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NUMERICAL ASYMPTOTIC METHOD FOR SINGULAR PERTURBATION PROBLEMS

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Abstract

In this paper a numerical asymptotic method is presented for solving singularly perturbed two point boundary value problems with boundary layer at left end of the underlying interval. The method is distinguished by the following fact: the original singularly perturbed two point boundary value problem is divided into two problems, namely inner and outer region problems. The terminal boundary condition is obtained from the solution of the reduced problem. We used the stretching transformations to construct a modified inner region problem. We implemented the method on five problems with left end boundary layer for different values of perturbation parameter . The proposed method is iterative on the terminal point. The process is to be repeated for various choices of the terminal point. The numerical solutions are compared with the exact solution. From the results, it is observed that the present method approximates the exact solution very well.

Keywords: Singular perturbation problems,inner region ,outer region ,boundary layer ,stretching transformation ,terminal point.

1 Introduction

Singular perturbation problems are of common occurrence in many branches of applied mathematics, such as fluid dynamics, elasticity, chemical reactor theory, aerodynamics, magneto hydrodynamics and plasma dynamics. A few notable examples are boundary layer problems, WKB problems, the modeling of steady and unsteady viscous flow problems with large Reynolds number, convective heat transport problems with large peclet numbers etc.

It is well known fact that the solutions of these problems exhibit a multi-scale character. That is there is /are thin layer(s) where the solution varies rapidly (non-uniformly), while away from the layer, the solution behaves regularly and varies slowly. Therefore, the numerical treatment of singularly perturbed boundary value problems gives major computational difficulties.

A wide variety of papers and books have been published in the recent years, describing various methods for solving singular perturbation problems, among these we mention Bender and Orszag[2], Hemker and Miller [3], Kevorkian and Cole [4], Nayfeh [5], O'Malley[6], Y.N.Reddy [7] and D.R. Smith [9]

In this paper a numerical asymptotic method is presented for solving singularly perturbed two point boundary value problems with boundary layer at left end of the underlying interval. The method is distinguished by the following fact: the original singularly perturbed two point boundary value problem is divided into two problems, namely inner and outer region problems. The terminal boundary condition is obtained from the solution of the reduced problem. We used the stretching transformations to construct a modified inner region problem. We implemented the method on five problems with left end boundary layer for different values of perturbation parameter . The proposed method is iterative on the terminal point. The process is to be repeated for various choices of the terminal point. The numerical solutions are compared with the exact solution.It is observed that the method approximates the exact solution very well.

2 Description of the Method

To describe the method, we consider a class of singularly perturbed two point boundary value problem of the form

$$''+ \ (\ , \ ) \ ' + \ (\ , \ ) \ = h(\ , \ ) \qquad [0.1] \qquad (1)$$

$$(0, \ ) = \ (\ ) \qquad (2.a)$$

$$(1, \ ) = \ (\ ) \qquad (2.b)$$

where is small positive parameter $(0 < \ll 1)$, , and h are sufficiently smooth functions and $(,) \geq > 0$, $(,) \leq ;$ $-4 \geq 0$, , for all $[0,1]$ and for all . These conditions provide the inverse monotonicity and the existence of a unique solution $( )$ which is stable, for each fixed . If $(,) \geq > 0$, the solution displays a boundary layer at the left end. If $(,) \leq - < 0$ the solution displays a boundary layer at the right end. The width of the layer is $(.)$

In order to evaluate the asymptotic solution for the problem(1), (2) we shall use the asymptotic power expansion for the functions , and h in the form

$$\begin{matrix} (,) \\ (,) \\ h(,) \end{matrix} = \begin{matrix} ( ) \\ ( ) \\ h ( ) \end{matrix}$$

Without loss of generality, we will assume that the boundary layer is at the left end of the interval [0,1]. We shall also represent the boundary conditions as the asymptotic power series

$$\begin{matrix} ( ) \\ ( ) \end{matrix} = {}^{\infty}$$

Thus the problem (1), (2) becomes

$$''+ {}^{\infty} ( ) \ '+ {}^{\infty} ( ) = {}^{\infty} h ( ) \quad ; \quad [0,1] \tag{3}$$

$$(0,) = {}^{\infty} \tag{4.a}$$

$$(1,) = {}^{\infty} \tag{4.b}$$

The reduced problem for (3), (4), which is same as the reduced problem for (1), (2) is

$$( ) \ ' + ( ) = h ( ) \quad [0,1] \tag{5}$$

With

$$(1) = \tag{6}$$

and its solution ( ) can be evaluated explicitly. For a detailed discussion we may refer to Vesna Varcel J. et.al. [10]. In this paper we shall describe two techniques to determine the asymptotic solution of (1), (2).

