M_{q'}^{\alpha}(y)=\sum_{s'=0}^{\infty}\frac{[(a_{p'})]_{s'}}{[(b_{q'})]_{s'}}\frac{y^{s'}}{\Gamma(\alpha s'+1)} \tag{1.8}$$

Here $\alpha \in \mathbb{C}, Re(\alpha) > 0$. $[(a_{p'})]_{s'} = (a_1)_{s'}\cdots(a_{p'})_{s'}$; $[(b_{q'})]_{s'} = (b_1)_{s'}\cdots(b_{q'})_{s'}$.
The serie (1.23) converge if $p' \leqslant q' and\, |y| < 1$.

In the document , we note :

$$a=\frac{(-N_1)_{\mathfrak{M}_1K_1}}{K_1!}\cdots\frac{(-N_u)_{\mathfrak{M}_uK_u}}{K_u!}A[N_1,K_1;\cdots;N_u,K_u] \tag{1.9}$$

$$b=\frac{(-E)_{F_1L_1+\cdots+F_\mathfrak{v}L_\mathfrak{v}}B(E;L_1,\cdots,L_\mathfrak{v})}{L_1!\cdots L_\mathfrak{v}!} \tag{1.10}$$

2. Multivariable Aleph-function

The function Aleph of several variables generalize the multivariable I-function recently study by C.K. Sharma and Ahmad [5] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We define : $\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{matrix}\mathrm{y}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{y}_r\end{matrix}\right|$

$$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}]\ ,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ ,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}]:$$

$$\left.\begin{matrix}[(\mathrm{c}_j^{(1)});\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;[(\mathrm{c}_j^{(r)});\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}]\\ [(\mathrm{d}_j^{(1)});\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;[(\mathrm{d}_j^{(r)});\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\end{matrix}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)y_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{2.2}$$

and $\theta_k(s_k) = \dfrac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]}$ (2.3)

Suppose , as usual , that the parameters

$a_j, j = 1,\cdots,p; b_j, j = 1,\cdots,q;$

$c_j^{(k)}, j = 1,\cdots,n_k; c_{ji^{(k)}}^{(k)}, j = n_k+1,\cdots,p_{i^{(k)}};$

$d_j^{(k)}, j = 1,\cdots,m_k; d_{ji^{(k)}}^{(k)}, j = m_k+1,\cdots,q_{i^{(k)}};$

with $k = 1\cdots,r, i = 1,\cdots,R \ , i^{(k)} = 1,\cdots,R^{(k)}$

are complex numbers , and the $\alpha's, \beta's, \gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)} + \tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} + \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)} - \tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)} - \sum_{j=1}^{m_k}\delta_j^{(k)}$$

$$-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{2.4}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)}s_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r}\alpha_j^{(k)}s_k)$ with $j = 1$ to n and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)}s_k)$ with $j = 1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)} - \tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)} - \tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} - \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)} - \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1\cdots,r, i = 1,\cdots,R \ , i^{(k)} = 1,\cdots,R^{(k)} \tag{2.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(y_1,\cdots,y_r) = 0(\ |y_1|^{\alpha_1}\ldots|y_r|^{\alpha_r}\ ), max(\ |y_1|\ldots|y_r|\ ) \rightarrow 0$

$\aleph(y_1,\cdots,y_r) = 0(\ |y_1|^{\beta_1}\ldots|y_r|^{\beta_r}\ ), min(\ |y_1|\ldots|y_r|\ ) \rightarrow \infty$

where, with $k = 1,\cdots,r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1,\cdots,m_k$ and

$\beta_k = max[Re((c_j^{(k)}-1)/\gamma_j^{(k)})], j=1,\cdots,n_k$

Serie representation of Aleph-function of several variables is given by

$$\aleph(y_1,\cdots,y_r) = \sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m_1}\cdots\sum_{g_r=0}^{m_r}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}\psi(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})$$

$$\times\ \theta_1(\eta_{G_1,g_1})\cdots\theta_r(\eta_{G_r,g_r})y_1^{-\eta_{G_1,g_1}}\cdots y_r^{-\eta_{G_r,g_r}} \tag{2.6}$$

Where $\psi(.,\cdots,.), \theta_i(.), i=1,\cdots,r$ are given respectively in (1.2), (1.3) and

$$\eta_{G_1,g_1} = \frac{d_{g_1}^{(1)}+G_1}{\delta_{g_1}^{(1)}},\cdots,\ \eta_{G_r,g_r} = \frac{d_{g_r}^{(r)}+G_r}{\delta_{g_r}^{(r)}}$$

which is valid under the conditions $\delta_{g_i}^{(i)}[d_j^i+p_i] \neq \delta_j^{(i)}[d_{g_i}^i+G_i]$ (2.7)

for $j \neq m_i, m_i = 1,\cdots \eta_{G_i,g_i}; p_i, n_i = 0,1,2,\cdots,; y_i \neq 0, i=1,\cdots,r$ (2.8)

Consider the Aleph-function of s variables

$$\aleph(z_1,\cdots,z_s) = \aleph^{0,N:M_1,N_1,\cdots,M_s,N_s}_{P_i,Q_i,\iota_i;r:P_{i^{(1)}},Q_{i^{(1)}},\iota_{i^{(1)}};r^{(1)};\cdots;P_{i^{(s)}},Q_{i^{(s)}};\iota_{i^{(s)}};r^{(s)}}\left(\begin{array}{c|}\mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_s\end{array}\right.$$

$$\begin{array}{ll}[(\mathrm{u}_j;\mu_j^{(1)},\cdots,\mu_j^{(r)})_{1,N}] & ,[\iota_i(u_{ji};\mu_{ji}^{(1)},\cdots,\mu_{ji}^{(r)})_{N+1,P_i}]:\\ \cdots\cdots\cdots\cdots\cdots\cdots & ,[\iota_i(v_{ji};\upsilon_{ji}^{(1)},\cdots,\upsilon_{ji}^{(r)})_{M+1,Q_i}]:\end{array}$$

