Int. J. of Mathematical Sciences and Applications,
Vol. 1, No. 3, September 2011

www.journalshub.com

SOME CLASSES OF LOWER TRIANGULAR MATRICES AND THEIR INVERSES

E.N ONWUCHEKWA

Mathematics Department, Rivers State University of Science and Technology

Port Harcourt, Nigeria.

ABSTRACT

This paper introduces some types of lower triangular matrices and shows their structure in order to obtain their inverses. These matrices, also called Pascal-Stirling matrices, include matrices formed from binomial coefficients (Pascal triangle) and Stirling numbers. The matrices have a number of uses in computations of moments, sum of powers and other applications in numerical analysis.

Many illustrative examples are included

Key Words/Phrases : Pascal's Triangle, Stirling Numbers, Lower Triangular Matrices

1. INTRODUCTION

Stirling numbers of the first type play an important role when we try to express the lower factorial polynomials $r_{(k)}$ in terms of the powers of r. Similarly, Stirling numbers of the second type are used when we deal with the inverse problem: namely, to express r^k in terms of the factorial polynomials. See See [Abramowitz, et al (1972), [Adamchik (1997)], [Butzer, et al (1991)], [Dickau (2011)], [Graham, et al (1994)], [Scheid (1989)], [Spiegel (1968)].

In this paper, we highlight a similarity between Stirling numbers and the numbers in Pascal's triangle (the table of binomial coefficients). We take the bold stance that these two types of numbers have a common origin. Apart from being similar in the way the numbers are generated, one notices other similarities. For instance, the matrices formed from these numbers are lower triangular matrices. Also, the diagonal elements are always equal to one (in the case of Pascal's triangle and Stirling numbers Type 1). This leads us to the introduction of Pascal-Stirling matrices).

The rest of the paper is devoted to how we can exploit their special structure to find the inverses of these and other types of lower triangular matrices.

2. PASCAL-STIRLING NUMBERS AND MATRICES

In this section we assume that we are given a finite sequence $Z = \{a_1, a_2, \dots, a_N\}$. **Definition 2.1** (Pascal-Stirling Polynomial)

We define the Pascal-Stirling Polynomials in *r* of order k by

$$p_k(r) = r(r - a_1)(r - a_2) \dots (r - a_{k-1}), \qquad \text{and}$$

$$q_k(r) = r(r + a_1)(r + a_2) \dots (r + a_{k-1})$$

Definition 2.2 (Pascal-Stirling Numbers Type 1 and 3)

The Pascal-Stirling numbers type 1 and order k are the coefficients $t_{kj}^{1,N}$ when $P_k(r)$ is expressed in terms of $r, r^2, \dots, r^k$. The Pascal-Stirling numbers Type 3 and order k are the coefficients $t_{kj}^{1,N}$ when $q_k(r)$ is expressed in terms of $r, r^2, \dots, r^k$. Thus

$$p_k(r) = \sum_{r=1}^{k} t_{k,j}^{1,N}\, r^j \quad , \qquad q_k(r) = \sum_{r=1}^{k} t_{k,j}^{3,N}\, r^j$$

Definition 2.3 (Pascal-Stirling Numbers Types 2 and 4)

The Pascal-Stirling numbers type 2 and order k are the coefficients $t_{k,j}^{2,N}$ when r^k is expressed in terms of $p_1(r), p_2(r), \ldots, p_k(r)$.

The Pascal-Stirling numbers type 4 and order k are the coefficients $t_{k,j}^{4,N}$ when r^k is expressed in terms of $q_1(r), q_2(r), \ldots, q_k(r)$.

Definition 2.4

The Pascal-Stirling matrix $P_1(1,N)$ (resp. $P_1(2,N)$, resp. $P_1(3,N)$ resp. $P_1(4,N)$ is the lower triangular $N \times N$ matrix whose (k,j)-th element is $t_{k,j}^{1,N}$ $\left(\text{resp. } t_{k,j}^{2,N}, \text{resp. } t_{k,j}^{3,N}, \text{resp. } t_{k,j}^{4,N}\right)$.

Note that $P_{2,N} = P_{1,N}^{-1}$ and $P_{4,N} = P_{3,N}^{-1}$

The $N \times N$ matrices above are said to be of order N or of size $N \times N$.

3. **EXAMPLES OF PASCAL-STIRLING NUMBERS AND MATRICES**

Example 3.1 (Pascal's Triangle or Table of binomial coefficients)

If we take $a_k = i$ *for all* k the Pascal-stirling matrix of type 3 is called Pascal's triangle (or table of binomial coefficients).

Example 3.2 (Stirling Numbers types 1 and 2)

Consider the case where $a_k = k$ **for all k.** Then the Pascal-Stirling numbers type 1are called Stirling Numbers type 1. (See [Scheid (1989)],for example). See also Appendix 1.

Also, the Pascal-Stirling numbers type 2 is called Stirling numbers type 2.

Example 3.3 (Pascal-Stirling Numbers Involving Squares of Natural Numbers)

Consider the case where $a_k = k^2$, **for all** $k = 1, 2, \ldots$ Here we obtain Pascal-Stirling numbers which involves the squared numbers 1, 4, 9, 16, …. See Appendix 2. See also [Onwuchekwa (2011:1,2,3)].