Technique I (O' Malley [6])

Let us consider an asymptotic solution of the form

$$(\, , \,) = \overset{\infty}{} \quad (\,) \tag{7}$$

which is given in O'Malley [6].

The solution does not satisfy the differential equation uniformly in x. So we look for two solutions, the outer solution (,) and the inner solution (,). Each of these two solutions satisfies the differential equation (1) only on a subinterval of [0,1]. The outer solution is of the form

$$(\, , \,) = \overset{\infty}{} \quad (\,) \tag{8}$$

If we take into account only the first term of the above expansion we obtain

$$(\, , \,) = \quad (\,) = \quad (\,) + \quad (\,) \tag{9}$$

According to O' Malley [6] function ( ) is determined as the solution of the problem

$$(\,) \; {}' (\,) + \quad (\,) \quad (\,) = h \; (\,) \qquad [0,1] \tag{10}$$

$$(1) = \tag{10.a}$$

It is obvious that

$$(\,) = \quad (\,)$$

The inner solution is of the form

$$(\ ,\ ) = \overset{\infty}{\ } \quad (\ ) \tag{11}$$

We take into account only the first term and obtain

$$(\ ,\ ) = \quad (\ ) = \quad (\ ) + \quad (\ ) \tag{12}$$

For determining ( )we introduce a new variable = / which will stretch out the boundary layer. We construct a new differential equation for the inner region solution from (1)

$$(\ ) = \quad (\ \ ) = \quad (\ )$$

$$'(\ ) = \frac{1}{\ } \quad '(\ \ ) = \frac{1}{\ } \quad '(\ )$$

$$''(\ ) = \frac{1}{\ } \quad ''(\ \ ) = \frac{1}{\ } \quad ''(\ )$$

Substituting in (1)

$$.\frac{1}{\ } \quad ''(\ ) + \quad (\ \ ,\ \ )\frac{1}{\ } \quad '(\ ) + \quad (\ \ ,\ \ )\ (\ ) = h(\ \ ,\ \ )$$

$$''(\ ) + \quad (\ \ ,\ \ ) \quad '(\ ) + \quad (\ \ ,\ \ )\ (\ ) = \quad h(\ \ ,\ \ ) \tag{13}$$

We approximate (, by) (0) and use the “matching” method from O’Malley [6] to obtain

$$(\ ) = \quad - \quad (0) \ \exp \ - \quad (0) - \tag{14}$$

Thus

$$(\ ) = \quad (\ ) + \quad - \quad (0) \ \exp \ - \quad (0) - \tag{15}$$

represents the asymptotic expansion of order ( )for the solution of the problem (1), (2)

Technique II (Smith [9])

We now describe the second asymptotic expansion from Smith [9]. Let us assume that it is of the form

$$(\ ) = \ (\ ) + \ (\ ) \exp\ -\ (\ ) \qquad (16)$$

Where

$$(\ ) = \frac{1}{\ } \quad (\ ) \qquad (17)$$

and ( ), ( ) are asymptotic series expansions

$$(\ ) = \sum^{\infty} \ (\ )$$

$$(\ ) = \sum^{\infty} \ (\ )$$

which are to be determined. Restricting these series to their first terms and substituting (16) in (3), (4) we obtain the equalities which determine ( ) and ( ) which are

$$(\ )\ '(\ ) + \ (\ )\ \ (\ ) = h\ (\ )$$

and

$$-\ \ (\ )\ \ (\ ) = \ (\ )(\ \ (\ ) - \ (\ )\ \ (\ ))$$

with the boundary conditions $(1) = \ ;\ \ (0) + \ (0) =$

It is well known that for these two asymptotic solutions the error estimate is

$$(\ ) - \ (\ ) \leq \qquad = 1,2,.. \qquad (18)$$

and is valid, uniformly for $[0,1]$ and > 0 Using (18) and (14) we get

$$|\ (\ ) - \ (\ )| = |\ (\ ) - \ (\ )| \leq \ + |\ - \ (0)| \exp(-\ (0) -) \qquad (19)$$