$$\left.\begin{array}{l}[(\mathrm{a}_j^{(1)});\alpha_j^{(1)})_{1,N_1}],[\iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)};\alpha_{ji^{(1)}}^{(1)})_{N_1+1,P_i^{(1)}}];\cdots;[(\mathrm{a}_j^{(s)});\alpha_j^{(s)})_{1,N_s}],[\iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)};\alpha_{ji^{(s)}}^{(s)})_{N_s+1,P_i^{(s)}}]\\ [(\mathrm{b}_j^{(1)});\beta_j^{(1)})_{1,M_1}],[\iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)};\beta_{ji^{(1)}}^{(1)})_{M_1+1,Q_i^{(1)}}];\cdots;[(\mathrm{b}_j^{(s)});\beta_j^{(s)})_{1,M_s}],[\iota_{i^{(s)}}(b_{ji^{(s)}}^{(s)};\beta_{ji^{(s)}}^{(s)})_{M_s+1,Q_i^{(s)}}]\end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^s}\int_{L_1}\cdots\int_{L_r}\zeta(t_1,\cdots,t_s)\prod_{k=1}^{s}\phi_k(t_k)z_k^{-t_k}\,\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{2.9}$$

with $\omega = \sqrt{-1}$

$$\zeta(t_1,\cdots,t_s) = \frac{\prod_{j=1}^{N}\Gamma(1-u_j+\sum_{k=1}^{s}\mu_j^{(k)}t_k)}{\sum_{i=1}^{r'}[\iota_i\prod_{j=N+1}^{P_i}\Gamma(u_{ji}-\sum_{k=1}^{s}\mu_{ji}^{(k)}t_k)\prod_{j=1}^{Q_i}\Gamma(1-\upsilon_{ji}+\sum_{k=1}^{s}\upsilon_{ji}^{(k)}t_k)]} \tag{2.10}$$

and $$\phi_k(t_k) = \frac{\prod_{j=1}^{M_k}\Gamma(b_j^{(k)}-\beta_j^{(k)}t_k)\prod_{j=1}^{N_k}\Gamma(1-a_j^{(k)}+\alpha_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{r^{(k)}}[\iota_{i^{(k)}}\prod_{j=M_k+1}^{Q_{i^{(k)}}}\Gamma(1-b_{ji^{(k)}}^{(k)}+\beta_{ji^{(k)}}^{(k)}t_k)\prod_{j=N_k+1}^{P_{i^{(k)}}}\Gamma(a_{ji^{(k)}}^{(k)}-\alpha_{ji^{(k)}}^{(k)}s_k)]} \tag{2.11}$$

Suppose , as usual , that the parameters

$u_j, j = 1, \cdots, P; v_j, j = 1, \cdots, Q;$

$a_j^{(k)}, j = 1, \cdots, N_k; a_{ji^{(k)}}^{(k)}, j = n_k + 1, \cdots, P_{i^{(k)}};$

$b_{ji^{(k)}}^{(k)}, j = m_k + 1, \cdots, Q_{i^{(k)}}; b_j^{(k)}, j = 1, \cdots, M_k;$

with $k = 1 \cdots, s, i = 1, \cdots, r', i^{(k)} = 1, \cdots, r^{(k)}$

are complex numbers , and the $\alpha' s, \beta' s, \gamma' s$ and $\delta' s$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{N} \mu_j^{(k)} + \iota_i \sum_{j=N+1}^{P_i} \mu_{ji}^{(k)} + \sum_{j=1}^{N_k} \alpha_j^{(k)} + \iota_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \alpha_{ji^{(k)}}^{(k)} - \iota_i \sum_{j=1}^{Q_i} \upsilon_{ji}^{(k)} - \sum_{j=1}^{M_k} \beta_j^{(k)}$$

$$- \iota_{i^{(k)}} \sum_{j=M_k+1}^{Q_{i^{(k)}}} \beta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{2.12}$$

The reals numbers ι_i are positives for $i = 1, \cdots, r$, $\iota_{i^{(k)}}$ are positives for $i^{(k)} = 1 \cdots r^{(k)}$

The contour L_k is in the t_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(b_j^{(k)} - \beta_j^{(k)} t_k)$ with $j = 1$ to M_k are separated from those of $\Gamma(1 - u_j + \sum_{i=1}^{s} \mu_j^{(k)} t_k)$with $j = 1$ to N and $\Gamma(1 - a_j^{(k)} + \alpha_j^{(k)} t_k)$ with $j = 1$ to N_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|arg z_k| < \frac{1}{2} B_i^{(k)} \pi$, where

$$B_i^{(k)} = \sum_{j=1}^{N} \mu_j^{(k)} - \iota_i \sum_{j=N+1}^{P_i} \mu_{ji}^{(k)} - \iota_i \sum_{j=1}^{Q_i} \upsilon_{ji}^{(k)} + \sum_{j=1}^{N_k} \alpha_j^{(k)} - \iota_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \alpha_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{M_k} \beta_j^{(k)} - \iota_{i^{(k)}} \sum_{j=M_k+1}^{q_{i^{(k)}}} \beta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1 \cdots, s, i = 1, \cdots, r, i^{(k)} = 1, \cdots, r^{(k)} \tag{2.13}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_s) = 0(|z_1|^{\alpha'_1} \dots |z_s|^{\alpha'_s}), max(|z_1| \dots |z_s|) \to 0$
$\aleph(z_1, \cdots, z_s) = 0(|z_1|^{\beta'_1} \dots |z_s|^{\beta'_s}), min(|z_1| \dots |z_s|) \to \infty$

where, with $k = 1, \cdots, z : \alpha'_k = min[Re(b_j^{(k)} / \beta_j^{(k)})], j = 1, \cdots, M_k$ and

