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SOME COMMON FIXED POINT THEOREMS IN D* - METRIC SPACE

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Abstract

In this paper we establish some common Fixed Point Theorems for contraction and generalized contraction mappings in D* - metric space which is introduced by Shaban Sedghi, Nabi Shobe and Haiyun Zhou [10]. In what follows (X, D*) will denote D* - metric space, N, the set of all natural number and R^+, the set of all positive real number.

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1. Introduction

There have been an number of generalization in generalized metric space (or D-Metric space) initated by Dhage [2] in 1992. He proved the existence of unique fixed point theorems of a self map satisfying a contractive conditions in complete and bounded D- Metric space. Dealing with D- Metric space, Ahmad etal. [1], Dhage [2, 3, 4] Rhoades [8], Singh and Sharma [9] and others made a significant contribution in fixed point theory of D-Metric space. Unfortunately almost all theorems in D-Metric space are not valid (See S.V.R Naidu and others [5-7]). Here our aim is to prove some common fixed point theorems using some generalized contractive conditions in D*- Metric space as a probable modification of the definition of D-Metric spaces introduced by Dhage [2].

Definition 1.1.

Let X be a non empty set. A generalized metric (or D* - metric) on X is a function $D^* : X^3 \to [0, \infty)$ that satisfies the following conditions for each x, y, z, a $\in$ X.

1. $D^*(x, y, z) \geq 0$
2. $D^*(x, y, z) = 0$ if and only if $x = y = z$
3. $D^*(x, y, z) = D^*(\rho\{x, y, z\})$ where ρ is permutation.
4. $D^*(x, y, a) \leq D^*(x, y, z) + D^*(a, z, z)$.

The pair (X, D*) is called generalized metric (or D* - metric) space.

Example 1.2:

(a) $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$,

(b) $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.

Here, d is the ordinary metric on X.

(c) If $X = R^n$ then we define

$D^*(x, y, z) = (\|x - y\|^p + \|y - z\|^p + \|z - x\|^p)^{1/p}$ for every $p \in R^+$

(d) If X = R then we define

$$D^*(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \operatorname{Max}\{x, y, z\} & \text{other wise,} \end{cases}$$

Remark 1.3.

In D* - metric space $D^*(x, y, y) = D^*(x, x, y)$

Definition 1.4.

A open ball in a D* - metric space X with centre x and radius r is denoted by $B_{D^*}(x, r)$ and is defined by $B_{D^*}(x,r) = \{y \in X : D^*(x, y, y) < r\}$

Example 1.5.

Let X=R Denote $D^*(x, y, z) = |x-y| + |y-z| + |z-x|$ for all $x, y, z \in R$.

Thus $B_{D^*}(0, 1) = \{y \in R / D^*(0, y, y) < 1\}$

$= \{ y \in R / |0 - y| + |y - y| + |y - 0| < 1\}$

$= \{ y \in R / |y| + |y| < 1\}$

$= \{ y \in R / |y| < ½ \}$

$= \{ y \in R / - ½ < y < ½ \}$

$= (-½, ½)$. (Open Interval)

Definition 1.6.

Let (X, D*) be a D* - metric space and $A \subseteq X$

1. If for every $x \in A$, there exist $r > o$ such that $B_{D^*}(x, r) \subseteq A$, then subset A is called open subset of X.
2. Subset A of A is said to be D* - bounded if there exist $r > o$ such that $D^*(x, y, y) < r$ for all $x, y \in A$.
3. A sequence $\{x_n\}$ in X converges to x if and only if $D^*(x_n, x_n, x) = D^*(x, x, x_n) \rightarrow 0$ as $n \rightarrow \infty$.
   That is, for each $\varepsilon > 0$ there exist $n_0 \in N$ such that for all $n \geq n_0$ implies $D^*(x, x, x_n) < \varepsilon$ This is equivalent for each $\varepsilon > 0$, there exist $n_0 \in N$ such that for all n, $\geq n_0$ implies $D^*(x, x_n, x_n) < \varepsilon$.
   It is also noted that $D^*(x_n, x_n, x) = D^*(x, x, x_n) < \varepsilon$ for all $n \geq n_0$, for some $n_0 \in N$.
4. A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$, there exist $n_0 \in N$ such that $D^*(x_n, x_n, x_m) < \varepsilon$ for each $n, m \geq n_0$ The D* - metric space (X, D*) is said to complete if every Cauchy sequence is convergent.

