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The Koebe's Differential Equation Derived from the Third Basic Univalent Meijer's G-function, $G^{1,1}_{1,1}$

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Abstract

This paper gives the relationship between the path, and series representation for the third basic univalent Meijer's G-function $G^{1,1}_{1,1}$. The Meijer's G-functions are complex functions including all elementary function and most of the well-known special functions. They satisfies the linear ordinary differential equation of the generalized hypergeometric type. An interesting finding is that the Koebe function, as the distinguished univalent function, is a Meijer's G-function. We derive the Koebe function from G-function $G^{1,1}_{1,1}$. Finally, we obtain Koebe's differential equation by using modified separation of variables method (MSV) introduced recently by Pishkoo and Darus.

Keywords: Univalent function; Meijer's G-function; Koebe function; Modified separation of variables.

1 Introduction.

It is well known that the Koebe function plays an important role in the theory of univalent functions. The Koebe function is in S and maps the unit disk E in one-to-one and onto the domain D that covers the entire complex plane except the branch cut for a slit along the negative real axis from $w = -\infty$ to $w = -\frac{1}{4}$. The series representation for the Koebe function is as follows [1, 2]:

$$K(z) = \frac{z}{(1-z)^2} = \sum_{n=0}^{\infty}(n+1)z^{n+1} = z + 2z^2 + 3z^3 + ... \quad . \tag{1.1}$$

In the first part of this manuscript we deduce the theorem representing the path integral representation for the G-function $G^{1,1}_{1,1}$ which is shown the Kobe function is its special case. Our previous work had focused on the introduction of three basic univalent G-functions. These functions are $G^{1,0}_{0,2}$, $G^{1,1}_{1,2}$, and $G^{1,1}_{1,1}$ (see [3]). We demonstrate that these G-functions can be as the solutions of some physical problems. For this intention in [4, 5, 6, 7] the newly method "Modified separation of variables method" (MSV)has been introduced, and applied to solve some partial differential equations; which led to representing its solution in terms of Meijer's G-functions. In the second part of this manuscript, the MSV method is used to obtain differential equation for the Koebe function.

The paper is organized as follows: Section 2 provides notations and the basic definition of Meijer's G-function. Section 3 introduces the Modified separation of variables (MSV) suggested by Pishkoo and Darus in [4]. Section 4 consists of the main results of the paper.

2 Meijer's G-functions

In mathematics, the G-function was introduced by Cornelis Simon Meijer (1936) as a very general function intended to include all elementary functions and most of the known special functions, for instance:

- $\sin z = \sqrt{\pi} G^{1,0}_{0,2}(\frac{z^2}{4}|^{-}_{\frac{1}{2},0}), \qquad -\frac{\pi}{2} < \arg z \leq \frac{\pi}{2}$
- $\cos z = \sqrt{\pi} G^{1,0}_{0,2}(\frac{x^2}{4}|^{-}_{0,\frac{1}{2}}), \qquad \forall z$
- $\ln z = G^{1,2}_{2,2}(x-1|^{1,1}_{1,0}), \qquad \forall z$
- $J_\nu(z) = G^{1,0}_{0,2}(\frac{z^2}{4}|^{-}_{\frac{\nu}{2},\frac{-\nu}{2}}), \qquad -\frac{\pi}{2} < \arg z \leq \frac{\pi}{2}$

A definition of the Meijer's G-function is given by the path integral in the complex plane, called Mellin-Barnes type integral see [7-11].

$$G^{m,n}_{p,q}(^{a_1,\ldots,a_p}_{b_1,\ldots,b_q}|z) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\ \prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\ \prod_{j=n+1}^{p}\Gamma(a_j - s)} z^s ds. \tag{2.1}$$

For the function

$$G^{m,n}_{p,q}(^{a_1,\ldots,a_p}_{b_1,\ldots,b_q}|z) \tag{2.2}$$

The integers $m; n; p; q$ are called orders of the G-function, or the components of the order $(m; n; p; q)$; a_p and b_q are called "parameters" and in general, they are complex numbers. The definition holds under the following assumptions: $0 \leq m \leq q$ and $0 \leq n \leq p$, where m, n, p, and q are integer numbers.

Based on the definition, the following basic properties are easily derived:

$$z^{\alpha}\, G^{m,n}_{p,q}\left({}^{\mathbf{a_p}}_{\mathbf{b_q}}|z\right) = G^{m,n}_{p,q}\left({}^{\mathbf{a_p}+\alpha}_{\mathbf{b_q}+\alpha}|z\right), \tag{2.3}$$

where the multiplying term z^{α} changes the parameters of the G-function.

In [7] all elementary functions can be expressed in terms of G-functions. For instance

$$(1-z)^{-\alpha} = \frac{1}{\Gamma(\alpha)} G^{1,1}_{1,1}\left({}^{1-\alpha}_{0}|z\right). \tag{2.4}$$

Wherein, if $\alpha = 2$, then (2.4) implies that

$$\frac{1}{(1-z)^2} = G^{1,1}_{1,1}\left({}^{-1}_{0}|z\right). \tag{2.5}$$

The Meijer's G-function $y(z) = G^{m,n}_{p,q}\left({}^{a_j}_{b_k}|z\right)$ satisfies the linear ordinary differential equation of the generalized hypergeometric type whose order is equal to $\max(p, q)$. (see [7,12-14])

$$[(-1)^{p-m-n} z \prod_{j=1}^{p}(z\frac{d}{dz} - a_j + 1) - \prod_{k=1}^{q}(z\frac{d}{dz} - b_k)]y(z) = 0. \tag{2.6}$$

3 "Modified Separation of Variables" method

The method proposed in [4],i.e. the "modified separation of variables", leads to obtain the solution of partial differential equations as a product of the Meijer's G-functions, with each one depending on one variable by making links between ordinary linear differential equation of the Meijer's G-functions and each one of the $ODEs$. This method consist of the following steps:

1) Choosing a convenient coordinates system by considering the boundary conditions and boundary surfaces.