$\beta'_k = max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \cdots, N_k$

We will use these following notations in this paper

$$U = P_i, Q_i, \iota_i; r' ; \ V = M_1, N_1; \cdots; M_s, N_s \tag{2.15}$$

$$\mathrm{W} = P_{i^{(1)}}, Q_{i^{(1)}}, \iota_{i^{(1)}}; r^{(1)}, \cdots, P_{i^{(r)}}, Q_{i^{(r)}}, \iota_{i^{(s)}}; r^{(s)} \tag{2.16}$$

$$A = \{(u_j;\mu_j^{(1)},\cdots,\mu_j^{(s)})_{1,N}\}, \{\iota_i(u_{ji};\mu_{ji}^{(1)},\cdots,\mu_{ji}^{(s)})_{N+1,P_i}\} \tag{2.17}$$

$$B = \{\iota_i(v_{ji};\upsilon_{ji}^{(1)},\cdots,\upsilon_{ji}^{(s)})_{M+1,Q_i}\} \tag{2.18}$$

$$C = (a_j^{(1)};\alpha_j^{(1)})_{1,N_1}, \iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)};\alpha_{ji^{(1)}}^{(1)})_{N_1+1,P_{i^{(1)}}}, \cdots, (a_j^{(s)};\alpha_j^{(s)})_{1,N_s}, \iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)};\alpha_{ji^{(s)}}^{(s)})_{N_s+1,P_{i^{(s)}}} \tag{2.19}$$

$$D = (b_j^{(1)};\beta_j^{(1)})_{1,M_1}, \iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)};\beta_{ji^{(1)}}^{(1)})_{M_1+1,Q_{i^{(1)}}}, \cdots, (b_j^{(s)};\beta_j^{(s)})_{1,M_s}, \iota_{i^{(s)}}(\beta_{ji^{(s)}}^{(s)};\beta_{ji^{(s)}}^{(s)})_{M_s+1,Q_{i^{(s)}}} \tag{2.20}$$

The multivariable Aleph-function write :

$$\aleph(z_1,\cdots,z_s) = \aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A} : \mathrm{C} \\ \cdot & \cdots \\ \cdot & \mathrm{B} : \mathrm{D} \\ \mathrm{z}_s & \end{array}\right) \tag{2.21}$$

In the document , we note :

$$G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r}) = \phi(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})\theta_1(\eta_{G_1,g_1})\cdots\theta_r(\eta_{G_r,g_r}) \tag{2.22}$$

3. Main results

We have the following formula:

$$I_{0,x}^{\mu,\mu',\upsilon,\upsilon',\eta}\left(t^{\rho-1}\ S_E^{F_1,\cdots,F_{\mathfrak{v}}}[z_1t^{n'_1},\cdots,z_{\mathfrak{v}}t^{n'_{\mathfrak{v}}}]_{p'}M_{q'}^{\alpha}(\tau t^{l'})S_{N_1,\cdots,N_u}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_u}[y_1t^{n_1},\cdots,y_ut^{n_u}]\right.$$

$$\left.\aleph(x'_1t^{\sigma'_1},\cdots,x'_rt^{\sigma'_r})\aleph(x_1t^{\sigma_1},\cdots,x_st^{\sigma_s})\right)$$

$$= x^{\rho+\eta-\mu-\mu'-1}\sum_{k,l=0}^{\infty}\ \sum_{G_1,\cdots,G_r=0}^{\infty}\ \sum_{g_1=0}^{m_1}\cdots\sum_{g_r=0}^{m_r}\ \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_u=0}^{[N_u/\mathfrak{M}_u]}\ \sum_{L_1,\cdots,L_{\mathfrak{v}}=0}^{F_1L_1+\cdots F_{\mathfrak{v}}L_{\mathfrak{v}}\leqslant E}\ \sum_{L=0}^{\infty} ab$$

$$\frac{(-)^l(\mu)_k(\mu')_l(\upsilon)_k(\upsilon')_l}{k!l!}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})(x'_1x^{\sigma'1})^{-\eta_{G_1,g_1}}\cdots(x'_rx^{\sigma'r})^{-\eta_{G_r,g_r}}$$

$$\frac{[(a_{p'})]_L}{[(b_{q'})]_L}\frac{\tau^L}{\Gamma(\alpha L+1)}\ (y_1x^{n_1})^{K_1}\cdots(y_ux^{n_u})^{K_u}\ (z_1x^{n'_1})^{L_1}\cdots(z_{\mathfrak{v}}x^{n'_{\mathfrak{v}}})^{L_{\mathfrak{v}}}\ \aleph_{U_{11}:W}^{0,N+1:V}\left(\begin{array}{c|c} \mathrm{x}_1x^{\sigma_1} & (1\text{-}\rho+l+\mu'+\sum_{i=1}^r\sigma'_i\eta_{G_i,g_i}-\sum_{i=1}^uK_in_i-\sum_{i=1}^{\mathfrak{v}}L_in'_i-Ll';\sigma_1,\cdots,\sigma_s),A:C \\ \cdot & \cdots \\ \cdot & (1\text{-}\rho-\eta-k+\mu'+\sum_{i=1}^r\sigma'_i\eta_{G_i,g_i}-\sum_{i=1}^uK_in_i-\sum_{i=1}^{\mathfrak{v}}L_in'_i-Ll';\sigma_1,\cdots,\sigma_s),B:D \\ \mathrm{x}_sx^{\sigma_s} & \end{array}\right) \tag{3.1}$$

where a and b, A, B, C, D are defined respectively in (1.9) and (1.10) ,(2.17), (2.18), (2.19) and (2.20) and

$U_{11} = P_i+1, Q_i+1, \iota_i; r'$; $G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})$ is defined by (2.22).

Provided that

a) $x>0, Re(\rho)>0, \mu,\mu',\upsilon,\upsilon',\eta\in\mathbb{C}; p'\leqslant q' and |\tau|<1, l'>0$

b) $Re[\rho_1+\sum_{i=1}^{r}\sigma'_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}+\sum_{i=1}^{s}\sigma_i\min_{1\leqslant j\leqslant M_i}\frac{b_j^{(i)}}{\beta_j^{(i)}}]>max\{0, Re(\mu+\mu'+\upsilon-\eta), Re(\mu'-\upsilon')\}$

where $\rho_1=\rho+\sum_{i=1}^{u}K_in_i+\sum_{i=1}^{v}L_in'_i+Ll'$

c) $z_i\in\mathbb{R}, i=1,\cdots,\mathfrak{v},\ y_i\in\mathbb{R}, i=1,\cdots,u$

d)$Re(n'_i)>0, i=1,\cdots,\mathfrak{v},\ Re(n_i)>0, i=1,\cdots,u, Re(\sigma'_i)>0, i=1,\cdots,r, Re(\eta)>0$