Example 3.4 (Pascal-Stirling Numbers Involving Triangular Numbers)

Consider the case where $a_k = \frac{1}{2}k(k+1)$ **for** $k = 1, 2, \ldots$ Here the Pascal-Stirling Numbers involve the triangular numbers. See Appendix 3. See also [Onwuchekwa (2011:1,2,3)].

4. **HALF-PASCAL-STIRLING NUMBERS**

Suppose that we are given a sequence (a_k). Its Half-Stirling numbers are obtained when we take one half of either the sum or the difference of two different types of Stirling numbers. Other half-stirling numbers are obtained when we take the inverses of the matrices generated by the above half-stirling numbers. More precisely we give the following definitions, where N= 2p.

$$h_{k,j}^{1,p} = t_{2k-1,2j-1}^{3}, 2p \qquad\qquad h_{k,j}^{3,p} = t_{2k,2j}^{3}, 2p$$

$$h_{k,j}^{7,p} = \frac{1}{t_{2k+1,2k}^{3}, 2p}\, t_{2k+1,2j}^{3}, 2p \qquad\qquad h_{k,j}^{5,p} = \frac{1}{t_{2k,2k-1}^{3}, 2p}\, t_{2k,2j-1}^{3}, 2p$$

$H_{1,p}, H_{3,p}, H_{5,p}$ and $H_{7,p}$ are the $p \times p$ matrices generated by $\left(h_{k,j}^{1,p}\right), \left(h_{k,j}^{3,p}\right), \left(h_{k,j}^{5,p}\right)$ and $\left(h_{k,j}^{7,p}\right)$ respectively. We define

$\mathbf{H}_{2,p} = \mathbf{H}_{1,p}^{-1}$, $\mathbf{H}_{4,p} = \mathbf{H}_{3,p}^{-1}$, $\mathbf{H}_{6,p} = \mathbf{H}_{5,p}^{-1}$, $\mathbf{H}_{8,p} = \mathbf{H}_{7,p}^{-1}$

are respectively, the k-j-th elements of the $p \times p$ matrices $\mathbf{H}_{2,p}$, $\mathbf{H}_{4,p}$, $\mathbf{H}_{6,p}$, and $\mathbf{H}_{8,p}$.

5. **RECURRENCE RELATIONS FOR PASCAL-STIRLING NUMBERS**

The elements of a Pascal-Stirling matrix can be found using its associated recurrence relations and boundary (and initial conditions)

For Type 1 Pascal-Stirling numbers we have

Recurrence relation: $\mathbf{t}_{k+1,j}^{1,N} = \mathbf{t}_{k,j-1}^{1,N} - \mathbf{a}_{k-1}\mathbf{t}_{k,j}^{1,N}$

Boundary conditions: $\mathbf{t}_{k,k}^{1,N} = 1$ **for all k**

Initial conditions: $\mathbf{t}_{k,1}^{1,N} = \mathbf{a}_1\mathbf{a}_2 \dots \mathbf{a}_{k-1}(-1)^{k-1}$ **for all k.**

For Type 2, we have

Recurrence relation: $\mathbf{t}_{k+1,j}^{2,N} = \mathbf{t}_{k,j-1}^{2,N} + \mathbf{a}_{j-1}\mathbf{t}_{k,j}^{2,N}$

Boundary conditions: $\mathbf{t}_{k,k}^{2,N} = 1$ **for all k**

Initial conditions: $\mathbf{t}_{k,1}^{2,N} = 1$ **for all k.**

For Type 3, we have

Recurrence relation: $\mathbf{t}_{k+1,j}^{3,N} = \mathbf{t}_{k,j-1}^{3,N} + \mathbf{a}_{k-1}\mathbf{t}_{k,j}^{1,N}$

Boundary conditions: $\mathbf{t}_{k,k}^{3,N} = 1$ **for all k**

Initial conditions: $\mathbf{t}_{k,1}^{3,N} = \mathbf{a}_1\mathbf{a}_2 \dots \mathbf{a}_{k-1}$ **for all k.**

For Type 4, we have

Recurrence relation: $\mathbf{t}_{k+1,j}^{4,N} = \mathbf{t}_{k,j-1}^{4,N} - \mathbf{a}_{k-1}\mathbf{t}_{k,j}^{4,N}$

Boundary conditions: $\mathbf{t}_{k,k}^{4,N} = 1$ **for all k**

Initial conditions: $\mathbf{t}_{k,1}^{1,N} = (-1)^{k-1}$ **for all k.**

6. **P-S-L MATRICES**

Definition 6.1

A matrix $A = (a_{ij})$ is said to be a P-S-L (resp. P-S-U) matrix of order N if it satisfies the following conditions:

(1) A is an N×N lower (resp. upper) triangular matrix,

(2) $a_{ii} = 1 \ for \ i = 1, 2, \dots, N.$

Remarks

(1) Triangular matrices derive much of their importance from the `triangular' factorization of matrices, which is helpful in matrix inversion and solution of linear equations.

(2) A lower (resp. upper) triangular matrix is invertible if and only if all its diagonal elements are non-zero. This means that an invertible lower (resp. upper) triangular matrix can be written as the product of a diagonal matrix (with only nonzero elements) and a P-S-L (resp. P-S-U) matrix.

(3) The transpose of a P-S-U matrix is a P-S-L (resp. P-S-U) matrix and vice-versa. Consequently, one can deduce the properties of a P-S-U matrix from those of a P-S-L matrix. We shall therefore limit all further discussion in this paper to P-S-L matrices and other lower triangular matrices.