2) Obtaining $ODEs$ from the PDE by writing the solution as a product form of different variables and putting it into the PDE.

3) Starting with (2.6), the orders m, n, p and q are chosen and the variable z is changed to $\acute{z}$, such that (2.6) converts into each of the $ODEs$, and this is 3 times for time independent problems (e.g., x, y, z coordinates)and 4 times for time dependent problems (e.g., x, y, z, t). If this particular step is done, solving $ODEs$ is not required because (2.6) is an equation that does not need to be solved. It is solved, and the solutions are $MGFs$

4) Using boundary conditions and obtaining the exact solution.

4 Main Results

First we deduce an important theorem for the third basic univalent G-function. This theorem relates the path integral representation to the series representation for univalent G-function $G^{1,1}_{1,1}$. Then it is shown that the Koebe function is made as the special case of $G^{1,1}_{1,1}$ by choosing special values for the parameters (a_1, b_1) of the G-function $G^{1,1}_{1,1}(^{a_1}_{b_1}|-z)$.

4.1 The Third Basic Univalent G-function $G^{1,1}_{1,1}$

Letting $m = 1; n = 1; p = 1; q = 1$ in (2.1) gives

$$G^{1,1}_{1,1}(^{a_1}_{b_1}| - z) = \frac{1}{2\pi i}\int_L \Gamma(b_1 - s)\Gamma(1 - a_1 + s)(-z)^s ds. \tag{4.1}$$

1. Position of poles $\Gamma(b_1 - s) : s = b_1 + n; \quad n = 0, 1, 2, ...$.
2. Position of poles $\Gamma(1 - a_1 + s) : s = a_1 - 1 - n; \quad n = 0, 1, 2, ...$.

Here we have:

Theorem 4.1 *Let $a_1 - b_1 \neq 1, 2, 3, ...$,which implies that no pole of $\Gamma(b_1 - s)$ coincides with any pole of $\Gamma(1 - a_1 + s)$, then*

$$G^{1,1}_{1,1}(^{a_1}_{b_1}| - z) = \sum_{n=0}^{\infty} \frac{\Gamma(1 - a_1 + b_1 + n)}{n!} z^{b_1+n} \tag{4.2}$$

Proof 1 *At a simple pole, the residue of function f is given by*

$$Res(f, c) = lim_{s\to c}(s - c)f(s).$$

L in (4.1) is a loop beginning and ending at $+\infty$, encircling all poles of $\Gamma(b_1-s)$ exactly once in the negative direction, but not encircling any pole of $\Gamma(1-a_1+s)$. So the residue is given with $\frac{(-1)^{n-1}}{n!}$. Then by putting $s = n + 1$ in $\Gamma(1 - a_1 + s)(-z)^s$ we obtain (4.2).

Multiplying (2.5) by z and taking $\alpha = 2$ in (2.4), we get $zG^{1,1}_{1,1}(^{-1}_{0}|z) = G^{1,1}_{1,1}(^{0}_{1}|z)$ which specifies the Koebe function as the G-function

$$K(z) = \frac{z}{(1 - z)^2} = G^{1,1}_{1,1}(^{0}_{1}|z). \tag{4.3}$$

Having Equation (4.3), if we put $a_1 = 0$ and $b_1 = 1$ in (4.1) then we get an integral representation for the Koebe function

$$K(z) = \frac{z}{(1 - z)^2} = G^{1,1}_{1,1}(^{0}_{1}| - z) = \frac{1}{2\pi i}\int_L \Gamma(1 - s)\Gamma(1 + s)z^s ds. \tag{4.4}$$

Corollary 4.2 *Putting $a_1 = 0$ and $b_1 = 1$ in (4.2) verifies (4.4)*

$$G^{1,1}_{1,1}(^{0}_{1}| - z) = \sum_{n=0}^{\infty} \frac{\Gamma(2 + n)}{n!} z^{n+1} = \sum_{n=0}^{\infty}(n + 1)z^{n+1} = z + 2z^2 + 3z^3 + ...$$

4.2 Koebe's differential equation

For the orders $m = 1, n = 1, p = 1, q = 1$, and the parameters $a_1 = 0, b_1 = 1$, Equation (2.6) is thus reduced to:

$$z(z-1)\frac{d}{dz}K(z) + (z+1)K(z) = 0. \tag{4.5}$$

Example 4.1 Find the orders and parameters of convex function $f = \frac{z}{1-z}$, as the G-function, and its differential equation

Putting $\alpha = 1$, and using again from (2.3), namely $zG_{1,1}^{1,1}\left(\begin{smallmatrix}0\\0\end{smallmatrix}\middle|z\right) = G_{1,1}^{1,1}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\middle|z\right)$, we get

$$f(z) = \frac{z}{(1-z)} = G_{1,1}^{1,1}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\middle|z\right]. \tag{4.6}$$

Applying the orders $m = 1, n = 1, p = 1, q = 1$, and the parameters $a_1 = 1, b_1 = 1$ in (2.3), one easily obtains

$$z(z-1)\frac{d}{dz}f(z) + f(z) = 0. \tag{4.7}$$

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