$Re(\sigma_i)>0, i=1,\cdots,s$

e) $|argx'_k|<\frac{1}{2}A_i^{(k)}\pi$; $|argx_k|<\frac{1}{2}B_i^{(k)}\pi$, where $A_i^{(k)}$ and $B_i^{(k)}$ are defined respectively by (2.5) and (2.13)

Proof of (3.1)

Let $M=\frac{1}{(2\pi\omega)^s}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_s)\prod_{k=1}^{s}\theta_k(s_k)$

Making an appeal to (1.1), (1.3), (1.6), (1.7), (1.8), (2.6) and (2.9), we get

$$I_{0,x}^{\mu,\mu',\upsilon,\upsilon',\eta}\big(t^{\rho-1}\ S_E^{F_1,\cdots,F_{\mathfrak{v}}}[z_1t^{n'_1},\cdots,z_{\mathfrak{v}}t^{n'_{\mathfrak{v}}}]_{p'}M_{q'}^{\alpha}(\tau t^l)S_{N_1,\cdots,N_u}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_u}[y_1t^{n_1},\cdots,y_ut^{n_u}]$$

$$\aleph(x'_1t^{\sigma'_1},\cdots,x'_rt^{\sigma'_r})\aleph(x_1t^{\sigma_1},\cdots,x_st^{\sigma_s})\big)$$

$$=\frac{x^{-\mu}}{\Gamma(\eta)}\int_0^x t^{\rho-1}(x-t)^{\eta-1}t^{-\mu'}\Big(\sum_{k,l=0}^{\infty}\frac{(-)^l(\mu)_k(\mu')_l(\upsilon)_k(\upsilon')_l}{(\eta)_{k+l}k!l!}(1-t/x)^k(1-x/t)^l\Big)$$

$$\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m_1}\cdots\sum_{g_r=0}^{m_r}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})(x'_1t^{\sigma'1})^{-\eta_{G_1,g_1}}\cdots(x'_rt^{\sigma'r})^{-\eta_{G_r,g_r}}$$

$$\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_u=0}^{[N_u/\mathfrak{M}_u]}\sum_{L_1,\cdots,L_{\mathfrak{v}}=0}^{F_1L_1+\cdots F_{\mathfrak{v}}L_{\mathfrak{v}}\leqslant E}\sum_{L=0}^{\infty}ab\frac{[(a_{p'})]_L}{[(b_{q'})]_L}\frac{\tau^L}{\Gamma(\alpha L+1)}\ y_1^{K_1}\cdots y_u^{K_u}z_1^{L_1}\cdots z_{\mathfrak{v}}^{L_{\mathfrak{v}}}$$

$$t^{\sum_{i=1}^{u}K_in_i+\sum_{i=1}^{v}L_in'_i+Ll'}\quad Mx_1^{-t_1}\cdots x_s^{-t_s}t^{\sum_{i=1}^{s}\sigma_it_i}\qquad \mathrm{d}t_1\cdots\mathrm{d}t_s\ \mathrm{d}t$$

Now changing the order of integrations and summation, under the given conditions, we obtain

$$I_{0,x}^{\mu,\mu',\upsilon,\upsilon',\eta}\big(t^{\rho-1}\ S_E^{F_1,\cdots,F_{\mathfrak{v}}}[z_1t^{n'_1},\cdots,z_{\mathfrak{v}}t^{n'_{\mathfrak{v}}}]_{p'}M_{q'}^{\alpha}(\tau t^l)S_{N_1,\cdots,N_u}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_u}[y_1t^{n_1},\cdots,y_ut^{n_u}]$$

$$\aleph(x'_1t^{\sigma'_1},\cdots,x'_rt^{\sigma'_r})\aleph(x_1t^{\sigma_1},\cdots,x_st^{\sigma_s})\big)$$

$$=\frac{x^{-\mu}}{\Gamma(\eta)}\Big(\sum_{k,l=0}^{\infty}\frac{(-)^l(\mu)_k(\mu')_l(\upsilon)_k(\upsilon')_l}{(\eta)_{k+l}k!l!}\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m_1}\cdots\sum_{g_r=0}^{m_r}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})$$

$$\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_u=0}^{[N_u/\mathfrak{M}_u]}\sum_{L_1,\cdots,L_\mathfrak{v}=0}^{F_1L_1+\cdots F_\mathfrak{v}L_\mathfrak{v}\leqslant E}\sum_{L=0}^{\infty}ab\frac{[(a_{p'})]_L}{[(b_{q'})]_L}\frac{\tau^L}{\Gamma(\alpha L+1)}x_1'^{-\eta_{G_1,g_1}}\cdots x_r'^{-\eta_{G_r,g_r}}y_1^{K_1}\cdots y_u^{K_u}$$

$$z_1^{L_1}\cdots z_\mathfrak{v}^{L_\mathfrak{v}}Mx_1^{-t_1}\cdots x_s^{-t_s}\Big(\int_0^x t^{\rho-\mu'+\sum_{i=1}^u K_in_i+\sum_{i=1}^v L_in_i'+Ll'-\sum_{i=1}^r\sigma_i'\eta_{G_i,g_i}-\sum_{i=1}^s\sigma_it_i-1}(x-t)^{\eta-1}$$

$$(1-t/x)^k(1-x/t)^l\,\mathrm{d}t\Big)\mathrm{d}t_1\cdots\mathrm{d}t_s$$

$$=\frac{x^{\eta-\mu-\mu'+\rho-1}}{\Gamma(\eta)}\sum_{k,l=0}^{\infty}\frac{(-)^l(\mu)_k(\mu')_l(\upsilon)_k(\upsilon')_l\Gamma(\eta+k+1)}{(\eta)_{k+l}k!l!}\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{M_1}\cdots\sum_{g_r=0}^{M_r}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}$$

$$G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})\ (x_1'x^{\sigma'1})^{-\eta_{G_1,g_1}}\cdots(x_r'x^{\sigma'r})^{-\eta_{G_r,g_r}}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_u=0}^{[N_u/\mathfrak{M}_u]}\sum_{L_1,\cdots,L_\mathfrak{v}=0}^{F_1L_1+\cdots F_\mathfrak{v}L_\mathfrak{v}\leqslant E}$$