Numerical Asymptotic Method

We now combine this asymptotic approach with a numerical procedure. We shall first determine the reduced solution $(\;)$ for the problem $(5), (6)$ by solving (10). We shall represent the solution as:

$$(\;) = \quad (\;) + \quad (\;) \tag{20}$$

where $(\;)$ satisfies the differential equation

$$''(\;) + \sum^{\infty} (\;) \quad (\;) + \sum^{\infty} (\;) \quad (\;) = - \quad ''(\;) + \quad (\;) \tag{21}$$

with the boundary conditions

$$(\;,\;) = \quad - \quad (0) + \quad (\;) \tag{22.a}$$

$$(1,\;) = \quad (\;) \tag{22.b}$$

Introducing the substitution

$$= \quad (0) -$$

and using the standard procedure for the first order asymptotic expansion the solution of $(21), (22)$ becomes

$$'' + \quad (\;) \quad ' = 0 \tag{23}$$

$$(0) = \quad - \quad (0) \tag{24.a}$$

$$\frac{(0)}{} = 0 \tag{24.b}$$

where $\qquad (\;) = \frac{\overline{\quad(\;)}}{(\;)}$

Both the techniques give us equation (23)-(24). In case of (15) we should take

$$(\frac{}{(0)}) \approx \quad (0)$$

Thus (23) becomes

$$'' + \quad ' = 0 \qquad (25)$$

$$(0) = \quad - \quad (0) \qquad (26.a)$$

$$\frac{(0)}{} \quad = 0 \qquad (26.b)$$

We solve (25), (26) numerically to provide the approximate layer solution .For that we divide the interval $[0, \frac{(\ )}{}]$, by a suitably chosen division point $\bar{}$, into two subintervals $[0, \ \bar{}]$ and $[\ \bar{}, \frac{(\ )}{}]$. On the second subinterval the solution of (3), (4) will be approximated by the reduced solution

$$(\ ) = \qquad / \quad (0)$$

And on $[0, \ \bar{}]$ we look for solution of the form

$$(\ ) = \quad (\ ) + \quad (\ ) \qquad (27)$$

Where

$$(\ ) = \quad (\qquad /(0))$$

$$. \quad (\ ) = \quad \frac{}{(0)} \qquad (28)$$

The function $(\ )$ satisfies the differential equation (23) and the boundary conditions

$$(0) = \quad - \quad (0) \qquad (29.a)$$

$$(\bar{}) = \quad (\bar{}) - y \ (\bar{}) \qquad (29.b)$$

We shall determine it numerically as the solution of the problem

$$''(\ ) + \quad (\ ) \ '(\ ) = 0 \qquad [0, \ ] \qquad (30)$$

$$(0) = \quad _0 - \quad (0) \qquad (31.a)$$

$$(\ ) = 0 \qquad (31.b)$$

The numerical solution is denoted by ( ) We now use finite difference method to obtain a three term recurrence relation. It is solved using Thomas Algorithm. The method is iterative on the terminal point ( ). We repeat the process for various values of ' '

3 Numerical Examples

To demonstrate the applicability of the method we have applied it to five singular perturbation problems. These examples have been chosen because they have been widely discussed in literature and because approximate solutions are available for comparison. The approximate solution is compared with the exact solution.

Example 3.1: Consider the following homogeneous SPP from Kevorkian and Cole [4, p.33, Eqs.(2.3.26) and (2.3.27)] with $= 0$

$$'' + \ '(\ ) = 0; \qquad [0,1]$$

$$h(0) = 0 \qquad (1) = 1.$$

The exact solution is given by $(\ ) = \frac{(\qquad (\ )\ /\ )}{(\qquad (\ )\ /\ )}$

The numerical results are presented in **Table 1(a)** and **1(b)** for $= 10$ $= 10$ respectively.

Example 3.2: Consider the following singular perturbation problem from Bender and Orszag [2, pp.480; problem 9.17 with α=0]

$$''(\ ) + \ '(\ ) - \ (\ ) = 0, \qquad \epsilon[0,1]$$

with y(0)=1 and y(1)=1

The exact solution is given by $(\ ) = \frac{[(e \qquad)e \qquad (\quad e\ )e \quad]}{[e \qquad e\ ]},$

where $= \frac{\sqrt{\quad}}{}$, $= \frac{\sqrt{\quad}}{}$

The numerical results are given in **Table 2(a)** and **2(b)** for ε= 10^{-2} and =10^{-3} respectively.