(4) All Pascal-Stirling matrices as well as Half-Pascal-Stirling matrices are P-S-L matrices.

(5) The set of all P-S-L matrices of a given order N is a group under the operation of matrix multiplication.

Inverse of a P-S-L Matrix

The inverse of a P-S-L matrix is easy to obtain. Infact we can prove the following theorem.

Theorem 6.2 (Inverse of a P-S-L matrix)

Let $\mathbf{B} = (b_{ij})$, the inverse of $A = (a_{ij})$ then

(1) $b_{i,i-1} = -a_{i,i-1}$

(2) $b_{i,i-2} = -a_{i,i-2} + a_{i,i-1} \cdot a_{i-1,i-2}$

(3) $b_{i,i-3} = -a_{i,i-3} + a_{i,i-2} \cdot a_{i-2,i-3} - a_{i,i-1} \cdot a_{i-1,i-3} + a_{i,i-1} \cdot a_{i-1,i-2} \cdot a_{i-2,i-3}$

(4) $b_{i,i-4} = -a_{i,i-4} + a_{i,i-3} \cdot a_{i-3,i-4} - a_{i,i-2} \cdot a_{i-2,i-4}$

$+a_{i,i-1} \cdot a_{i-1,i-4} + a_{i,i-2} \cdot a_{i-2,i-3} \cdot a_{i-3,i-4}$

$+a_{i,i-1} \cdot a_{i-1,i-3} \cdot a_{i-3,i-4} - a_{i,i-1} \cdot a_{i-1,i-2} \cdot a_{i-2,i-4}$

$+a_{i,i-1} \cdot a_{i-1,i-2} \cdot a_{i-2,i-3} \cdot a_{i-3,i-4}$

(5) $b_{i,i-5} = -a_{i,i-5} + a_{i,i-4} \cdot a_{i-4,i-5} - a_{i,i-3} \cdot a_{i-3,i-5} + a_{i,i-2} \cdot a_{i-2,i-5} - a_{i,i-1} \cdot a_{i-1,i-5}$

$+a_{i,i-3} \cdot a_{i-3,i-4} \cdot a_{i-4,i-5} - a_{i,i-2} \cdot a_{i-2,i-4} \cdot a_{i-4,i-5}$

$+a_{i,i-2} \cdot a_{i-2,i-3} \cdot a_{i-3,i-5} - a_{i,i-1} \cdot a_{i-1,i-4} \cdot a_{i-4,i-5}$

$+a_{i,i-1} \cdot a_{i-1,i-3} \cdot a_{i-3,i-5} - a_{i,i-1} \cdot a_{i-1,i-2} \cdot a_{i-2,i-5}$

$+a_{i,i-2} \cdot a_{i-2,i-3} \cdot a_{i-3,i-4} \cdot a_{i-4,i-5} - a_{i,i-1} \cdot a_{i-1,i-3} \cdot a_{i-3,i-4} \cdot a_{i-4,i-5}$

$+a_{i,i-1} \cdot a_{i-1,i-2} \cdot a_{i-2,i-4} \cdot a_{i-4,i-5} - a_{i,i-1} \cdot a_{i-1,i-2} \cdot a_{i-2,i-3} \cdot a_{i-3,i-5}$

$+a_{i,i-1} \cdot a_{i-1,i-2} \cdot a_{i-2,i-3} \cdot a_{i-2,i-3} \cdot a_{i-3,i-4} \cdot a_{i-4,i-5}$

Theorem 6.3

If a P-S matrix A is partitioned such that

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

Then $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are also P-S matrices.

Corollary 6.1 Suppose that $= A^{-1}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & \\ a_{21} & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & \\ b_{21} & 1 \end{bmatrix}$$ then $b_{21} = -a_{21}$

Corollary 6.2 Suppose that $= A^{-1}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & & \\ a_{21} & 1 & \\ a_{31} & a_{32} & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & & \\ a_{21} & 1 & \\ a_{31} & a_{32} & 1 \end{bmatrix}$$

then $b_{21} = -a_{21}, b_{32} = -a_{32}, b_{31} = -a_{31} + a_{32}a_{21}$

Corollary 6.3 Suppose that $= A^{-1}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & & & \\ a_{21} & 1 & & \\ a_{31} & a_{32} & 1 & \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & & & \\ a_{21} & 1 & & \\ a_{31} & a_{32} & 1 & \\ b_{41} & b_{42} & b_{43} & 1 \end{bmatrix}$$

then $b_{21} = -a_{21}, b_{32} = -a_{32}, b_{43} = -a_{43}, \quad b_{31} = -a_{31} + a_{32}a_{21}$

$b_{42} = -a_{42} + a_{43}a_{32}, \quad b_{41} = -a_{41} + (a_{42}a_{21} + a_{43}a_{31}) - a_{43}a_{32}a_{21}$

Corollary 6.4 Suppose that $= A^{-1}$, where

$$A = \begin{bmatrix} 1 & & & & \\ a_{21} & 1 & & & \\ a_{31} & a_{32} & 1 & & \\ a_{41} & a_{42} & a_{43} & 1 & \\ a_{51} & a_{52} & a_{53} & a_{54} & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & & & & \\ a_{21} & 1 & & & \\ a_{31} & a_{32} & 1 & & \\ b_{41} & b_{42} & b_{43} & 1 & \\ b_{51} & b_{52} & b_{53} & b_{54} & 1 \end{bmatrix}$$

then

(i) $b_{21} = -a_{21}, b_{32} = -a_{32}, b_{43} = -a_{43}, b_{54} = -a_{54}$

(ii) $b_{31} = -a_{31} + a_{32}a_{21}$, $b_{42} = -a_{42} + a_{43}a_{32}$, $b_{53} = -a_{53} + a_{54}a_{43}$