$$\sum_{L=0}^{\infty}ab\frac{[(a_{p'})]_L}{[(b_{q'})]_L}\frac{(x\tau)^L}{\Gamma(\alpha L+1)}(y_1x^{n_1})^{K_1}\cdots(y_ux^{n_u})^{K_u}(z_1x^{n_1'})^{L_1}\cdots(z_\mathfrak{v}x^{n_\mathfrak{v}'})^{L_\mathfrak{v}}M(x_1x)^{-t_1}\cdots(x_sx)^{-t_s}$$

$$\frac{\Gamma(\rho-\mu'-1+\sum_{i=1}^u k_in_i+\sum_{i=1}^\mathfrak{v}L_in_i'+Ll'-\sum_{i=1}^r\sigma_i'\eta_{G_i,g_i}-\sum_{i=1}^s\sigma_it_i)}{\Gamma(\rho+\eta+k-\mu'-1+\sum_{i=1}^u K_in_i+\sum_{i=1}^\mathfrak{v}L_in_i'+Ll'-\sum_{i=1}^r\sigma_i'\eta_{G_i,g_i}-\sum_{i=1}^s\sigma_it_i)}\mathrm{d}t_1\cdots\mathrm{d}t_s$$

Now use the equation (2.9), we obtain the desired result.

Remark

We note $g(x)=\dfrac{x}{a(1-\alpha)}$ and $\eta*=1+\eta/(1-\alpha)$ (3.2)

Putting $k=l=0,\mu=0,\mu'=0,\upsilon=0,\upsilon'=0,\eta=1+\eta/(1-\alpha)$ in the equation (3.1), we get the following result

$$P_{0+}^{\eta,\alpha}\big(\,t^{\rho-1}\ S_E^{F_1,\cdots,F_\mathfrak{v}}[z_1t^{n_1'},\cdots,z_\mathfrak{v}t^{n_\mathfrak{v}'}]_{p'}M_{q'}^{\alpha}(\tau t^l)S_{N_1,\cdots,N_u}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_u}[y_1t^{n_1},\cdots,y_ut^{n_u}]$$

$$\aleph(x_1't^{\sigma_1'},\cdots,x_r't^{\sigma_r'})\aleph(x_1t^{\sigma_1},\cdots,x_st^{\sigma_s})\big)$$

$$=x^\eta(g(x))^\rho\ \Gamma(1+\eta*)\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m_1}\cdots\sum_{g_r=0}^{m_r}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_u=0}^{[N_u/\mathfrak{M}_u]}\sum_{L_1,\cdots,L_\mathfrak{v}=0}^{F_1L_1+\cdots F_\mathfrak{v}L_\mathfrak{v}\leqslant E}$$

$$\sum_{L=0}^{\infty}ab\quad\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})\frac{[(a_{p'})]_L}{[(b_{q'})]_L}\frac{\tau^L}{\Gamma(\alpha L+1)}\quad y_1^{K_1}\cdots y_u^{K_u}$$

$$z_1^{L_1}\cdots z_\mathfrak{v}^{L_\mathfrak{v}}\quad\big(g(x)\big)^{\sum_{i=1}^u K_in_i+\sum_{i=1}^{\mathfrak{L}}{}_in_i'-\sum_{i=1}^r\sigma_i'\eta_{G_i,g_i}}\quad\aleph_{U_{11}:W}^{0,N+1:V}\left(\begin{array}{c}\mathrm{x}_1(g(x))^{\sigma_1}\\ \cdot\\ \cdot\\ \mathrm{x}_s(g(x))^{\sigma_s}\end{array}\right|$$

$$\left.\begin{array}{c}(1\text{-}\rho+\sum_{i=1}^r\sigma_i'\eta_{G_i,g_i}-\sum_{i=1}^u K_in_i-\sum_{i=1}^\mathfrak{v}L_in_i'-Ll';\sigma_1,\cdots,\sigma_s),A:C\\ \cdots\\ (\text{-}\rho-\eta*+\sum_{i=1}^r\sigma_i'\eta_{G_i,g_i}-\sum_{i=1}^u K_in_i-\sum_{i=1}^\mathfrak{v}L_in_i'-Ll';\sigma_1,\cdots,\sigma_s),B:D\end{array}\right)\quad(3.3)$$

where $P_{0+}^{\eta,\alpha}$ is the Pathway operator, see Nair [3] for more details and $G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})$ is defined by (2.22).

4. Particular cases

a) If $\iota_i=\iota_{i^{(1)}}=\cdots=\iota_{i^{(s)}}=1$,and $\tau_i=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=1$ then the multivariable Aleph-functions degenere to the multivariable I-functions defined by Sharma et al [5]. And we have the following result.

$$I_{0,x}^{\mu,\mu',\upsilon,\upsilon',\eta}\Big(t^{\rho-1}\ S_E^{F_1,\cdots,F_{\mathfrak{v}}}[z_1t^{n'_1},\cdots,z_{\mathfrak{v}}t^{n'_{\mathfrak{v}}}]_{p'}M_{q'}^{\alpha}(\tau t^{l'})S_{N_1,\cdots,N_u}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_u}[y_1t^{n_1},\cdots,y_ut^{n_u}]$$

$$I(x'_1t^{\sigma'_1},\cdots,x'_rt^{\sigma'_r})I(x_1t^{\sigma_1},\cdots,x_st^{\sigma_s})\Big)$$

$$=x^{\rho+\eta-\mu-\mu'-1}\sum_{k,l=0}^{\infty}\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m_1}\cdots\sum_{g_r=0}^{m_r}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_u=0}^{[N_u/\mathfrak{M}_u]}\sum_{L_1,\cdots,L_{\mathfrak{v}}=0}^{F_1L_1+\cdots F_{\mathfrak{v}}L_{\mathfrak{v}}\leqslant E}\sum_{L=0}^{\infty}ab$$

$$\frac{(-)^l(\mu)_k(\mu')_l(\upsilon)_k(\upsilon')_l}{k!l!}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}G_1(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})(x'_1x^{\sigma'1})^{-\eta_{G_1,g_1}}\cdots(x'_rx^{\sigma'r})^{-\eta_{G_r,g_r}}$$

