Int. Jr. of Mathematical Sciences & Applications
Vol. 8, No. 1, (January-June, 2018)
Copyright: Mind Reader Publications
ISSN No: 2230-9888
www.ijmsa.yolasite.com

A multiple integral involving general class of polynomials, sequence of functions, Konhauser biorthogonal polynomials and multivariable Aleph-functions

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ABSTRACT
In this present paper, we evaluate a multiple integral with integrand involving the product of a multivariable general class of polynomials, sequence of functions, Konhauser biorthogonal polynomials and the multivariable Aleph-functions. This integral is quite general in nature and yields several (known and new) results as its special cases. Further, the integral is applied to establish an expansion formula for the products of a multivariable general class of polynomials, sequence of functions, Konhauser biorthogonal polynomials and the multivariable Aleph-functions in series of biorthogonal polynomials. Some special cases of our results are also discussed here briefly.

KEYWORDS : Aleph-function of several variables, multiple integral, Aleph-function, Aleph-function of two variables , general class of polynomials, sequence of functions, biorthogonal polynomials, expansion serie.

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20

1.Introduction and preliminaries.

Konhauser [3] has considered following pair of the biorthogonal polynomials

$$Z_n^{(\alpha)}(x;k)=\frac{\Gamma(\alpha+kn+1)}{n!}\sum_{j=0}^{n}(-)^j\binom{n}{j}\frac{x^{kj}}{\Gamma(\alpha+kj+1)} \tag{1.1}$$

and

$$Y_n^{(\alpha)}(x;k)=\frac{1}{n!}\sum_{\omega=0}^{n}\frac{x^{\omega}}{\omega!}\sum_{j=0}^{\omega}(-)^j\binom{\omega}{j}\left(\frac{\alpha+j+1}{k}\right)_n \tag{1.2}$$

which were actually suggested by the Laguerre polynomials, k being a positive integer . Indeed for $k=1$, each of these polynomials reduces to the Laguerre polynomials $L_n^{(\alpha)}(x)$, and their special cases when $k=2$ were encountered early by Spencer and Fano [6] in certain analytical calculations involving the penetration of Gamma rays through matter and were studied subsequently by Preiser [4].

For this study, we need the following series formula for the general sequence of functions introduced by Agrawal and Chaubey [1] and was established by Salim [5].

$$R_n^{\alpha,\beta}[x;E,F,g,h;p,q;\gamma;\delta;e^{-\mathfrak{s}x^{\tau}}]=\sum_{w,v,u,t',e,k_1,k_2}\psi(w,v,u,t',e,k_1,k_2)x^R \tag{1.3}$$

where $\psi(w,v,u,t',e,k_1,k_2)=\dfrac{(-)^{t'+w+k_2}(-v)_u(-t')_e(\alpha)_t l^n}{w!v!u!t'!e!l'_n k_1!k_2!}\dfrac{\mathfrak{s}^{w+k_1}F^{\gamma n-t'}}{(1-\alpha-t')_e}(-\alpha-\gamma n)_e(-\beta-\delta n)_v$

$$g^{v+k_2}h^{\delta n-v-k_2}\,(v-\delta n)_{k_2}E^{t'}\left(\frac{pe+\tau w+\lambda+qu}{l}\right)_n \tag{1.4}$$

and $\displaystyle\sum_{w,v,u,t',e,k_1,k_2}=\sum_{w=0}^{\infty}\sum_{v=0}^{n}\sum_{u=0}^{v}\sum_{t'=0}^{n}\sum_{e=0}^{t'}\sum_{k_1,k_2=0}^{\infty}$

The infinite series in the right hand side of (1.3) is absolutely convergent and $R=ln+qv+pt'+\tau w+\tau k_1+k_2q$

We shall note $R_n^{\alpha,\beta}[x;E,F,g,h;p,q;\gamma;\delta;e^{-\mathfrak{s}x^{\tau}}]=R_n^{\alpha,\beta}(x)$ (1.5)

The generalized polynomials of multivariables defined by Srivastava [9], is given in the following manner :

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[y_1,\cdots,y_v]=\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\frac{(-N_1)_{\mathfrak{M}_1K_1}}{K_1!}\cdots\frac{(-N_v)_{\mathfrak{M}_vK_v}}{K_v!}A[N_1,K_1;\cdots;N_v,K_v]y_1^{K_1}\cdots y_v^{K_v} \quad (1.6)$$

where $\mathfrak{M}_1,\cdots,\mathfrak{M}_v$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_v,K_v]$ are arbitrary constants, real or complex.

We shall note $a_v=\dfrac{(-N_1)_{\mathfrak{M}_1K_1}}{K_1!}\cdots\dfrac{(-N_v)_{\mathfrak{M}_vK_v}}{K_v!}A[N_1,K_1;\cdots;N_v,K_v]$

The Aleph-function of several variables is an extension the multivariable I-function defined by Sharma and Ahmad [8] , itself is a generalisation of G and H-functions of several variables defined by Srivastava et Panda [11,12]. The multiple Mellin-Barnes integral occurring in this paper will be referred to as the multivariables Aleph-function of $r-$variables throughout our present study and will be defined and represented as follows (see Ayant [2]).

We have : $\aleph(z_1,\cdots,z_r)=\aleph_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}\left(\begin{array}{c|c} \mathrm{z}_1 & [(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}], \\ \cdot & \cdot \\ \cdot & \cdots\cdots\cdots\cdots\cdots, \\ \cdot & \\ \mathrm{z}_r & \end{array}\right.$

$$[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:\ [(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;$$
$$[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{1,q_i}]:\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;$$

$$\left.\begin{array}{c}[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}]\\ \cdot \\ [(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\end{array}\right)=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \quad (1.7)$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \quad (1.8)$$

and $$\theta_k(s_k)=\frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \quad (1.9)$$

For more details, see Ayant [2]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}-\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}+\sum_{j=1}^{m_k}\delta_j^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}>0,$$

with $k=1,\cdots,r, i=1,\cdots,R \; , i^{(k)}=1,\cdots,R^{(k)}$ (1.10)

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

If all the poles of (1.9) are simple ,then the integral (1.7) can be evaluated with the help of the residue theorem to give

$$\aleph(z_1,\cdots,z_r)=\sum_{G_K=1}^{m_K}\sum_{g_K=0}^{\infty}\phi\frac{\prod_{K=1}^{r}\phi_K z_K^{\eta_{G_K,g_K}}(-)^{\sum_{K=1}^{r}g_K}}{\prod_{K=1}^{r}\delta_{G^{(K)}}^{(K)}\prod_{K=1}^{r}g_K!} \tag{1.11}$$

where

$$\phi=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{K=1}^{r}\alpha_j^{(i)}S_K)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{K=1}^{r}\alpha_{ji}^{(K)}S_K)\prod_{j=1}^{q}\Gamma(1-b_{ji}+\sum_{K=1}^{r}\beta_{ji}^{(K)}S_K)]} \tag{1.12}$$

$$\phi_K=\frac{\prod_{j=1}^{m_K}\Gamma(d_j^{(K)}-\delta_j^{(K)}S_K)\prod_{j=1}^{n_K}\Gamma(1-c_j^{(K)}+\gamma_j^{(K)}S_K)}{\sum_{i^{(i)}=1}^{R^{(i)}}[\tau_{i^{(i)}}\prod_{j=m_K+1}^{q_{i(K)}}\Gamma(1-d_{ji^{(K)}}^{(K)}+\delta_{ji^{(K)}}^{(K)}S_K)\prod_{j=n_K+1}^{p_{i(K)}}\Gamma(c_{ji^{(K)}}^{(K)}-\gamma_{ji^{(K)}}^{(K)}S_K)]} \tag{1.13}$$

and

$$S_K=\eta_{G_K,g_K}=\frac{d_{g_K}^{(K)}+G_K}{\delta_{g_K}^{(K)}}\ \texttt{for}\ K=1,\cdots,r \tag{1.14}$$

which is valid under the following conditions : $\epsilon_{M_K}^{(k)}[p_j^{(K)}+p'_K]\neq\epsilon_j^{(K)}[p_{M_K}+g_K]$