(iii) $b_{41} = -a_{41} + (a_{42}a_{21} + a_{43}a_{31}) - a_{43}a_{32}a_{21}$,

$b_{52} = -a_{52} + (a_{53}a_{32} + a_{54}a_{42}) - a_{54}a_{43}a_{32}$

(iv) $b_{51} = -a_{51} + (a_{52}a_{21} + a_{53}a_{31} + a_{54}a_{41}) - (a_{53}a_{32}a_{21} + a_{54}a_{42}a_{21} + a_{54}a_{43}a_{31}) + a_{54}a_{43}a_{32}a_{21}$

7. **B-P-S-L Matrices**

A matrix A is called a B-P-S-L matrix if it has a partition of the form

$$A = \begin{bmatrix} A_{11} & \dots & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix}$$

with $A_{ii} = I,$ an identity matrix and $A_{ij} = 0 \; if \; i < j.$ Similarly, A matrix is called a B-P-S-U matrix if its transpose is a B-P-S-L matrix.

Inverse of a B-P-S-L Matrix

The inverse of a B-P-S-L matrix is the same as in section 6 except that a_{ij} is replaced with A_{ij} and b_{ij} is replaced with B_{ij} for all i and j .

Theorem 7.6

Suppose that $= A^{-1}$, and

$$A = \begin{bmatrix} A_{11} & & & & \\ a_{21} & A_{22} & & & \\ a_{31} & a_{32} & A_{33} & & \\ a_{41} & a_{42} & a_{43} & A_{44} & \\ a_{51} & a_{52} & a_{53} & a_{54} & A_{55} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & & & & \\ a_{21} & B_{22} & & & \\ a_{31} & a_{32} & B_{33} & & \\ b_{41} & b_{42} & b_{43} & B_{44} & \\ b_{51} & b_{52} & b_{53} & b_{54} & B_{55} \end{bmatrix}$$

where A and B are partitioned as above. Then

(i) $B_{ii} = A_{ii}^{-1} \;\; (i = 1, 2, \dots, 5)$

(ii) $B_{21} = -B_{22}A_{21}B_{11}$, $B_{32} = -B_{33}A_{32}B_{22}$, $B_{43} = -B_{44}A_{43}B_{33}$, $B_{54} = -B_{55}A_{54}B_{44}$

(iii) $B_{31} = -B_{33}A_{31}B_{11} + B_{33}A_{32}B_{22}A_{21}B_{11}$, $B_{42} = -B_{44}A_{42}B_{22} + B_{44}A_{43}B_{33}A_{32}B_{22}$

$B_{53} = -B_{55}A_{53}B_{33} + B_{55}A_{54}B_{44}A_{43}B_{33}$

(iv) $B_{41} = -B_{44}A_{41}B_{11} + B_{44}A_{42}B_{22}A_{21}B_{11} + B_{44}A_{43}B_{33}A_{31}B_{11} - B_{44}A_{43}B_{33}A_{32}B_{22}A_{21}B_{11}$

(v) $B_{1}51 = -B_{\downarrow}55\ A_{1}51\ B_{1}11 + (B_{1}55\ A_{1}52\ B_{1}22\ A_{1}21\ B_{\downarrow}11 + B_{1}55\ A_{1}53\ B_{1}33\ A_{1}31\ B_{1}11 + B_{1}55\ A_{1}54\ B_{\downarrow}44\ A_{1}4$

8. **LOWER TRIANGULAR MATRICES**

In this section we discuss lower triangular matrices and their properties. Upper triangular matrices also have similar properties.

Definition 8.1

An $\mathbf{N \times N}$ **matrix** $\mathbf{A} = (a_{ij})$ is said to be a lower triangular matrix if $a_{ii} = 0$ whenever $i < j$ $(i = 1,2, \dots, N)$. The order of the matrix is N.

Lemma 8.1

An $\mathbf{N \times N}$ lower triangular matrix is invertible if and only if $a_{ij} \neq 0$ **for all** $i = 1, \dots, N$.

Theorem 8.1

The set of all lower triangular matrices of order N is a group under the operation of matrix multiplication

Inverse of a lower triangular Matrix

The inverse of an invertible lower matrix (resp. upper) triangular matrix is often easy to obtain. It is usually obtained using the method of undetermined coefficients and backward substitution. (Other methods of course exist). See [Conte et al(1972)], [Dass, H.K (2007)], [Ralston (1965)], [Salstry (1990)] for example. The use of partitioning (or decomposition of matrices) is also common.

We have the following result.

Theorem 8.2

Suppose that C= DA, where A is a P-S matrix. If B is the inverse of a, then $C^{-1} = BD^{-1}$.