Example 3.3: Now consider the following non homogeneous singular perturbation problem from fluid dynamics for fluid of small viscosity (Reinhardt [8],example 2)

$$"(\ ) + \ '(\ ) = 1 + 2 \qquad \epsilon[0,1]$$

with $(0) = 1$ and $(1) = 1$.

The exact solution is given by $(\ ) = \ (\ \ + 1 - 2\ ) + \frac{(\quad)(\quad e^{\ /\ })}{(\quad e\quad)}$

The numerical results are given in **Table 3(a)** and **3(b)** for ε= 10^{-2} and ε=10^{-3} respectively.

Example 3.4: We consider the following variable coefficients singular perturbation problem from Kevorkian and Cole [4, p.33; eqs (2.3.26) and (2.3.27) with α=-1/2]

$$"(\ ) + (1 - \frac{}{2})y'(x) - \frac{1}{2}y(x) = 0, \qquad \epsilon[0,1]$$

with $(0) = 1$ and $(1) = 1$.

We have chosen to use uniformly valid approximation, which is obtained by the method given by Nayfeh [5],p.148;eq (4.2.32) as our exact solution.

$$(\ ) = \frac{}{} - -e^{(\ \frac{\overline{\ }}{}\ )}$$

The numerical results are given in **Table 4(a)** and **4(b)** for ε= 10^{-2} and =10^{-3} respectively.

Example 3.5: Consider the following non linear example from Bender and Orszag [2, p.463, Eq (9.7.1)]. We have used quasilinearisation process and applied our numerical asymptotic method.

$$"(\ ) + 2\ '(\ ) + \quad = 0; \qquad \epsilon[0,1]$$

with $(0) = 0$ and $(1) = 0$.

We have chosen to use Bender and Orszags uniformly valid approximation [2, p 463, Eq.(9.7.6)] for comparison.

$$(\ ) = \quad \frac{2}{1+} - \quad \exp \frac{2}{} \qquad\qquad 2$$

For this example, we have boundary layer of width O( ) at x=0 (cf Bender and Orszag [2])

The numerical results are given in **Table 5(a)** and **5(b)** for $\varepsilon = 10^{-2}$ and $=10^{-3}$ respectively.

3.1 Discussion and Conclusions

We have presented a numerical asymptotic method for solving singularly perturbed two – point boundary value problems with boundary layer at left end point. The solution of the given problem is computed numerically by dividing it into inner and outer region problems. A terminal boundary condition is obtained from the solution of the reduced problem. A new inner region problem is constructed and solved as a two point boundary value problem. The proposed method is iterative on the terminal point. The process is to be repeated for various choices of , the terminal point, until the solution profiles do not differ materially from iteration to iteration. Numerical results are presented in tables. It is observed that the present method approximates the exact solution very well.

Table 1 (a) Numerical results for **Example 3.1** with = 10

	( )		Exact Solution
	=5	=10	
0.000	0.0000000	0.0000000	0.0000000
0.005	0.3937668	0.3937668	0.3934693
0.010	0.6324781	0.6324781	0.6321206
0.015	0.7772043	0.7772043	0.7768699
0.020	0.8649502	0.8649502	0.8646648
0.025	0.9181435	0.9181435	0.9179150
0.030	0.9503918	0.9503918	0.9502130
0.035	0.9699421	0.9699421	0.9698026
0.040	0.9817948	0.9817948	0.9816844
0.045	0.9889803	0.9889803	0.9888910
0.050	0.9933367	0.9933367	0.9932621
0.200	1.0000000	1.0000000	1.0000000
0.400	1.0000000	1.0000000	1.0000000
0.600	1.0000000	1.0000000	1.0000000
0.800	1.0000000	1.0000000	1.0000000
1.000	1.0000000	1.0000000	1.0000000