We shall note $\aleph(z_1,\cdots,z_r)$ = $\aleph_1(z_1,\cdots,z_r)$ and

$$\aleph(z'_1,\cdots,z'_s)=\aleph^{0,n':m'_1,n'_1,\cdots,m'_s,n'_s}_{p'_i,q'_i,\iota_i;r':p'_{i^{(1)}},q'_{i^{(1)}},\iota_{i^{(1)}};r^{(1)};\cdots;p'_{i^{(s)}},q'_{i^{(s)}};\iota_{i^{(s)}};r^{(s)}}\left(\begin{array}{c}z'_1\\ \cdot\\ \cdot\\ \cdot\\ z'_s\end{array}\middle|\begin{array}{c}[(\mathrm{u}_j;\mu_j^{(1)},\cdots,\mu_j^{(s)})_{1,n'}],\\ \cdot\\ \cdots\cdots\cdots\cdots\cdots\cdots ,\end{array}\right.$$

$$\begin{array}{l}[\iota_i(u_{ji};\mu_{ji}^{(1)},\cdots,\mu_{ji}^{(s)})_{n'+1,p'_i}]:[(\mathrm{a}_j^{(1)});\alpha_j^{(1)})_{1,n'_1}],[\iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)};\alpha_{ji^{(1)}}^{(1)})_{n'_1+1,p_i'^{(1)}}];\cdots;\\ \cdot\\ [\iota_i(v_{ji};\upsilon_{ji}^{(1)},\cdots,\upsilon_{ji}^{(s)})_{1,q'_i}]:[(\mathrm{b}_j^{(1)});\beta_j^{(1)})_{1,m'_1}],[\iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)};\beta_{ji^{(1)}}^{(1)})_{m'_1+1,q_i'^{(1)}}];\cdots;\end{array}$$

$$\left.\begin{array}{l}[(\mathrm{a}_j^{(s)});\alpha_j^{(s)})_{1,n'_s}],[\iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)};\alpha_{ji^{(s)}}^{(s)})_{n'_s+1,p_i'^{(s)}}]\\ \cdot\\ [(\mathrm{b}_j^{(s)});\beta_j^{(s)})_{1,m'_s}],[\iota_{i^{(s)}}(b_{ji^{(s)}}^{(s)};\beta_{ji^{(s)}}^{(s)})_{m'_s+1,q_i'^{(s)}}]\end{array}\right)=\frac{1}{(2\pi\omega)^s}\int_{L'_1}\cdots\int_{L'_s}\zeta(t_1,\cdots,t_s)\prod_{k=1}^{s}\phi_k(t_k){z'_k}^{t_k}\,\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.15}$$

with $\omega=\sqrt{-1}$

$$\zeta(t_1,\cdots,t_s)=\frac{\prod_{j=1}^{n'}\Gamma(1-u_j+\sum_{k=1}^{s}\mu_j^{(k)}t_k)}{\sum_{i=1}^{r'}[\iota_i\prod_{j=n'+1}^{p'_i}\Gamma(u_{ji}-\sum_{k=1}^{s}\mu_{ji}^{(k)}t_k)\prod_{j=1}^{q'_i}\Gamma(1-v_{ji}+\sum_{k=1}^{s}\upsilon_{ji}^{(k)}t_k)]} \tag{1.16}$$

and $\phi_k(t_k) = \dfrac{\prod_{j=1}^{m'_k}\Gamma(b_j^{(k)}-\beta_j^{(k)}t_k)\prod_{j=1}^{n'_k}\Gamma(1-a_j^{(k)}+\alpha_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{r^{(k)}}[\iota_{i^{(k)}}\prod_{j=m'_k+1}^{q'_{i^{(k)}}}\Gamma(1-b_{ji^{(k)}}^{(k)}+\beta_{ji^{(k)}}^{(k)}t_k)\prod_{j=n'_k+1}^{p'_{i^{(k)}}}\Gamma(a_{ji^{(k)}}^{(k)}-\alpha_{ji^{(k)}}^{(k)}s_k)]}$ (1.17)

For more details, see Ayant [2]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable H-function given by as :
The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz'_k| < \frac{1}{2}B_i^{(k)}\pi$, where

$$B_i^{(k)} = \sum_{j=1}^{n'}\mu_j^{(k)} - \iota_i\sum_{j=n'+1}^{p'_i}\mu_{ji}^{(k)} - \iota_i\sum_{j=1}^{q'_i}\upsilon_{ji}^{(k)} + \sum_{j=1}^{n'_k}\alpha_j^{(k)} - \iota_{i^{(k)}}\sum_{j=n'_k+1}^{p'_{i^{(k)}}}\alpha_{ji^{(k)}}^{(k)} + \sum_{j=1}^{m'_k}\beta_j^{(k)} - \iota_{i^{(k)}}\sum_{j=m'_k+1}^{q'_{i^{(k)}}}\beta_{ji^{(k)}}^{(k)} > 0,$$

with $k = 1,\cdots,s, i = 1,\cdots,r'\ , i^{(k)} = 1,\cdots,r^{(k)}$ (1.18)

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1,\cdots,z_s) = 0(\ |z_1|^{\alpha'_1},\cdots,|z_s|^{\alpha'_s}\ )\ , max(\ |z_1|,\cdots,|z_s|\ ) \rightarrow 0$$

$$\aleph(z_1,\cdots,z_s)\ = 0(\ |z_1|^{\beta'_1},\cdots,|z_s|^{\beta'_s}\ )\ , min(\ |z_1|,\cdots,|z_s|\ ) \rightarrow \infty$$

where $k = 1,\cdots,z : \alpha'_k = min[Re(b_j^{(k)}/\beta_j^{(k)})], j = 1,\cdots,m'_k$ and

$$\beta'_k = max[Re((a_j^{(k)}-1)/\alpha_j^{(k)})], j = 1,\cdots,n'_k$$

We shall note $\aleph(z'_1,\cdots,z'_s) = \aleph_2(z'_1,\cdots,z'_s)$

2. Main integral

We have the following general multiple integral

Theorem

$$\int_{a_1}^{\infty}\cdots\int_{a_t}^{\infty}\prod_{i=1}^{t}\left[(x_i-a_i)^{\mu_i-1}e^{-\eta'_i x_i}Y_{\mathfrak{n}_i}^{\alpha_i}(x_i;k)\right]\aleph_1\left(\begin{array}{c}\mathrm{z}_1\prod_{i=1}^{t}\left[(x_i-a_i)^{b_i^{(1)}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\prod_{i=1}^{t}\left[(x_i-a_i)^{b_i^{(r)}}\right]\end{array}\right)R_n^{\alpha,\beta}\left[\prod_{i=1}^{t}\left[(x_i-a_i)^{\zeta_i}\right]\right]$$