$$\frac{[(a_{p'})]_L}{[(b_{q'})]_L}\frac{\tau^L}{\Gamma(\alpha L+1)}\quad(y_1x^{n_1})^{K_1}\cdots(y_ux^{n_u})^{K_u}\ (z_1x^{n'_1})^{L_1}\cdots(z_{\mathfrak{v}}x^{n'_{\mathfrak{v}}})^{L_{\mathfrak{v}}}\ I_{U_{11}:W}^{0,N+1:V}\left(\begin{array}{c|}\mathrm{x}_1x^{\sigma_1}\\ \cdot\\ \cdot\\ \mathrm{x}_sx^{\sigma_s}\end{array}\right.$$

$$\left.\begin{array}{c}(1\text{-}\rho+l+\mu'+\sum_{i=1}^r\sigma'_i\eta_{G_i,g_i}-\sum_{i=1}^uK_in_i-\sum_{i=1}^{\mathfrak{v}}L_in'_i-Ll';\sigma_1,\cdots,\sigma_s),A_1:C_1\\ \cdots\\ (1\text{-}\rho-\eta-k+\mu'+\sum_{i=1}^r\sigma'_i\eta_{G_i,g_i}-\sum_{i=1}^uK_in_i-\sum_{i=1}^{\mathfrak{v}}L_in'_i-Ll';\sigma_1,\cdots,\sigma_s),B_1:D_1\end{array}\right)\quad(4.1)$$

valid under the same conditions as needed for (3.1) with $\tau_i=\cdots=\tau_{i^{(r)}}=1$

Where : $U_{11}=P_i+1,Q_i+1;r'$

$G_1(\eta_{G_1,g_1},\cdots\eta_{G_r,g_r})=G(\eta_{G_1,g_1},\cdots\eta_{G_r,g_r})_{\tau=\tau_{i^{(1)}}=\cdots,\tau_{i^{(r)}}=1}$

$A_1=A_{\iota=\iota_{i^{(1)}}=\cdots=\iota_{i^{(s)}}=1}$; $B_1=B_{\iota=\iota_{i^{(1)}}=\cdots=\iota_{i^{(s)}}=1}$

$C_1=C_{\iota=\iota_{i^{(1)}}=\cdots=\iota_{i^{(s)}}=1}$; $D_1=D_{\iota=\iota_{i^{(1)}}=\cdots=\iota_{i^{(s)}}=1}$

b) If $\iota_i=\iota_{i^{(1)}}=\cdots=\iota_{i^{(s)}}=1$ and $r=r^{(1)}=\cdots=r^{(s)}=1$, then the multivariable Aleph-function degenere to the multivariable H-function defined by Srivastava et al [12]. And we have the following result.

$$I_{0,x}^{\mu,\mu',\upsilon,\upsilon',\eta}\Big(t^{\rho-1}\ {}_{p'}M_{q'}^{\alpha}(\tau t^{l'})S_{N_1,\cdots,N_u}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_u}[y_1t^{n_1},\cdots,y_ut^{n_u}]S_E^{F_1,\cdots,F_{\mathfrak{v}}}[z_1t^{n'_1},\cdots,z_{\mathfrak{v}}t^{n'_{\mathfrak{v}}}]$$

$$\aleph(x'_1t^{\sigma'_1},\cdots,x'_rt^{\sigma'_r})H(x_1t^{\sigma_1},\cdots,x_st^{\sigma_s})\Big)$$

$$=x^{\rho+\eta-\mu-\mu'-1}\sum_{k,l=0}^{\infty}\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m_1}\cdots\sum_{g_r=0}^{m_r}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_u=0}^{[N_u/\mathfrak{M}_u]}\sum_{L_1,\cdots,L_{\mathfrak{v}}=0}^{F_1L_1+\cdots F_{\mathfrak{v}}L_{\mathfrak{v}}\leqslant E}\sum_{L=0}^{\infty}ab$$

$$\frac{(-)^l(\mu)_k(\mu')_l(\upsilon)_k(\upsilon')_l}{k!l!}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}g(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})(x'_1x^{\sigma'1})^{-\eta_{G_1,g_1}}\cdots(x'_rx^{\sigma'r})^{-\eta_{G_r,g_r}}$$

$$\frac{[(a_{p'})]_L}{[(b_{q'})]_L}\frac{\tau^L}{\Gamma(\alpha L+1)}\quad (y_1x^{n_1})^{K_1}\cdots(y_ux^{n_u})^{K_u}\ (z_1x^{n'_1})^{L_1}\cdots(z_{\mathfrak{v}}x^{n'_{\mathfrak{v}}})^{L_{\mathfrak{v}}}\ \ H^{0,N+1:V}_{P+1,Q+1:W}\left(\begin{array}{c}\mathrm{x}_1x^{\sigma_1}\\ \cdot\\ \cdot\\ \mathrm{x}_sx^{\sigma_s}\end{array}\right|$$

$$\left.\begin{array}{c}(1\text{-}\rho+l+\mu'+\sum_{i=1}^r\sigma'_i\eta_{G_i,g_i}-\sum_{i=1}^uK_in_i-\sum_{i=1}^{\mathfrak{v}}L_in'_i-Ll';\sigma_1,\cdots,\sigma_s),A':C'\\ \cdot\ \cdot\ \cdot\\ (1\text{-}\rho-\eta-k+\mu'+\sum_{i=1}^r\sigma'_i\eta_{G_i,g_i}-\sum_{i=1}^uK_in_i-\sum_{i=1}^{\mathfrak{v}}L_in'_i-Ll';\sigma_1,\cdots,\sigma_s),B':D'\end{array}\right)\quad (4.2)$$

valid under the same conditions as needed for (3.1) with $\tau_i=\cdots=\tau_{i^{(r)}}=1$ and $r=r^{(1)}=\cdots=r^{(s)}=1$

c) If $r=s=2$, we obtain two Aleph-functions of two variables defined by K. Sharma [7]. We get.