Table 1 (b): Numerical results for **Example 3.1** with = 10

	( )		Exact Solution
	=5	=10	
0.0000	0.0000000	0.0000000	0.0000000
0.0005	0.3963991	0.3937668	0.3934693
0.0010	0.6367077	0.6324781	0.6321206
0.0015	0.7854032	0.7772043	0.7768699
0.0020	0.8707378	0.8649502	0.8646648
0.0025	0.9242895	0.9181435	0.9179150
0.0030	0.9567561	0.9503918	0.9502130
0.0035	0.9764402	0.9699421	0.9698026
0.0040	0.9883751	0.9817948	0.9816844
0.0045	0.9956117	0.9889803	0.9888910
0.0050	1.0000000	0.9933367	0.9932621
0.2000	1.0000000	1.0000000	1.0000000
0.4000	1.0000000	1.0000000	1.0000000
0.6000	1.0000000	1.0000000	1.0000000
0.8000	1.0000000	1.0000000	1.0000000
1.0000	1.0000000	1.0000000	1.0000000

Table 2 (a) : Numerical results for **Example 3.2** with = 10

	( )		Exact Solution
	=5	=10	
0.000	0.9998795	0.9998795	1.0000000
0.005	0.7511992	0.7528629	0.7526670
0.010	0.6011775	0.6038505	0.6041340
0.015	0.5109603	0.5142460	0.5152320
0.020	0.4570048	0.4606625	0.4623240
0.025	0.4250414	0.4289256	0.4311440
0.030	0.4064132	0.4104354	0.4130810
0.035	0.3958730	0.3999797	0.4029370
0.040	0.3902396	0.3943985	0.3975780
0.045	0.3875856	0.3917766	0.3951100
0.050	0.3867410	0.3909523	0.3943900
0.200	0.4493290	0.4493290	0.4528670
0.400	0.5488116	0.5488116	0.5520500
0.600	0.6703200	0.6703200	0.6729540
0.800	0.8187308	0.8187308	0.8203380
1.000	1.0000000	1.0000000	1.0000000

Table 2 (b) : Numerical results for **Example 3.2** with = 10

	( )		Exact Solution
	=5	=10	
0.0000	0.9998795	0.9998795	1.0000000
0.0005	0.7495391	0.7512029	0.7514170
0.0010	0.5978482	0.6005213	0.6007920
0.0015	0.5059528	0.5092384	0.5095510
0.0020	0.4503096	0.4539673	0.4543110
0.0025	0.4166493	0.4205335	0.4208980
0.0030	0.3963148	0.4003371	0.4007110
0.0035	0.3840591	0.3881658	0.3885470
0.0040	0.3767008	0.3808596	0.3812460
0.0045	0.3723121	0.3765031	0.3768930
0.0050	0.3697234	0.3739347	0.3743260
0.2000	0.4493289	0.4493290	0.4496880
0.4000	0.5488116	0.5488116	0.5491400
0.6000	0.6703200	0.6703200	0.6705880
0.8000	0.8187301	0.8187308	0.8188940
1.0000	1.0000000	1.0000000	1.0000000

Table 3 (a): Numerical results for **Example 3.3** with $= 10$

	( )		Exact Solution
	=5	=10	
0.000	0.0000000	0.0000000	0.0000000
0.005	-0.3913740	-0.3887410	-0.3806750
0.010	-0.6266077	-0.6223782	-0.6095780
0.015	-0.7671782	-0.7619793	-0.7464070
0.020	-0.8503379	-0.8445502	-0.8273710
0.025	-0.8986645	-0.8925185	-0.8744320
0.030	-0.9258562	-0.9194918	-0.9009090
0.035	-0.9402152	-0.9337171	-0.9148820
0.040	-0.9467751	-0.9401948	-0.9212510
0.045	-0.9485866	-0.9419553	-0.9229880
0.050	-0.9475001	-0.9408357	-0.9218970
0.200	-0.7600000	-0.7600000	-0.7440000
0.400	-0.4400000	-0.4400000	-0.4280000
0.600	-0.0400000	-0.0400000	-0.0320000
0.800	0.4400000	0.4400000	0.4440000
1.000	1.0000000	1.0000000	1.0000000

Table 3 (b): Numerical results for **Example 3.3** with $= 10$

	( )		Exact Solution
	=5	=10	
0.0000	0.0000000	0.0000000	0.0000000
0.0005	-0.3958980	-0.3932660	-0.3921830
0.0010	-0.6357060	-0.6314772	-0.6298570
0.0015	-0.7809010	-0.7757021	-0.7738170
0.0020	-0.8687339	-0.8629462	-0.8609350
0.0025	-0.9217833	-0.9156374	-0.9135780
0.0030	-0.9537472	-0.9473827	-0.9453100
0.0035	-0.9729280	-0.9664299	-0.9643560
0.0040	-0.9843591	-0.9777789	-0.9757130
0.0045	-0.9910914	-0.9844601	-0.9824020
0.0050	-0.9949751	-0.9883117	-0.9862600
0.2000	-0.7600000	-0.7600000	-0.7584000
0.4000	-0.4400000	-0.4400000	-0.4388000
0.6000	-0.0400000	-0.0400000	-0.0392000
0.8000	0.4400000	0.4400000	0.4404000
1.0000	1.0000000	1.0000000	1.0000000