$$\aleph_2\left(\begin{array}{c}\mathrm{z}'_1\prod_{i=1}^{t}\left[(x_i-a_i)^{c_i^{(1)}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_s\prod_{i=1}^{t}\left[(x_i-a_i)^{c_i^{(s)}}\right]\end{array}\right)S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\left(\begin{array}{c}\mathrm{z}''_1\prod_{i=1}^{t}\left[(x_i-a_i)^{d_i^{(1)}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}''_v\prod_{i=1}^{t}\left[(x_i-a_i)^{d_i^{(v)}}\right]\end{array}\right)\mathrm{d}x_1\cdots\mathrm{d}x_t =$$

$$\sum_{w,v,u,t',e,k_1,k_2}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_\mathfrak{v}]}\sum_{G_K=1}^{m_K}\sum_{g_K=0}^{\infty}\phi\frac{\prod_{K=1}^r\phi_K z_K^{\eta_{G_K,g_K}}(-)^{\sum_{K=1}^r g_K}}{\prod_{K=1}^r\delta_{G^{(K)}}^{(K)}\prod_{K=1}^r g_K!}a_v z_1''^{K_1}\cdots z_v''^{K_v}\psi(w,v,u,t',e,k_1,k_2)$$

$$\prod_{i=1}^t\left\{\frac{e^{-\eta_i' a_i}}{\mathfrak{n}_i!}\sum_{\omega_i=0}^{\mathfrak{n}_i}(-)^{\mathfrak{n}_i}a_i^{\omega_i}\sum_{j_i=0}^{\omega_i}\frac{(-\omega_i)_{j_i}}{\omega_i!j_i!}\right\}\aleph_{p_l'+2t;q_l'+t,\iota_l;r':W}^{0,n'+2t:V}\left(\begin{array}{c|c}\mathrm{z}_1' & \mathbf{A}(\mu_i')\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_s' & \mathbf{B}(\mu_i')\end{array}\right)\tag{2.1}$$

where

$$V=m_1',n_1';\cdots;m_s',n_s'\tag{2.2}$$

$$W=p_{l^{(1)}}',q_{l^{(1)}}',\iota_{l^{(1)}};r^{(1)};\cdots;p_{l^{(s)}}',q_{l^{(s)}}',\iota_{l^{(s)}};r^{(s)}\tag{2.3}$$

$$\mathbf{A}(\mu_i')=\left[1-\mu_i';c_i^{(1)},\cdots,c_i^{(s)}\right]_{1,t},\left[\frac{1+\alpha_i-\mu_i'-j_i}{k};\frac{c_i^{(1)}}{k},\cdots,\frac{c_i^{(s)}}{k}\right]_{1,t},$$

$$\{(u_j;\mu_j^{(1)},\cdots,\mu_j^{(s)})_{1,n'}\},\ \{\iota_l(u_{jl};\mu_{jl}^{(1)},\cdots,\mu_{jl}^{(s)})_{n'+1,p_l'}\}:\{(a_j^{(1)};\alpha_j^{(1)})_{1,n_1'},\iota_{l^{(1)}}(a_{jl^{(1)}}^{(1)};\alpha_{jl^{(1)}}^{(1)})_{n_1'+1,p_{l^{(1)}}'}\}$$

$$;\cdots;\{(a_j^{(s)};\alpha_j^{(s)})_{1,n_s'},\iota_{l^{(s)}}(a_{jl^{(s)}}^{(s)};\alpha_{jl^{(s)}}^{(s)})_{n_s'+1,p_{l^{(s)}}'}\}\tag{2.4}$$

$$\mathbf{B}(\mu_i')=\left[\frac{1+\alpha_i-\mu_i'-j_i}{k}+n_i;\frac{c_i^{(1)}}{k},\cdots,\frac{c_i^{(s)}}{k}\right]_{1,t},\{\iota_l(v_{jl};\upsilon_{jl}^{(1)},\cdots,\upsilon_{jl}^{(s)})_{m'+1,q_l'}\}:$$

$$\{(b_j^{(1)};\beta_j^{(1)})_{1,m_1'},\iota_{l^{(1)}}(b_{jl^{(1)}}^{(1)};\beta_{jl^{(1)}}^{(1)})_{m_1'+1,q_{l^{(1)}}'}\};\cdots;\{(b_j^{(s)};\beta_j^{(s)})_{1,m_s'},\iota_{l^{(s)}}(\beta_{jl^{(s)}}^{(s)};\beta_{jl^{(s)}}^{(s)})_{m_s'+1,q_{l^{(s)}}'}\}\tag{2.5}$$

$$\mu_i'=\mu_i+\zeta_i R+\sum_{j=1}^v K_j d_i^{(j)}+\sum_{K=1}^r\eta_{G_K,g_K}b_i^{(K)}\tag{2.6}$$

and

$$\sum_{G_K=1}^{m_k}\sum_{g_K=0}^{\infty}=\sum_{G_1,\cdots,G_r=1}^{m_1,\cdots,m_r}\sum_{g_1,\cdots,g_r=0}^{\infty}\tag{2.7}$$

Provided that

1)$Re(\eta_i')>0,Re(\alpha_i)>0,;d_i^{(j)}>0,b_i^{(K)}>0,c_i^{(l)}>0;i=1,\cdots,t;j=1,\cdots,v;K=1,\cdots,r;l=1,\cdots,s$

2)$Re(\mu_i')+\sum_{l=1}^s c_i^{(l)}\min_{1\leqslant j\leqslant m_l'}Re\left(\frac{b_j^{(l)}}{\beta_j^{(l)}}\right)>0$ where $\mu_i'=\mu_i+\zeta_i R+\sum_{j=1}^v K_j d_i^{(j)}+\sum_{K=1}^r\eta_{G_K,g_K}b_i^{(K)}$

3) $|argz_i|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}-\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)}-\ \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}-b_l^{(i)}>0\quad(i=1,\cdots,r;l=1,\cdots,t)$$

4) $|argz'_l|<\frac{1}{2}B_l^{(k)}\pi$, where $B_l^{(k)}\ =\ \sum_{j=1}^{n'}\mu_j^{(k)}-\iota_l\sum_{j=n'+1}^{p'_l}\mu_{jl}^{(k)}\ -\iota_l\sum_{j=1}^{q'_l}\upsilon_{jl}^{(k)}+\sum_{j=1}^{n'_k}\alpha_j^{(k)}-\iota_{l^{(k)}}\sum_{j=n'_k+1}^{p'_{l^{(k)}}}\alpha_{jl^{(k)}}^{(k)}$

$$+\sum_{j=1}^{m'_k}\beta_j^{(k)}\ -\ \iota_{l^{(k)}}\sum_{j=m'_k+1}^{q'_{l^{(k)}}}\beta_{jl^{(k)}}^{(k)}-d_i^{(l)}>0(i=1,\cdots,t;l=1,\cdots,s)$$