Theorem 8.3

Let $A = (a_{ij})$. $B = (b_{ij})$. Suppose that A is a lower triangular matrix and that $b = A^{-1}$. Then

1) $b_{i,i} = a_{i,i}^{-1}$
2) $b_{i,i-1} = -b_{i,i}a_{i,i-1}b_{i-1,i}$
3) $b_{i,i-2} = b_{i,i}\left[-a_{i,i-2} + a_{i,i-1}b_{i-1,i-1}a_{i-1,i-2}\right]b_{i-2,i-2}$
4) $b_{i,i-3} = b_{i,i}\left[-a_{i,i-3} + \left(a_{i,i-2}b_{i-2,i-2}a_{i-2,i-3} + a_{i,i-1}b_{i-1,i-1}a_{i-1,i-3}\right) - a_{i,i-1}b_{i-1,i-1}a_{i-1,i-2}b_{i-2,i-2}\right.$
5)

9. **BLOCK LOWER TRIANGULAR MATRICES**

Definition

A matrix A is called a block lower triangular matrix (resp. block upper triangular matrix) if it has a partition of the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} & \dots & \mathbf{A_{1n}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} & \dots & \mathbf{A_{2n}} \\ \dots & \dots & \dots & \dots \\ \mathbf{A_{n1}} & \mathbf{A_{n2}} & \dots & \mathbf{A_{nn}} \end{bmatrix}$$

with $A_{ij} = 0 \ \ if \ i < j$

A matrix is called a block upper triangular matrix if its transpose is a block lower triangular matrix.

Remark

A block lower triangular matrix is not necessarily a triangular matrix.

Inverse of a Block Lower Triangular Matrix

The inverse of a block lower triangular matrix is the same as section 8 except that a_{ij} is replaced with A_{ij} and b_{ij} is replaced with B_{ij} .

10. **A GUIDED TOUR**

The procedure for obtaining the inverse of a triangular matrix or even a block lower matrix can be described in the language of Operations Research. For a block lower triangular matrix we can proceed along the following lines.

(1) Compute $B_{ii} = A_{ii}^{-1} \ for \ all \ i$

(2) Replace A_{ii} with B_{ii} for all i

(3) Write $B_{ij} = 0$ *whenever* $i < j$

(4) To compute B_{ij} where $i > j$ we proceed as follows

(4. 1) Consider all feasible tours cell (i,j) to cell (j,j).

(4. 2) The sign of a tour is $(-1)^k$ where 2k is the number of moves from (i,j) to (j,j).

(4. 3) The numerical value of the move is the product of the entries in the cells, where the product is taken in the order in which the moves were made.

(4. 4) The signed value of the move is the product of the answers in (4.3) and(4.4).

(4. 5) B_{ij} is the sum of all the signed values of the various moves, where the sum is taken over all the feasible moves.

Remark In the above procedure, a feasible move is a path of the form

$(i,i) \to (i,j) \to (j,j) \to (i_1,j_2) \to \square .. \to (i_k,j_{k+1}) \dots \to (j,j)$. Thus a feasible move is a sequence of alternating horizontal and vertical moves from (i,i) to (j,j) where each odd-numbered node is on the principal diagonal.

Example 10.1

The feasible moves from (5, 5) to (1, 1) are tabulated below with their signs and nodes for the block lower triangular matrix $A = (A_{ij})$. We assume that $B = A^{-1}$ where $B = (B_{ij})$. Feasible move

(a) (5, 5)→ (5, 1)→ (1, 1)

(b) (5, 5)→(5, 2)→(2, 2)→(2,1)→(1, 1)

(c) (5, 5)→(5, 3)→(3, 3)→(3, 1)→(1,1)

(d) (5, 5)→(5, 4)→(4, 4)→(4, 1)→(1, 1)

(e) (5, 5)→(5, 3)→(3, 3)→(3, 2→(2, 2)→(2,1)→(1, 1)

(f) (5, 5)→(5, 4)→(4, 4)→(4, 2)→(2, 2)→(2, 1)→(1, 1)

(g) (5, 5)→(5, 4)→(4, 4)→(4, 3)→(3, 3)→(3, 1)→(1, 1)

(h) (5, 5)→(5, 4)→(4, 4)→(4, 3)→(3, 3)→(3, 3)→(2, 2)→(2, 1)→(1, 1)

We observe that here we have one tour of length 2, three tours of length 4, three tours of length 6 and one tour of length 8, as expected. The signs of those tours are respectively $-, +, +, +, -, -, -, +$. Using the nodes as subscripts of the B_{ij} and A_{ij}, and taking the signs into consideration, we obtain the expression

$$B_{15} = B_{55}[-A_{51} + (A_{51}B_{22}A_{21} + A_{53}B_{33}A_{31} + A_{54}B_{44}A_{41}) - (A_{53}B_{33}A_{32}B_{22}A_{21} + A_{54}B_{44}A_{42}B_{22}A_{21} + A_{54}B_{44}A_{43}B_{33}A_{31})$$

11. INVERSIONOF MATRICES AND SOLUTION OF LINEAR SYSTEMS

One of the well-known methods of matrix inversion and solution of linear systems of equations is the factorization of matrices. See [Conte et al (1972)], [Ralston (1965)], [Salstry (1990)] for example. Our contribution here is that the process can often be speeded up if we take full advantage of the properties of P-S-L matrices as we shall see below.

Theorem 11.1 Suppose that A= BC where $A = (a_{ij}), B = (b_{ij})$ *and* $C = (c_{ij})$ are n×n matrices. Suppose that B is a matrix (i.e a lower triangular matrix with ones on the main diagonal), and C is an upper triangular matrix. Suppose that $c_{11} \neq 0$. Then

$c_{ij} = a_{ij}$ $(j = 1,2,\dots n)$ and $b_{ii} = \frac{a_{i1}}{c_{11}} = \frac{a_{i1}}{a_{11}}$ $(i = 1,2,\dots,n)$

Theorem 11.2

Suppose $C = (c_{ij})$ is an n×n upper triangular matrix and E is a P-S-U matrix.