Table 4 (a): Numerical results for **Example 3.4** with = 10

	( )		Exact Solution
	=5	=10	
0.000	0.0000000	0.0000000	0.0000000
0.005	0.1985219	0.1970857	0.1977980
0.010	0.3195580	0.3172495	0.3181120
0.015	0.3936391	0.3908002	0.3915840
0.020	0.4392346	0.4360726	0.4367030
0.025	0.4675259	0.4641668	0.4646400
0.030	0.4852942	0.4818145	0.4821540
0.035	0.4966566	0.4931030	0.4933380
0.040	0.5041157	0.5005167	0.5006720
0.045	0.5091943	0.5055167	0.5056660
0.050	0.5128205	0.5091762	0.5092340
0.200	0.5555555	0.5555555	0.5555560
0.400	0.6250000	0.6250000	0.6250000
0.600	0.7142857	0.7145857	0.7142860
0.800	0.8333333	0.8333333	0.8333330
1.000	1.0000000	1.0000000	1.0000000

Table 4 (b): Numerical results for **Example 3.4** with = 10

	( )		Exact Solution
	=5	=10	
0.0000	0.0000000	0.0000000	0.0000000
0.0005	0.1981500	0.1968240	0.1968410
0.0010	0.3183900	0.3162461	0.3162640
0.0015	0.3913686	0.3887357	0.3887470
0.0020	0.4356920	0.4327600	0.4327650
0.0025	0.4626338	0.4595207	0.4595190
0.0030	0.4790314	0.4758081	0.4758020
0.0035	0.4890329	0.4857426	0.4857320
0.0040	0.4951532	0.4918221	0.4918070
0.0045	0.4989179	0.4955618	0.4955450
0.0050	0.5012531	0.4978818	0.4978630
0.2000	0.5555555	0.5555555	0.5555560
0.4000	0.6250000	0.6250000	0.6250000
0.6000	0.7142857	0.7142857	0.7142860
0.8000	0.8333333	0.8333333	0.8333333
1.0000	1.0000000	1.0000000	1.0000000

Table 5 (a): Numerical results for **Example 3.5** with = 10

	( )		Exact Solution
	=5	=10	
0.000	0.0000000	0.0000000	0.0000000
0.005	0.4060336	0.4060152	0.4331650
0.010	0.5527760	0.5527506	0.5893900
0.015	0.6041580	0.6041302	0.6437490
0.020	0.6204758	0.6204470	0.6606490
0.025	0.6239043	0.6238752	0.6637840
0.030	0.6225988	0.6225695	0.6618700
0.035	0.6195585	0.6195292	0.6581140
0.040	0.6158866	0.6158573	0.6536940
0.045	0.6119888	0.6119595	0.6490450
0.050	0.6080142	0.6079849	0.6443260
0.200	0.4918246	0.4318246	0.5108260
0.400	0.3498588	0.3498588	0.3566750
0.600	0.2214027	0.2214027	0.2231440
0.800	0.1051709	0.1051709	0.1053600
1.000	0.0000000	0.0000000	0.0000000

Table 5 (b): Numerical results for **Example 3.5** with = 10

	( )		Exact Solution
	= 5	=10	
0.0000	0.0000000	0.000000	0.0000000
0.0025	0.6423255	0.642296	0.6859800
0.0030	0.6446738	0.644644	0.6884340
0.0035	0.6452773	0.645248	0.6896210
0.0040	0.6452393	0.645210	0.6889230
0.0045	0.6449654	0.644936	0.6885720
0.0050	0.6446046	0.644575	0.6881260
0.2000	0.4918247	0.491824	0.5108260
0.4000	0.3498588	0.349858	0.3566750
0.6000	0.2214028	0.221402	0.2231440
0.8000	0.1051710	0.105170	0.1053600
1.0000	0.0000000	0.000000	0.0000000

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