We shall note

$$\aleph_{p'_l+2t;q'_l+t,\iota_l;r':W}^{0,n'+2t:V}\left(\begin{array}{c|c}\mathrm{z}'_1 & \mathbf{A}(\mu'_i)\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}'_s & \mathbf{B}(\mu'_i)\end{array}\right)=\ \ \mu'_i\aleph_{p'_l+2t,q'_l+t,\iota_l;r';W}^{0,n'+2t;V}\left(\begin{array}{c}\mathrm{z}'_1\\ \cdot\\ \cdot\\ \mathrm{z}'_s\end{array}\right)\tag{2.8}$$

Proof

To evaluate the multiple integral (2.1), we first express the class of multivariable polynomials $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_\mathfrak{v}}[.]$ in series with the help of (1.6), the multivariable Aleph-function $\aleph_1(z_1,\cdots,z_r)$ in serie with the help of (1.11), the sequence of functions in series with the help of (1.4), use contour integral representation with the help of (1.15) for the multivariable Aleph-function $\aleph_2(z'_1,\cdots,z'_s)$ occuring in its left-hand side and change the order of integrations and summations, which is permissible under the conditions stated with (2.1), we get the left-hand side of (2.1) as

$$\text{L.H.S}=\sum_{w,v,u,t',e,k_1,k_2}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_\mathfrak{v}]}\sum_{G_K=1}^{m_K}\sum_{g_K=0}^{\infty}\phi\frac{\prod_{K=1}^r\phi_K z_K^{\eta_{G_K,g_K}}(-)^{\sum_{K=1}^r g_K}}{\prod_{K=1}^r\delta_{G^{(K)}}^{(K)}\prod_{K=1}^r g_K!}a_v z_1''^{K_1}\cdots z_v''^{K_v}\psi(w,v,u,t',e,k_1,k_2)$$

$$\frac{1}{(2\pi\omega)^s}\int_{L'_1}\cdots\int_{L'_s}\zeta(t_1,\cdots,t_s)\prod_{l=1}^s\phi_l(t_l)z_l'^{t_l}\left[\prod_{i=1}^t\int_{a_i}^{\infty}(x_i-a_i)^{\mu'_i+\sum_{l=1}^s c_i^{(l)}t_l-1}\left[e^{-\eta'_i x_i}Y_{\mathfrak{n}_i}^{\alpha_i}(x_i;k)\right]\mathrm{d}x_i\right]$$

$$\mathrm{d}t_1\cdots\mathrm{d}t_s\tag{2.9}$$

where μ'_i is defined by (2.6). Now for evaluation of the inner x_i-integral, using the result ([10, p.43, Eq. (3.9)) and expressing the multiple contour integral as the multivariable Aleph-function, we get the right-had side of (2.1).

3. Particular cases

a) Taking $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_\mathfrak{v}}\left(\begin{array}{c}\mathrm{z}''_1\prod_{i=1}^t\left[(x_i-a_i)^{d_i^{(1)}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}''_v\prod_{i=1}^t\left[(x_i-a_i)^{d_i^{(v)}}\right]\end{array}\right)\rightarrow Z_N^{\beta}\left[\prod_{i=1}^t(x_i-a_i);g\right]$, the integral (2.1) writes

$$\int_{a_1}^{\infty}\cdots\int_{a_t}^{\infty}\prod_{i=1}^t\left[(x_i-a_i)^{\mu_i-1}e^{-\eta'_i x_i}Y_{\mathfrak{n}_i}^{\alpha_i}(x_i;k)\right]\aleph_1\left(\begin{array}{c}\mathrm{z}_1\prod_{i=1}^t\left[(x_i-a_i)^{b_i^{(1)}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\prod_{i=1}^t\left[(x_i-a_i)^{b_i^{(r)}}\right]\end{array}\right)R_n^{\alpha,\beta}\left[\prod_{i=1}^t\left[(x_i-a_i)^{\zeta_i}\right]\right]$$

$$\aleph_2\left(\begin{array}{c}\mathrm{z}'_1\prod_{i=1}^t\left[(x_i-a_i)^{c_i^{(1)}}\right]\\ .\\ .\\ .\\ \mathrm{z}'_s\prod_{i=1}^t\left[(x_i-a_i)^{c_i^{(s)}}\right]\end{array}\right) Z_N^{\beta}\left[z''\prod_{i=1}^t(x_i-a_i);g\right]\,\mathrm{d}x_1\cdots\mathrm{d}x_t=$$

$$\sum_{w,v,u,t',e,k_1,k_2}\sum_{K'=0}^{N}\sum_{G_K=1}^{m_K}\sum_{g_K=0}^{\infty}\phi\frac{\prod_{K=1}^r\phi_K z_K^{\eta_{G_K,g_K}}(-)^{\sum_{K=1}^r g_K}}{\prod_{K=1}^r\delta_{G^{(K)}}^{(K)}\prod_{K=1}^r g_K!}\,z''^{K'}\,\frac{(-N)_{K'}\Gamma(1+\beta+gN)}{K'!\Gamma(1+\beta+gK')}$$

$$\psi(w,v,u,t',e,k_1,k_2)\prod_{i=1}^t\left\{\frac{e^{-\eta'_i a_i}}{\mathfrak{n}_i!}\sum_{\omega_i=0}^{\mathfrak{n}_i}(-)^{\mathfrak{n}_i}a_i^{\omega_i}\sum_{j_i=0}^{\omega_i}\frac{(-\omega_i)_{j_i}}{\omega_i!j_i!}\right\}\ {}_{\mu_i^*}\aleph^{0,n'+2t;V}_{p'_l+2t,q'_l+t,\iota_l;r';W}\left(\begin{array}{c}\mathrm{z}'_1\\ .\\ .\\ \mathrm{z}'_s\end{array}\right)\tag{3.1}$$

where $\mu_i^*=\mu_i+\zeta_iR+K'g+\sum_{K=1}^r\eta_{G_K,g_K}b_i^{(K)}$

We obtain the same notations that (2.1) with $\mu'_i=\mu_i^*$. We have the same validity conditions 1), 3), 4) that (2.1) and

2') $Re(\mu_i^*)+\sum_{l=1}^s c_i^{(l)}\min_{1\leqslant j\leqslant m'_l}Re\left(\frac{b_j^{(l)}}{\beta_j^{(l)}}\right)>0$

b) Taking $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\left(\begin{array}{c}\mathrm{z}''_1\prod_{i=1}^t\left[(x_i-a_i)^{d_i^{(1)}}\right]\\ .\\ .\\ .\\ \mathrm{z}''_v\prod_{i=1}^t\left[(x_i-a_i)^{d_i^{(v)}}\right]\end{array}\right)\to 1$, the integral (2.1) writes

$$\int_{a_1}^{\infty}\cdots\int_{a_t}^{\infty}\prod_{i=1}^t\left[(x_i-a_i)^{\mu_i-1}e^{-\eta'_ix_i}Y_{\mathfrak{n}_i}^{\alpha_i}(x_i;k)\right]\aleph_1\left(\begin{array}{c}\mathrm{z}_1\prod_{i=1}^t\left[(x_i-a_i)^{b_i^{(1)}}\right]\\ .\\ .\\ .\\ \mathrm{z}_r\prod_{i=1}^t\left[(x_i-a_i)^{b_i^{(r)}}\right]\end{array}\right) R_n^{\alpha,\beta}\left[\prod_{i=1}^t\left[(x_i-a_i)^{\zeta_i}\right]\right]$$