The application of these theorems, which are easy to prove, are illustrated in the following example.

12. **EXAMPLES**

EXAMPLE 1

Find the inverse of the P-S matrix

$$A = \begin{bmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ 4 & -5 & 1 & & & \\ -36 & 49 & -14 & 1 & & \\ 576 & -820 & 273 & -30 & 1 & \\ -14400 & 21076 & -7645 & 1023 & -55 & 1 \end{bmatrix}$$

Solution

Let us write $B = A^{-1}$

$$A = \begin{bmatrix} A_{11} & \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & \\ B_{21} & B_{22} \end{bmatrix}$$

where $A_{11}, A_{21}, A_{22}, B_{11}, B_{21}, B_{22}$ are all 3×3 matrices. Then

$$B_{11} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 5 & 1 \end{bmatrix} = , \quad B_{22} = \begin{bmatrix} 1 & & \\ 30 & 1 & \\ 627 & 55 & 1 \end{bmatrix}$$

since $(-1)(-5) - 4 = 1$ and $(-30)(-55) - 1023 = 627$

Consequently, we obtain

$$B_{21} = -B_{22}A_{21}B_{11}$$

$$= -\begin{bmatrix} 1 & & \\ 30 & 1 & \\ 627 & 55 & 1 \end{bmatrix}\begin{bmatrix} -36 & 49 & -14 \\ 576 & -820 & 273 \\ -14400 & 21076 & -7645 \end{bmatrix}\begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 21 & 14 \\ 1 & 85 & 147 \\ 1 & 341 & 1408 \end{bmatrix}$$

Hence $$B = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 5 & 1 & & & \\ 1 & 21 & 14 & 1 & & \\ 1 & 85 & 147 & 30 & 1 & \\ 1 & 341 & 1408 & 627 & 55 & 1 \end{bmatrix}$$

Alternative solution 1

Let us write $B = A^{-1}$

$$A = \begin{bmatrix} A_{11} & & \\ A_{21} & A_{22} & \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & & \\ B_{21} & B_{22} & \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

where $A_{11}, A_{21}, A_{22}, A_{31}, A_{32}, A_{33}$ as well as $B_{11}, B_{21}, B_{22}, B_{31}, B_{32}, B_{33}$ are all 2×2 matrices. Then

$$B_{11} = \begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix}, B_{22} = \begin{bmatrix} 1 & \\ 14 & 1 \end{bmatrix}, B_{33} = \begin{bmatrix} 1 & \\ 55 & 1 \end{bmatrix}$$

Consequently we obtain $B_{21} = -B_{22}A_{21}B_{11}$

$$= \begin{bmatrix} 1 & \\ 14 & 1 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ -36 & 49 \end{bmatrix}\begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 1 & 21 \end{bmatrix}$$

Consequently, we obtain $B_{32} = -B_{33}A_{32}B_{22}$

$$= -\begin{bmatrix} 1 & \\ 55 & 1 \end{bmatrix}\begin{bmatrix} 273 & -30 \\ -7645 & 1023 \end{bmatrix}\begin{bmatrix} 1 & \\ 14 & 1 \end{bmatrix} = \begin{bmatrix} 147 & 30 \\ 1408 & 627 \end{bmatrix}$$

$B_{31} = -B_{33}A_{31}B_{11} + B_{33}A_{32}B_{22}A_{21}B_{11}$

$$= -\begin{bmatrix} 1 & \\ 55 & 1 \end{bmatrix}\begin{bmatrix} 576 & -820 \\ -14400 & 21076 \end{bmatrix}\begin{bmatrix} 1 & \\ 14 & 1 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ -36 & 49 \end{bmatrix}\begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \\ 1 & 341 \end{bmatrix}$$

Hence $$B = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 5 & 1 & & & \\ 1 & 21 & 14 & 1 & & \\ 1 & 85 & 147 & 30 & 1 & \\ 1 & 341 & 1408 & 627 & 55 & 1 \end{bmatrix}$$

<u>Alternative solution 2</u>

<u>L</u>et us write $B = A^{-1}$ where $A = (a_{ij}), B = (b_{ij})$. We note that $b_{ij} = 0$ whenever $i < j$.

Also,

$b_{ii} = 1 \quad (i = 1, 2, \ldots, 6)$ $\qquad b_{21} = 1, b_{32} = 5, b_{43} = 14, b_{54} = 30, b_{65} = 55$

$b_{31} = -4 + (-5)(-1) = 1$ $\qquad b_{42} = -49 + (-14)(-5) = 21$

$b_{53} = -273 + (-30)(-14) = 147$ $\qquad b_{64} = -1023 + (-55)(-30) = 627$

$b_{41} = -(-36\ \ 49\ \ -14).(1\ \ 1\ \ 1) = 1$ $\qquad b_{52} = -(820\ \ 273\ \ -30).(1\ \ 5\ \ 2) = 85$

$b_{63} = -(7645\ \ 1023\ \ -55).(1\ \ 14\ \ 147) = 1408$

$b_{51} = -(576\ \ -820\ \ 273\ \ -30).(1\ \ 1\ \ 1\ \ 1) = 1$

$b_{62} = -(21076\ \ -7645\ \ 1023\ \ -55).(1\ \ 5\ \ 21\ \ 85) = 341$

$b_{61} = -(-1440 - 0\ \ 21076\ \ -7645\ \ 1023\ \ -55).(1\ \ 5\ \ 21\ \ 85) = 1$

Hence $$B = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 5 & 1 & & & \\ 1 & 21 & 14 & 1 & & \\ 1 & 85 & 147 & 30 & 1 & \\ 1 & 341 & 1408 & 627 & 55 & 1 \end{bmatrix}$$

<u>Example 2</u> Solve the system equations

$2x + 3y + z = 9$

$x + 2y + z = 6$

$3x + y + 2z = 8$

by the factorization method.