$$I^{\mu,\mu',\upsilon,\upsilon',\eta}_{0,x}\left(t^{\rho-1}\ S^{F_1,\cdots,F_{\mathfrak{v}}}_E[z_1t^{n'_1},\cdots,z_{\mathfrak{v}}t^{n'_{\mathfrak{v}}}]_{p'}M^{\alpha}_{q'}(\tau t^{l'})S^{\mathfrak{M}_1,\cdots,\mathfrak{M}_u}_{N_1,\cdots,N_u}[y_1t^{n_1},\cdots,y_ut^{n_u}]\right.$$

$$\left.\aleph(x'_1t^{\sigma'_1},x'_2t^{\sigma'_2})\aleph(x_1t^{\sigma_1},x_2t^{\sigma_2})\right)$$

$$=x^{\rho+\eta-\mu-\mu'-1}\sum_{k,l=0}^{\infty}\ \sum_{G_1,G_2=0}^{\infty}\sum_{g_1=0}^{m_1}\sum_{g_2=0}^{m_2}\ \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_u=0}^{[N_u/\mathfrak{M}_u]}\ \sum_{L_1,\cdots,L_{\mathfrak{v}}=0}^{F_1L_1+\cdots F_{\mathfrak{v}}L_{\mathfrak{v}}\leqslant E}\ \sum_{L=0}^{\infty}ab$$

$$\frac{(-)^l(\mu)_k(\mu')_l(\upsilon)_k(\upsilon')_l}{k!l!}\ \frac{(-)^{G_1+G_2}}{\delta_{g_1}G_1!\delta_{g_2}G_2!}G(\eta_{G_1,g_1},\eta_{G_2,g_2})(x'_1x^{\sigma'1})^{-\eta_{G_1,g_1}}(x'_2x^{\sigma'2})^{-\eta_{G_2,g_2}}$$

$$\frac{[(a_{p'})]_L}{[(b_{q'})]_L}\frac{\tau^L}{\Gamma(\alpha L+1)}\quad (y_1x^{n_1})^{K_1}\cdots(y_ux^{n_u})^{K_u}\ (z_1x^{n'_1})^{L_1}\cdots(z_{\mathfrak{v}}x^{n'_{\mathfrak{v}}})^{L_{\mathfrak{v}}}\ \ \aleph^{0,N+1:V}_{U_{11}:W}\left(\begin{array}{c}\mathrm{x}_1x^{\sigma_1}\\ \cdot\\ \cdot\\ \mathrm{x}_2x^{\sigma_2}\end{array}\right|$$

$$\left.\begin{array}{c}(1\text{-}\rho+l+\mu'+\sum_{i=1}^2\sigma'_i\eta_{G_i,g_i}-\sum_{i=1}^uK_in_i-\sum_{i=1}^{\mathfrak{v}}L_in'_i-Ll';\sigma_1,\sigma_2),A_2:C_2\\ \cdot\ \cdot\ \cdot\\ (1\text{-}\rho-\eta-k+\mu'+\sum_{i=1}^2\sigma'_i\eta_{G_i,g_i}-\sum_{i=1}^uK_in_i-\sum_{i=1}^{\mathfrak{v}}L_in'_i-Ll';\sigma_1,\sigma_2),B_2:D_2\end{array}\right)\quad (4.3)$$

valid under the same conditions as needed for (3.1) with $r=2$

d) If $r=s=2$ and $\tau=\tau'=\tau''=\iota=\iota'=\iota''=1$, we obtain two I-functions of two variables defined by Sharma and Mishra [6] and we have.

$$I^{\mu,\mu',\upsilon,\upsilon',\eta}_{0,x}\left(t^{\rho-1}\ S^{F_1,\cdots,F_{\mathfrak{v}}}_E[z_1t^{n'_1},\cdots,z_{\mathfrak{v}}t^{n'_{\mathfrak{v}}}]_{p'}M^{\alpha}_{q'}(\tau t^{l'})S^{\mathfrak{M}_1,\cdots,\mathfrak{M}_u}_{N_1,\cdots,N_u}[y_1t^{n_1},\cdots,y_ut^{n_u}]\right.$$

$$\left.I(x'_1t^{\sigma'_1},x'_2t^{\sigma'_2})I(x_1t^{\sigma_1},x_2t^{\sigma_2})\right)$$

$$=x^{\rho+\eta-\mu-\mu'-1}\sum_{k,l=0}^{\infty}\ \sum_{G_1,G_2=0}^{\infty}\sum_{g_1=0}^{m_1}\sum_{g_2=0}^{m_2}\ \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_u=0}^{[N_u/\mathfrak{M}_u]}\ \sum_{L_1,\cdots,L_{\mathfrak{v}}=0}^{F_1L_1+\cdots F_{\mathfrak{v}}L_{\mathfrak{v}}\leqslant E}\ \sum_{L=0}^{\infty}ab$$

$$\frac{(-)^l(\mu)_k(\mu')_l(\upsilon)_k(\upsilon')_l}{k!l!}\ \frac{(-)^{G_1+G_2}}{\delta_{g_1}G_1!\delta_{g_2}G_2!}G(\eta_{G_1,g_1},\eta_{G_2,g_2})(x'_1x^{\sigma'1})^{-\eta_{G_1,g_1}}(x'_2x^{\sigma'2})^{-\eta_{G_2,g_2}}$$

$$\frac{[(a_{p'})]_L}{[(b_{q'})]_L}\frac{\tau^L}{\Gamma(\alpha L+1)} \quad (y_1x^{n_1})^{K_1}\cdots(y_ux^{n_u})^{K_u} \ (z_1x^{n'_1})^{L_1}\cdots(z_{\mathfrak{v}}x^{n'_{\mathfrak{v}}})^{L_{\mathfrak{v}}} \ \ I_{U_{11}:W}^{0,N+1:V}\left(\begin{array}{c} \mathrm{x}_1x^{\sigma_1} \\ \cdot \\ \cdot \\ \mathrm{x}_2x^{\sigma_2} \end{array}\right|$$

$$\left.\begin{array}{c}(1\text{-}\rho+l+\mu'+\sum_{i=1}^2\sigma'_i\eta_{G_i,g_i}-\sum_{i=1}^u K_in_i-\sum_{i=1}^{\mathfrak{v}}L_in'_i-Ll';\sigma_1,\sigma_2),A'_2:C'_2 \\ \cdots \\ (1\text{-}\rho-\eta-k+\mu'+\sum_{i=1}^2\sigma'_i\eta_{G_i,g_i}-\sum_{i=1}^u K_in_i-\sum_{i=1}^{\mathfrak{v}}L_in'_i-Ll';\sigma_1,\sigma_2),B'_2:D'_2\end{array}\right) \tag{4.4}$$