$$\aleph_2\left(\begin{array}{c}\mathrm{z}'_1\prod_{i=1}^t\left[(x_i-a_i)^{c_i^{(1)}}\right]\\ .\\ .\\ .\\ \mathrm{z}'_s\prod_{i=1}^t\left[(x_i-a_i)^{c_i^{(s)}}\right]\end{array}\right)\,\mathrm{d}x_1\cdots\mathrm{d}x_t=\sum_{w,v,u,t',e,k_1,k_2}\sum_{G_K=1}^{m_K}\sum_{g_K=0}^{\infty}\phi\frac{\prod_{K=1}^r\phi_K z_K^{\eta_{G_K,g_K}}(-)^{\sum_{K=1}^r g_K}}{\prod_{K=1}^r\delta_{G^{(K)}}^{(K)}\prod_{K=1}^r g_K!}$$

$$\psi(w,v,u,t',e,k_1,k_2)\prod_{i=1}^t\left\{\frac{e^{-\eta'_i a_i}}{\mathfrak{n}_i!}\sum_{\omega_i=0}^{\mathfrak{n}_i}(-)^{\mathfrak{n}_i}a_i^{\omega_i}\sum_{j_i=0}^{\omega_i}\frac{(-\omega_i)_{j_i}}{\omega_i!j_i!}\right\}\ {}_{\mu_i^0}\aleph^{0,n'+2t;V}_{p'_l+2t,q'_l+t,\iota_l;r';W}\left(\begin{array}{c}\mathrm{z}'_1\\ .\\ .\\ \mathrm{z}'_s\end{array}\right)\tag{3.2}$$

where $\mu_i^0=\mu_i+\zeta_i R+\sum_{K=1}^{r}\eta_{G_K,g_K}b_i^{(K)}$

We obtain the same notations that (2.1) with $\mu'_i=\mu_i^0$. We have the same validity conditions 1), 3), 4) that (2.1) and

2") $Re(\mu_i^0)+\sum_{l=1}^{s}c_i^{(l)}\min_{1\leqslant j\leqslant m'_l} Re\left(\frac{b_j^{(l)}}{\beta_j^{(l)}}\right)>0$

c) If $r=s=2$, then the multivariable Aleph-functions reduce to Aleph-functions of two variables defined by Sharma [7], we obtain

$$\int_{a_1}^{\infty}\cdots\int_{a_t}^{\infty}\prod_{i=1}^{t}\left[(x_i-a_i)^{\mu_i-1}e^{-\eta'_i x_i}Y_{\mathfrak{n}_i}^{\alpha_i}(x_i;k)\right]\aleph_1\left(\begin{array}{c}\mathrm{z}_1\prod_{i=1}^{t}\left[(x_i-a_i)^{b_i^{(1)}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_2\prod_{i=1}^{t}\left[(x_i-a_i)^{b_i^{(2)}}\right]\end{array}\right)R_n^{\alpha,\beta}\left[\prod_{i=1}^{t}\left[(x_i-a_i)^{\zeta_i}\right]\right]$$

$$\aleph_2\left(\begin{array}{c}\mathrm{z}'_1\prod_{i=1}^{t}\left[(x_i-a_i)^{c_i^{(1)}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_2\prod_{i=1}^{t}\left[(x_i-a_i)^{c_i^{(2)}}\right]\end{array}\right)S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\left(\begin{array}{c}\mathrm{z}''_1\prod_{i=1}^{t}\left[(x_i-a_i)^{d_i^{(1)}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}''_v\prod_{i=1}^{t}\left[(x_i-a_i)^{d_i^{(v)}}\right]\end{array}\right)\mathrm{d}x_1\cdots\mathrm{d}x_t=$$

$$\sum_{w,v,u,t',e,k_1,k_2}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_1=1}^{m_1}\sum_{G_2=1}^{m_2}\sum_{g_1,g_2=0}^{\infty}\phi_2\frac{\prod_{K=1}^{2}\phi_{2K}z_K^{\eta_{G_K,g_K}}(-)^{\sum_{K=1}^{2}g_K}}{\prod_{K=1}^{2}\delta_{G^{(K)}}^{(K)}\prod_{K=1}^{2}g_K!}a_v z_1''^{K_1}\cdots z_v''^{K_v}$$

$$\psi(w,v,u,t',e,k_1,k_2)\prod_{i=1}^{t}\left\{\frac{e^{-\eta'_i a_i}}{\mathfrak{n}_i!}\sum_{\omega_i=0}^{\mathfrak{n}_i}(-)^{\mathfrak{n}_i}a_i^{\omega_i}\sum_{j_i=0}^{\omega_i}\frac{(-\omega_i)_{j_i}}{\omega_i!j_i!}\right\}_{\mu'_i}\aleph_{p'_l+2t,q'_l+t,\iota_l;r';W}^{0,n'+2t;V}\left(\begin{array}{c}\mathrm{z}'_1\\ \cdot\\ \cdot\\ \mathrm{z}'_2\end{array}\right)\qquad(3.3)$$

under the same notations (2.1) to (2.6) and validity conditions that (2.1) with $r=s=2$.

c) If $r=s=1$, then the multivariable Aleph-functions reduce to Aleph-functions of one variable defined by Sudland [13,14], we obtain

$$\int_{a_1}^{\infty}\cdots\int_{a_t}^{\infty}\prod_{i=1}^{t}\left[(x_i-a_i)^{\mu_i-1}e^{-\eta'_i x_i}Y_{\mathfrak{n}_i}^{\alpha_i}(x_i;k)\right]\aleph_1\left(\mathrm{z}\prod_{i=1}^{t}\left[(x_i-a_i)^{b_i}\right]\right)R_n^{\alpha,\beta}\left[\prod_{i=1}^{t}\left[(x_i-a_i)^{\zeta_i}\right]\right]$$

$$\aleph_2\left(\mathrm{z}'\prod_{i=1}^{t}\left[(x_i-a_i)^{c_i}\right]\right)S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\left(\begin{array}{c}\mathrm{z}''_1\prod_{i=1}^{t}\left[(x_i-a_i)^{d_i^{(1)}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}''_v\prod_{i=1}^{t}\left[(x_i-a_i)^{d_i^{(v)}}\right]\end{array}\right)\mathrm{d}x_1\cdots\mathrm{d}x_t=$$

$$\sum_{w,v,u,t',e,k_1,k_2}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G=1}^{m}\sum_{g=0}^{\infty}\phi_1\frac{z^{\eta_{G,g}}(-)^g}{\delta_G g!}a_v z_1''^{K_1}\cdots z_v''^{K_v}$$

$$\psi(w,v,u,t',e,k_1,k_2)\prod_{i=1}^{t}\left\{\frac{e^{-\eta_i' a_i}}{\mathfrak{n}_i!}\sum_{\omega_i=0}^{\mathfrak{n}_i}(-)^{\mathfrak{n}_i}a_i^{\omega_i}\sum_{j_i=0}^{\omega_i}\frac{(-\omega_i)_{j_i}}{\omega_i!j_i!}\right\}{}_{\mu_i'}\aleph^{0,n'+2t}_{p_l'+2t,q_l'+t,\iota_l;r'}(\text{ z' }) \tag{3.4}$$

under the same notations (2.1) to (2.6) and validity conditions that (2.1) with $r=s=1$.