<u>Solution</u>Let us write

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ \frac{1}{2} & 1 & \\ \frac{3}{2} & b_{32} & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ & c_{22} & c_{23} \\ & & c_{33} \end{bmatrix}$$

Then $\frac{1}{2}(3) + c_{22} = 2$ $\quad , \frac{1}{2} + c_{23} =$

Then on multiplying out we obtain

$\frac{1}{2}(3) + c_{22} = 2$, $\quad \frac{1}{2} + c_{23} = 3$, $\quad \frac{3}{2}(3) + b_{32}c_{22} = 1$, $\frac{3}{2} + b_{32}c_{23} + c_{33} = 2$

These imply $c_{22} = \frac{1}{2}$, $c_{23} = \frac{5}{2}$, $\quad b_{32} = -7$, $\quad c_{33} = 18$

It follows that

$$\begin{bmatrix}2&3&1\\1&2&3\\3&1&2\end{bmatrix}=\begin{bmatrix}1&&\\\frac{1}{2}&1&\\\frac{3}{2}&-7&1\end{bmatrix}\begin{bmatrix}2&3&1\\&\frac{1}{2}&\frac{5}{2}\\&&18\end{bmatrix}=\begin{bmatrix}1&&\\\frac{1}{2}&1&\\\frac{3}{2}&-7&1\end{bmatrix}\begin{bmatrix}2&&\\&\frac{1}{2}&\\&&18\end{bmatrix}\begin{bmatrix}1&\frac{3}{2}&\frac{1}{2}\\&1&\frac{5}{2}\\&&1\end{bmatrix}$$

Taking the inverses using Theorem … , we have

$$\begin{bmatrix}2&3&1\\1&2&3\\3&1&2\end{bmatrix}^{-1}=\begin{bmatrix}1&-\frac{3}{2}&7\\&1&-5\\&&1\end{bmatrix}\begin{bmatrix}\frac{1}{2}&&\\&2&\\&&\frac{1}{18}\end{bmatrix}\begin{bmatrix}1&&\\-\frac{1}{2}&1&\\-5&7&1\end{bmatrix}=\frac{1}{18}\begin{bmatrix}1&-5&7\\7&1&-5\\-5&7&1\end{bmatrix}$$

Hence the solution of the system is given by

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{18}\begin{bmatrix}1&-5&7\\7&1&-5\\-5&7&1\end{bmatrix}\begin{bmatrix}9\\6\\8\end{bmatrix}=\frac{1}{18}\begin{bmatrix}35\\29\\5\end{bmatrix}$$

And so $x=\frac{35}{18}, y=\frac{29}{18}$ *and* $z=\frac{5}{18}$

Alternatively we note that

$$\begin{bmatrix}2&3&1\\1&2&3\\3&1&2\end{bmatrix}^{-1}=\begin{bmatrix}\frac{1}{2}&f_1&f_3\\&2&f_2\\&&\frac{1}{18}\end{bmatrix}\begin{bmatrix}1&&\\\frac{1}{2}&1&\\-5&7&1\end{bmatrix}$$

where $f_1=\frac{2}{2\left(\frac{1}{2}\right)}=3$, $f_2=\frac{\frac{5}{2}}{\left(\frac{1}{2}\right)(9)}=-\frac{5}{9}$, $f_3=\frac{1}{2(18)}=+\frac{3\left(\frac{5}{2}\right)}{2\left(\frac{1}{2}\right)(18)}=-\frac{1}{36}+\frac{15}{36}=\frac{14}{36}=\frac{7}{18}$

Thus

$$\begin{bmatrix}2&3&1\\1&2&3\\3&1&2\end{bmatrix}^{-1}=\begin{bmatrix}\frac{1}{2}&-3_1&\frac{7}{18}\\&2&-\frac{5}{9}\\&&\frac{1}{18}\end{bmatrix}\begin{bmatrix}1&&\\\frac{1}{2}&1&\\-5&7&1\end{bmatrix}=\frac{1}{18}\begin{bmatrix}1&-5&7\\7&1&-5\\-5&-7&1\end{bmatrix}$$

The rest of the solution follows as before. See also [10].

13. SUMMARY AND CONCLUSION

In this paper, the following types of lower triangular matrices were defined and discussed: (1) Pascal-Stirling matrices (2) Half-Pascal-Stirling matrices (3) P-S-L matrices. Reference was also made to B-P-S-L and block lower triangular matrices.

It was emphasized that Pascal's triangle and Stirling's matrices (derived from Stirling numbers) are members of the same subfamily in lower triangular matrices. The subfamily is the set of P-S-L matrices as they are called in this paper. The subfamily is actually a multiplicative subgroup of the set of all invertible lower triangular matrices. They have many other interesting properties, including the fact that their inverses can be found quite easily using `a guided tour'.

The paper also discusses various extensions and applications.