Where $U_{11}=P_i+1,Q_i+1;2$

e) If $B(E;L_1,\cdots,L_{\mathfrak{v}})=\dfrac{\prod_{j=1}^A(a_j)_{L_1\theta'_j+\cdots+L_v\theta_j^{(\mathfrak{v})}}\prod_{j=1}^{B'}(b'_j)_{L_1\phi'_j}\cdots\prod_{j=1}^{B^{(\mathfrak{v})}}(b_j^{(\mathfrak{v})})_{L_v\phi_j^{(\mathfrak{v})}}}{\prod_{j=1}^C(c_j)_{m_1\psi'_j+\cdots+m_{\mathfrak{v}}\psi_j^{(\mathfrak{v})}}\prod_{j=1}^{D'}(d'_j)_{L_1\delta'_j}\cdots\prod_{j=1}^{D^{(\mathfrak{v})}}(d_j^{(\mathfrak{v})})_{L_r\delta_j^{(\mathfrak{v})}}}$ (4.5)

then the general class of multivariable polynomial $S_E^{F_1,\cdots,F_{\mathfrak{v}}}[z_1,\cdots,z_{\mathfrak{v}}]$ reduces to generalized Lauricella function defined by Srivastava et al [10].

$$F_{C:D';\cdots;D^{(\mathfrak{v})}}^{1+A:B';\cdots;B^{(\mathfrak{v})}}\left(\begin{array}{c}\mathrm{z}_1\\ \cdots \\ \mathrm{z}_{\mathfrak{v}}\end{array}\right|\left.\begin{array}{c}[(\text{-E}):\ \mathrm{F}_1,\cdots,F_{\mathfrak{v}}],[(a):\theta',\cdots,\theta^{(\mathfrak{v})}];[(b'):\phi'];\cdots;[(b^{(\mathfrak{v})}):\phi^{(\mathfrak{v})}] \\ [(\mathrm{c}):\ \psi',\cdots,\psi^{(\mathfrak{v})}];[(d'):\delta'];\cdots;[(b)^{(\mathfrak{v})}:\delta^{(\mathfrak{v})}]\end{array}\right) \tag{4.6}$$

We obtain the following result

$$I_{0,x}^{\mu,\mu',\upsilon,\upsilon',\eta}\Big(t^{\rho-1}F_{C:D';\cdots;D^{(\mathfrak{v})}}^{1+A:B';\cdots;B^{(\mathfrak{v})}}\left(\begin{array}{c}\mathrm{z}_1\\ \cdots \\ \mathrm{z}_{\mathfrak{v}}\end{array}\right|\left.\begin{array}{c}[(\text{-E}):\ \mathrm{F}_1,\cdots,F_{\mathfrak{v}}],[(a):\theta',\cdots,\theta^{(\mathfrak{v})}];[(b'):\phi'];\cdots;[(b^{(\mathfrak{v})}):\phi^{(\mathfrak{v})}] \\ [(\mathrm{c}):\ \psi',\cdots,\psi^{(\mathfrak{v})}];[(d'):\delta'];\cdots;[(b)^{(\mathfrak{v})}:\delta^{(\mathfrak{v})}]\end{array}\right)$$

$${}_{p'}M_{q'}^{\alpha}(\tau t^{l'})S_{N_1,\cdots,N_u}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_u}[y_1t^{n_1},\cdots,y_ut^{n_u}]\,\aleph(x'_1t^{\sigma'_1},\cdots,x'_rt^{\sigma'_r})\aleph(x_1t^{\sigma_1},\cdots,x_st^{\sigma_s})\Big)$$

$$=x^{\rho+\eta-\mu-\mu'-1}\sum_{k,l=0}^{\infty}\ \sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m_1}\cdots\sum_{g_r=0}^{m_r}\ \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_u=0}^{[N_u/\mathfrak{M}_u]}\ \sum_{L_1,\cdots,L_{\mathfrak{v}}=0}^{F_1L_1+\cdots F_{\mathfrak{v}}L_{\mathfrak{v}}\leqslant E}\ \sum_{L=0}^{\infty}ab$$

$$\frac{(-)^l(\mu)_k(\mu')_l(\upsilon)_k(\upsilon')_l}{k!l!}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})(x'_1x^{\sigma'1})^{-\eta_{G_1,g_1}}\cdots(x'_rx^{\sigma'r})^{-\eta_{G_r,g_r}}$$

$$\frac{[(a_{p'})]_L}{[(b_{q'})]_L}\frac{\tau^L}{\Gamma(\alpha L+1)} \quad (y_1x^{n_1})^{K_1}\cdots(y_ux^{n_u})^{K_u} \ (z_1x^{n'_1})^{L_1}\cdots(z_{\mathfrak{v}}x^{n'_{\mathfrak{v}}})^{L_{\mathfrak{v}}} \ \ \aleph_{U_{11}:W}^{0,N+1:V}\left(\begin{array}{c} \mathrm{x}_1x^{\sigma_1} \\ \cdot \\ \cdot \\ \mathrm{x}_sx^{\sigma_s} \end{array}\right|$$

$$\left.\begin{array}{c}(1\text{-}\rho+l+\mu'+\sum_{i=1}^r\sigma'_i\eta_{G_i,g_i}-\sum_{i=1}^u K_in_i-\sum_{i=1}^{\mathfrak{v}}L_in'_i-Ll';\sigma_1,\cdots,\sigma_s),A:C \\ \cdots \\ (1\text{-}\rho-\eta-k+\mu'+\sum_{i=1}^r\sigma'_i\eta_{G_i,g_i}-\sum_{i=1}^u K_in_i-\sum_{i=1}^{\mathfrak{v}}L_in'_i-Ll';\sigma_1,\cdots,\sigma_s),B:D\end{array}\right) \tag{4.7}$$

valid under the same conditions as needed for (3.1)

where b is defined by (1.10) and $B(E;L_1,\cdots,L_v)$ is defined by (4.5)

5. Conclusion

The aleph-function of several variables presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions such as , multivariable H-function , defined by Srivastava et al [12] , the Aleph-function of two variables defined by K.sharma [7].

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