4. Expansion series

We have the following formula

$$\prod_{i=1}^{t}x_i^{\mu_i-1}\aleph_1\begin{pmatrix}\text{z}_1\prod_{i=1}^{t}x_i^{b_i^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ \text{z}_r\prod_{i=1}^{t}x_i^{b_i^{(r)}}\end{pmatrix}\aleph_2\begin{pmatrix}\text{z}_1'\prod_{i=1}^{t}x_i^{c_i^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ \text{z}_s'\prod_{i=1}^{t}x_i^{c_i^{(s)}}\end{pmatrix}S^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}_{N_1,\cdots,N_v}\begin{pmatrix}\text{z}_1''\prod_{i=1}^{t}x_i^{d_i^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ \text{z}_v''\prod_{i=1}^{t}x_i^{d_i^{(v)}}\end{pmatrix}R_n^{\alpha,\beta}\left[\prod_{i=1}^{t}x_i^{\zeta_i}\right]$$

$$=\sum_{\mathbf{n}_1,\cdots,\mathbf{n}_t=0}^{\infty}\sum_{w,v,u,t',e,k_1,k_2}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_K=1}^{m_K}\sum_{g_K=0}^{\infty}\phi\frac{\prod_{K=1}^{r}\phi_K z_K^{\eta_{G_K,g_K}}(-)^{\sum_{K=1}^{r}g_K}}{\prod_{K=1}^{r}\delta^{(K)}_{G^{(K)}}\prod_{K=1}^{r}g_K!}a_v z_1''^{K_1}\cdots z_v''^{K_v}$$

$$\psi(w,v,u,t',e,k_1,k_2)\prod_{i=1}^{t}\left\{\sum_{\omega_i=0}^{\mathbf{n}_i}\frac{(-)^{\mathbf{n}_i}}{\omega_i!\Gamma(\alpha_i+\mathbf{n}_ik+1)}\sum_{j_i=0}^{\omega_i}\frac{(-\omega_i)_{j_i}}{j_i!}\right\}{}_{\mu_i'+\alpha_i}\aleph^{0,n'+2t;V}_{p_l'+2t,q_l'+t,\iota_l;r';W}\begin{pmatrix}\text{z}_1'\\ \cdot\\ \cdot\\ \text{z}_s'\end{pmatrix}\prod_{i=1}^{t}Z_{\mathbf{n}_i}^{\alpha_i}[x_i;k] \tag{4.1}$$

where $\aleph^{0,n'+2t:V}_{p_l'+2t;q_l'+t,\iota_l;r':W}\left(\begin{array}{c|c}\text{z}_1' & \mathbf{A}(\mu_i'+\alpha_i)\\ \cdot & \cdot\\ \cdot & \cdot\\ \text{z}_s' & \mathbf{B}(\mu_i'+\alpha_i)\end{array}\right)={}_{\mu_i'+\alpha_i}\aleph^{0,n'+2t;V}_{p_l'+2t,q_l'+t,\iota_l;r';W}\begin{pmatrix}\text{z}_1'\\ \cdot\\ \cdot\\ \text{z}_s'\end{pmatrix}$

Provided that

1)$Re(\eta_i')>0,Re(\alpha_i)>-1,;\,d_i^{(j)}>0,b_i^{(k)}>0,c_i^{(l)}>0;i=1,\cdots,t;j=1,\cdots,v;k=1,\cdots,r;l=1,\cdots,s$

2)$Re(\mu_i')+\sum_{l=1}^{s}c_i^{(l)}\min_{1\leqslant j\leqslant m_l'}Re\left(\frac{b_j^{(l)}}{\beta_j^{(l)}}\right)>-1$ where $\mu_i'=\mu_i+\zeta_iR+\sum_{j=1}^{v}K_jd_i^{(j)}+\sum_{K=1}^{r}\eta_{G_K,g_K}b_i^{(K)}$

3) $|argz_i|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}-\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)}-\ \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}-b_l^{(i)}>0\quad(i=1,\cdots,r;l=1,\cdots,t)$$

4) $|arg z'_l| < \frac{1}{2}B_l^{(k)}\pi$, where $B_l^{(k)} = \sum_{j=1}^{n'}\mu_j^{(k)} - \iota_l \sum_{j=n'+1}^{p'_l}\mu_{jl}^{(k)} - \iota_l \sum_{j=1}^{q'_l} \upsilon_{jl}^{(k)} + \sum_{j=1}^{n'_k}\alpha_j^{(k)} - \iota_{l^{(k)}} \sum_{j=n'_k+1}^{p'_{l^{(k)}}} \alpha_{jl^{(k)}}^{(k)}$

$+\sum_{j=1}^{m'_k}\beta_j^{(k)} - \iota_{l^{(k)}} \sum_{j=m'_k+1}^{q'_{l^{(k)}}} \beta_{jl^{(k)}}^{(k)} - d_i^{(l)} > 0 (i = 1, \cdots, t; l = 1, \cdots, s)$

Proof

Let $\prod_{i=1}^{t} x_i^{\mu_i - 1} \aleph_1 \begin{pmatrix} z_1 \prod_{i=1}^t x_i^{b_i^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z_r \prod_{i=1}^t x_i^{b_i^{(r)}} \end{pmatrix} \aleph_2 \begin{pmatrix} z'_1 \prod_{i=1}^t x_i^{c_i^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z'_s \prod_{i=1}^t x_i^{c_i^{(s)}} \end{pmatrix} S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_\mathfrak{v}} \begin{pmatrix} z''_1 \prod_{i=1}^t x_i^{d_i^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z''_v \prod_{i=1}^t x_i^{d_i^{(v)}} \end{pmatrix} R_n^{\alpha,\beta}\left[\prod_{i=1}^t x_i^{\zeta_i}\right]$

$$\sum_{\mathbf{n}_1,\cdots,\mathbf{n}_t=0}^{\infty} L_{\mathbf{n}_1,\cdots,\mathbf{n}_t} \prod_{i=1}^{t} Z_{\mathbf{n}_i}^{\alpha_i}[x_i;k] \tag{4.2}$$

Multiplying both sides of (4.2) by $\prod_{i=1}^{t} x_i^{\alpha_i} e^{-x_i} Y_{\upsilon_i}^{\alpha_i}[x_i;k]$and integrating with respect to $x_1, \cdots, x_t$ from $0, \cdots, 0$ respectively to ∞, we get

$$\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{t} \left[(x_i^{\mu_i+\alpha_i-1} e^{-x_i} Y_{\upsilon_i}^{\alpha_i}(x_i;k)\right] \aleph_1 \begin{pmatrix} z_1 \prod_{i=1}^t x_i^{b_i^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z_r \prod_{i=1}^t x_i^{b_i^{(r)}} \end{pmatrix} \aleph_2 \begin{pmatrix} z'_1 \prod_{i=1}^t x_i^{c_i^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z'_s \prod_{i=1}^t x_i^{c_i^{(s)}} \end{pmatrix}$$