REFERENCES

1. Abramowitz, M and Stegun, I.A (eds):"Stirling Numbers of the First Kind".§24.1.3 in Handbook of Mathematical functions and Formulas, Graphs and Mathematical Tables, 9th Printing, New York, p.824, 1972.
2. Adamchik, V. "On Stirling and Euler Sums', J. Computer Appl. Math. 79, 119-130, 1997
3. Butzer, P.L and Hauss, M.:"Stirling functions of the First kinds: Some new Applications". Israel Mathematical conference Proceedings: approximation, Interpolation, and Summability, In Honor of Ammon Jakimovski on his Sixty-Fifth birthday. (Ed. S. Baron and D. Leviatan), Ramat Gan, Israel: IMCP pp.89-108, 1991.

4. Conte S.D and Carl de Boor, Elementary Numerical Analysis, 2nd ed., McGraw-Hill, Kogakusha, Ltd, Tokyo, London, 1972.
5. Dass, H.K. advanced Engineering Mathematics, S. Chand & Co. Ltd, New Delhi, 2007
6. Dickau, R.M. "Stirling Numbers of the First Kind", http://mathforum.org/advanced/robertd/stirlingl.html, 2011.
7. Graham, R.L; Knuth, D.E; and Patashnik, O, "Stirling Numbers'. §6.1 in Concrete Mathematics: A Foundation for Computer science, 2nd ed, Published by Addison-Wesley, Reading, Mass., U.S.A. pp 257, 1994.
8. Krishnamurthy P.V .and S.K. Sen, Numerical Analysis: Computations in Science and Engineering, 2nd ed, Affiliated East-west Press Pvt Ltd, New Delhi, 1989.
9. Onwuchekwa, E.N., "Factorial Polynomials and Pascal-Stirling Matrices'. International Journal of Mathematical Sciences and Applications. Cited as [Onwuchekwa (2011)].
10. Onwuchekwa, E.N., "Moments and Backward Repeated Cumulatives". Accepted for Publication in International Journal of Numerical Mathematics. Cited as [Onwuchekwa (2011)].
11. Onwuchekwa, E.N., "Some Matrix Solutions of the Ordinary Moments Problem". International Journal of Mathematical Sciences and Applications. Cited as [Onwuchekwa (2011)].
12. Ralston, Anthony. A First Course in Numerical Analysis, McGraw-Hill, Kogakusha, Ltd, Tokyo, London, 1965.
13. Salstry, S.S. . Introductory Methods of Numerical Analysis, 2nd ed., Prentice-Hall of India Pvt Ltd, New Delhi, 1990.
14. Scheid, Francis: Schaum's Outline of Theory and Problems of Numerical Analysis, 2nd ed., McGraw-Hill, New York and London, 1989, pp.30-61, 163-183, 241-274, 354-404.
15. Spiegel, M.R., Mathematics Handbook (Schaum's Series), McGraw Hill, New York and London, 1968 pp. 107-108, 114-115.

APPENDIX

A1 PASCAL-STIRLING AND HALF-PASCAL-STIRLING MATRICES PASCAL-STIRLING MATRICES FOR THE SEQUENCE $a_k=k$

$$T_1.6 = \begin{bmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ 2 & -3 & 1 & & & \\ -6 & 11 & -6 & 1 & & \\ 24 & -50 & 35 & -10 & 1 & \\ -120 & 274 & -225 & 85 & -15 & 1 \end{bmatrix}, \quad T_2.6 = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 7 & 6 & 1 & & \\ 1 & 15 & 25 & 10 & 1 & \\ 1 & 31 & 90 & 65 & 15 & 1 \end{bmatrix}$$

$$T_3.6 = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 3 & 1 & & & \\ 6 & 11 & 6 & 1 & & \\ 24 & 50 & 35 & 10 & 1 & \\ 120 & 274 & 225 & 85 & 15 & 1 \end{bmatrix}, \quad T_4.6 = \begin{bmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ 1 & -3 & 1 & & & \\ -1 & 7 & -6 & 1 & & \\ 1 & -15 & 25 & -10 & 1 & \\ -1 & 31 & -90 & 65 & -15 & 1 \end{bmatrix}$$

A.2: PASCAL-STIRLING MATRICES FOR THE SEQUENCE $a_k=k^2$

$$T_1.3 = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ 4 & -5 & 1 \end{bmatrix}, \quad T_2.3 = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 5 & 1 \end{bmatrix}$$

A.3: HALF-PASCAL-STIRLING MATRICES FOR THE SEQUENCE $a_k=k$

$$H_1.3 = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 24 & 35 & 1 \end{bmatrix}, \quad H_2.3 = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ 46 & -35 & 1 \end{bmatrix}$$

$$H_3.3 = \begin{bmatrix} 1 & & \\ 11 & 1 & \\ 274 & 85 & 1 \end{bmatrix}, \quad H_4.3 = \begin{bmatrix} 1 & & \\ -11 & 1 & \\ 661 & -85 & 1 \end{bmatrix}$$

$$H_5.3 = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 8 & 15 & 1 \end{bmatrix}, \quad H_6.3 = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ 7 & -15 & 1 \end{bmatrix}$$

$$H_7, 3 = \begin{bmatrix} 1 & & \\ 5 & 1 & \\ 84 & 35 & 1 \end{bmatrix}, \qquad H_8, 3 = \begin{bmatrix} 1 & & \\ -5 & 1 & \\ 91 & -35 & 1 \end{bmatrix}$$