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_\mathfrak{v}} \begin{pmatrix} z''_1 \prod_{i=1}^t x_i^{d_i^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z''_v \prod_{i=1}^t x_i^{d_i^{(v)}} \end{pmatrix} R_n^{\alpha,\beta}\left[\prod_{i=1}^t [(x_i - a_i)^{\zeta_i}]\right] \mathrm{d}x_1 \cdots \mathrm{d}x_t =$$

$$\sum_{\mathbf{n}_1,\cdots,\mathbf{n}_t=0}^{\infty} L_{\mathbf{n}_1,\cdots,\mathbf{n}_t} \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{t} \left[x_i^{\alpha_i} e^{-x_i} Z_{\mathbf{n}_i}^{\alpha_i}[x_i;k] Y_{\upsilon_i}^{\alpha_i}[x_i;k]\right] \mathrm{d}x_1 \cdots \mathrm{d}x_t \tag{4.3}$$

using the integral (2.1) and the following orthogonal property ([3],p.303)

$$\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{t} \left[x_i^{\alpha_i} e^{-x_i} Z_{\mathbf{n}_i}^{\alpha_i}[x_i;k] Y_{\upsilon_i}^{\alpha_i}[x_i;k]\right] \mathrm{d}x_1 \cdots \mathrm{d}x_t = \prod_{i=1}^{t} \frac{\Gamma(\alpha_i + k\mathbf{n}_i + 1)}{\mathbf{n}_i!} \delta_{\upsilon_i,\mathbf{n}_i} \tag{4.4}$$

where $Re(\alpha_i) > -1, k, \upsilon_i, \mathbf{n}_i (i = 1, \cdots, t)$ are positive integers. Also $\delta_{\upsilon_i,\mathbf{n}_i}$ is the Kronecker delta symbol.

Subtituting (4.4) in (4.3), we find that

$$L_{\mathbf{n}_1,\cdots,\mathbf{n}_t} = \sum_{w,v,u,t',e,k_1,k_2} \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} \sum_{G_K=1}^{m_K} \sum_{g_K=0}^{\infty} \phi \frac{\prod_{K=1}^r \phi_K z_K^{\eta_{G_K,g_K}} (-)^{\sum_{K=1}^r g_K}}{\prod_{K=1}^r \delta_{G^{(K)}}^{(K)} \prod_{K=1}^r g_K!} a_v z_1''^{K_1} \cdots z_v''^{K_v}$$

$$\psi(w,v,u,t',e,k_1,k_2)\prod_{i=1}^t \left\{ \sum_{\omega_i=0}^{\mathbf{n}_i} \frac{(-)^{\mathbf{n}_i}}{\omega_i!\Gamma(\alpha_i+\mathbf{n}_i k+1)} \sum_{j_i=0}^{\omega_i} \frac{(-\omega_i)_{j_i}}{j_i!} \right\}_{\mu_i'+\alpha_i} \aleph^{0,n'+2t;V}_{p_l'+2t,q_l'+t,\iota_l;r';W} \begin{pmatrix} \mathrm{z}_1' \\ \cdot \\ \cdot \\ \mathrm{z}_s' \end{pmatrix} \tag{4.5}$$

Subtituting (4.5) in (4.2), we arrive at the required expansion formula (4.1), where

$$\text{where} \quad \aleph^{0,n'+2t:V}_{p_l'+2t;q_l'+t,\iota_l;r':W} \left(\begin{array}{c|c} \mathrm{z}_1' & \mathbf{A}(\mu_i'+\alpha_i) \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_s' & \mathbf{B}(\mu_i'+\alpha_i) \end{array} \right) = {}_{\mu_i'+\alpha_i}\aleph^{0,n'+2t;V}_{p_l'+2t,q_l'+t,\iota_l;r';W} \begin{pmatrix} \mathrm{z}_1' \\ \cdot \\ \cdot \\ \mathrm{z}_s' \end{pmatrix} \tag{4.6}$$

and $\mu_i' = \mu_i + \zeta_i R + \sum_{j=1}^{v} K_j d_i^{(j)} + \sum_{K=1}^{r} \eta_{G_K,g_K} b_i^{(K)}$ (see the notation in (2.4), (2.5) and (2.8)).

5. Particular expansions

a) Taking $S^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}_{N_1,\cdots,N_v} \begin{pmatrix} \mathrm{z}_1'' \prod_{i=1}^t x_i^{d_i^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_v'' \prod_{i=1}^t x_i^{d_i^{(v)}} \end{pmatrix} \rightarrow Z_N^\beta \left[z'' \prod_{i=1}^t x_i ; g \right]$, the expansion formula (4.1) writes

$$\prod_{i=1}^t x_i^{\mu_i-1} \aleph_1 \begin{pmatrix} \mathrm{z}_1 \prod_{i=1}^t x_i^{b_i^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \prod_{i=1}^t x_i^{b_i^{(r)}} \end{pmatrix} \aleph_2 \begin{pmatrix} \mathrm{z}_1' \prod_{i=1}^t x_i^{c_i^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_s' \prod_{i=1}^t x_i^{c_i^{(s)}} \end{pmatrix} Z_N^\beta \left[z'' \prod_{i=1}^t x_i ; g \right] R_n^{\alpha,\beta} \left[\prod_{i=1}^t x_i^{\zeta_i} \right]$$

$$= \sum_{\mathbf{n}_1,\cdots,\mathbf{n}_t=0}^{\infty} \sum_{w,v,u,t',e,k_1,k_2} \sum_{K'=0}^{N} \sum_{G_K=1}^{m_K} \sum_{g_K=0}^{\infty} \phi \frac{\prod_{K=1}^r \phi_K z_K^{\eta_{G_K,g_K}} (-)^{\sum_{K=1}^r g_K}}{\prod_{K=1}^r \delta_{G^{(K)}}^{(K)} \prod_{K=1}^r g_K!} z''^K \frac{(-N)_{K'} \Gamma(1+\beta+gN)}{K'!\Gamma(1+\beta+gK')}$$

$$\psi(w,v,u,t',e,k_1,k_2)\prod_{i=1}^t \left\{ \sum_{\omega_i=0}^{\mathbf{n}_i} \frac{(-)^{\mathbf{n}_i}}{\omega_i!\Gamma(\alpha_i+\mathbf{n}_i k+1)} \sum_{j_i=0}^{\omega_i} \frac{(-\omega_i)_{j_i}}{j_i!} \right\}_{\mu_i^*+\alpha_i} \aleph^{0,n'+2t;V}_{p_l'+2t,q_l'+t,\iota_l;r';W} \begin{pmatrix} \mathrm{z}_1' \\ \cdot \\ \cdot \\ \mathrm{z}_s' \end{pmatrix} \prod_{i=1}^t Z_{\mathbf{n}_i}^{\alpha_i}[x_i;k] \tag{5.1}$$

where $\mu_i^* = \mu_i + \zeta_i R + K'g + \sum_{K=1}^{r} \eta_{G_K,g_K} b_i^{(K)}$

b) Taking $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\left(\begin{array}{c} \mathrm{z}''_1\prod_{i=1}^t x_i^{d_i^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}''_v\prod_{i=1}^t x_i^{d_i^{(v)}} \end{array}\right) \rightarrow 1$, the expansion formula (4.1) writes

$$\prod_{i=1}^t x_i^{\mu_i-1}\aleph_1\left(\begin{array}{c} \mathrm{z}_1\prod_{i=1}^t x_i^{b_i^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r\prod_{i=1}^t x_i^{b_i^{(r)}} \end{array}\right)\aleph_2\left(\begin{array}{c} \mathrm{z}'_1\prod_{i=1}^t x_i^{c_i^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}'_s\prod_{i=1}^t x_i^{c_i^{(s)}} \end{array}\right)R_n^{\alpha,\beta}\left[\prod_{i=1}^t x_i^{\zeta_i}\right]$$

$$=\sum_{\mathbf{n}_1,\cdots,\mathbf{n}_t=0}^{\infty}\sum_{w,v,u,t',e,k_1,k_2}\sum_{G_K=1}^{m_K}\sum_{g_K=0}^{\infty}\phi\frac{\prod_{K=1}^r\phi_K z_K^{\eta_{G_K,g_K}}(-)^{\sum_{K=1}^r g_K}}{\prod_{K=1}^r\delta_{G^{(K)}}^{(K)}\prod_{K=1}^r g_K!}\psi(w,v,u,t',e,k_1,k_2)$$

$$\prod_{i=1}^t\left\{\sum_{\omega_i=0}^{\mathbf{n}_i}\frac{(-)^{\mathbf{n}_i}}{\omega_i!\Gamma(\alpha_i+\mathbf{n}_ik+1)}\sum_{j_i=0}^{\omega_i}\frac{(-\omega_i)_{j_i}}{j_i!}\right\}_{\mu_i^0+\alpha_i}\aleph_{p_l'+2t,q_l'+t,\iota_l;r';W}^{0,n'+2t;V}\left(\begin{array}{c} \mathrm{z}'_1 \\ \cdot \\ \cdot \\ \mathrm{z}'_s \end{array}\right)\prod_{i=1}^t Z_{\mathbf{n}_i}^{\alpha_i}[x_i;k] \tag{5.2}$$

where $\mu_i^0=\mu_i+\zeta_iR+\sum_{K=1}^r\eta_{G_K,g_K}b_i^{(K)}$

c) If $r=s=2$, then the multivariable Aleph-functions reduce to Aleph-functions of two variables defined by Sharma [7], we obtain the following expansion

$$\prod_{i=1}^t x_i^{\mu_i-1}\aleph_1\left(\begin{array}{c} \mathrm{z}_1\prod_{i=1}^t x_i^{b_i^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_2\prod_{i=1}^t x_i^{b_i^{(2)}} \end{array}\right)\aleph_2\left(\begin{array}{c} \mathrm{z}'_1\prod_{i=1}^t x_i^{c_i^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}'_2\prod_{i=1}^t x_i^{c_i^{(2)}} \end{array}\right)S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\left(\begin{array}{c} \mathrm{z}''_1\prod_{i=1}^t x_i^{d_i^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}''_v\prod_{i=1}^t x_i^{d_i^{(v)}} \end{array}\right)R_n^{\alpha,\beta}\left[\prod_{i=1}^t x_i^{\zeta_i}\right]$$

$$=\sum_{\mathbf{n}_1,\cdots,\mathbf{n}_t=0}^{\infty}\sum_{w,v,u,t',e,k_1,k_2}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_1=1}^{m_1}\sum_{G_2=1}^{m_2}\sum_{g_1,g_2=0}^{\infty}\phi_2\frac{\prod_{K=1}^2\phi_{2K}z_K^{\eta_{G_K,g_K}}(-)^{\sum_{K=1}^2 g_K}}{\prod_{K=1}^2\delta_{G^{(K)}}^{(K)}\prod_{K=1}^2 g_K!}a_v z_1''^{K_1}\cdots z_v''^{K_v}$$

$$\psi(w,v,u,t',e,k_1,k_2)\prod_{i=1}^t\left\{\sum_{\omega_i=0}^{\mathbf{n}_i}\frac{(-)^{\mathbf{n}_i}}{\omega_i!\Gamma(\alpha_i+\mathbf{n}_ik+1)}\sum_{j_i=0}^{\omega_i}\frac{(-\omega_i)_{j_i}}{j_i!}\right\}_{\mu_i'+\alpha_i}\aleph_{p_l'+2t,q_l'+t,\iota_l;r';W}^{0,n'+2t;V}\left(\begin{array}{c} \mathrm{z}'_1 \\ \cdot \\ \cdot \\ \mathrm{z}'_2 \end{array}\right)\prod_{i=1}^t Z_{\mathbf{n}_i}^{\alpha_i}[x_i;k] \tag{5.3}$$

under the same notations (2.1) to (2.6) and validity conditions that (2.1) with $r=s=2$.

c) If $r=s=1$, then the multivariable Aleph-functions reduce to Aleph-functions of one variables defined by Sudland [13,14], we obtain the following formula

$$\prod_{i=1}^{t} x_i^{\mu_i-1}\aleph_1\left(\text{ z}\prod_{i=1}^{t} x_i^{b_i}\right)\aleph_2\left(\text{ z'}\prod_{i=1}^{t} x_i^{c_i}\right)S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix} \text{z}_1''\prod_{i=1}^{t} x_i^{d_i^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \text{z}_v''\prod_{i=1}^{t} x_i^{d_i^{(v)}} \end{pmatrix} R_n^{\alpha,\beta}\left[\prod_{i=1}^{t} x_i^{\zeta_i}\right]$$

$$= \sum_{\mathbf{n}_1,\cdots,\mathbf{n}_t=0}^{\infty} \sum_{w,v,u,t',e,k_1,k_2} \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} \sum_{G=1}^{m}\sum_{g=0}^{\infty}\phi_1 \frac{z^{\eta_{G,g}}(-)^g}{\delta_G g!}\, a_v z_1''^{K_1}\cdots z_v''^{K_v}$$

$$\psi(w,v,u,t',e,k_1,k_2)\prod_{i=1}^{t}\left\{\sum_{\omega_i=0}^{\mathbf{n}_i}\frac{(-)^{\mathbf{n}_i}}{\omega_i!\Gamma(\alpha_i+\mathbf{n}_i k+1)}\sum_{j_i=0}^{\omega_i}\frac{(-\omega_i)_{j_i}}{j_i!}\right\}_{\mu_i'+\alpha_i}\aleph_{p_l'+2t,q_l'+t,\iota_l;r}^{0,n'+2t}\left(\text{ z'}\right)\prod_{i=1}^{t} Z_{\mathbf{n}_i}^{\alpha_i}[x_i;k] \qquad (5.4)$$

under the same notations (2.1) to (2.6) and validity conditions that (2.1) with $r = s = 1$.

Remarks :

By the similar procedure, the results of this document can be extented to product of any finite number of multivariable Aleph-functions and a class of multivariable polynomials defined by Srivastava [9].

We obtain the similar formulae with the multivariable H-functions [11,12].

6. Conclusion

A number of other integrals and expansion serie formulae involving product of elementary special functions of one and several variables can be obtained from (2.1), (3.1), (3.2), (3.4), (4.1), (5.1), (5.2), (5.3) and (5.4) as special cases. This can be done by specializing the parameters of the sequence of functions, the class of multivariable polynomials and multivariable Aleph functions in a suitable manner.

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