Remark 1.7.

(1) D* is continuous function X^3

(2) If sequence $\{x_n\}$ in X converges to x, then x is unique.

(3) Any convergent sequence in (X, D*) is a Cauchy sequence.

Definition 1.8.

A point x in X is a fixed point of the map $T : X \rightarrow X$ if $Tx = x$.

Definition 1.9.

A point x in X is a common fixed point of the two maps $T_1, T_2 : X \rightarrow X$ if $T_1(x) = T_2(x) = x$.

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Theorem 1

Let X be a D* - complete metric space and $T_1, T_2 : X \rightarrow X$ be any two maps such that $D^*(T_1x, T_2y, z) \leq \alpha D^*(x, y, z)$ for all $x, y, z \in X$ and $o \leq \alpha < ½$ Then T_1 & T_2 have a unique common fixed point.

Proof

Let $x_o \in X$ be any fixed arbitrary element Define a sequence $\{x_n\}$ in X as.

$x_{n+1} = T_1 x_n$ and

$x_{n+2} = T_2 x_{n+1}$ for n = 0, 1, 2, ….

Let $d_n = D^*(x_n, x_{n+1}, x_{n+1})$ for all n = 0, 1, 2, ….

Now $d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+2})$

$= D^*(T_1x_n, T_2x_{n+1}, x_{n+2})$

$\leq \alpha D^*(x_n, x_{1n+1}, x_{n+2})$

$\leq \alpha D^*(x_n, x_{1n+1}, x_{n+1}) + \alpha D^*(x_{n+1}, x_{1n+2}, x_{n+2})$

$= \alpha d_n + \alpha d_{n+1}$

$(1 - \alpha) d_{n+1} \leq \alpha d_n$

$$d_{n+1} \leq \frac{\alpha}{1-\alpha} d_n$$

$d_{n+1} \leq k d_n$ for all n =0,1,2…, where $k = \dfrac{\alpha}{1-\alpha} < 1$ (Since $\alpha < ½$)

$d_n \leq k d_{n-1}$

.

$\leq k^n d_0 \rightarrow 0$ as $n \rightarrow \infty$

There fore $\lim_{n\to\alpha} d_n = 0$ Thus $\lim_{n\to\alpha} D^*(x_n, x_{n+1}, x_{n+1}) = 0$

Now we shall prove that $\{x_n\}$ is a D* - Cauchy sequence in X.

Let m > n >, n_0 for some $n_0 \in$ N.

Now $D^*(x_n, x_n, x_m) \le D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m)$

$$\le \sum_{k=\infty}^{m-1} D^*(x_k, x_k, x_{k+1}) \rightarrow 0 \text{ as } m, n \rightarrow \infty$$

Thus $\lim_{n,m\to\alpha} D^*(x_n, x_n, x_m) = 0$

There fore $\{x_n\}$ is D* - Cauchy sequence in X.

Since X is D* - Complete $x_n \rightarrow x$ in X. we prove that x is a fixed point of T_1 supper $x \ne T_1x$.

Then $D^*(T_1x, x, x) = \lim_{n\to\infty} D^*(T_1 x, x_{n+2}, x)$

$$= \lim_{n\to\infty} D^*(T_1x, T_2x_{n+1}, x)$$

$$\le \alpha \lim_{n\to\infty} D^*(x, x_{n+1}, x)$$

$$= 0.$$

There fore $D^*(T_1x, x, x) = 0$, Therefore $T_1 x=x$ Simillarly we can prove that $T_2x=x$.

Hence $T_1x = T_2x = x$..Thus x is common fixed point of T_1 and T_2

Uniqueness

Supper $x \ne y$ such that $T_1y = T_2y = y$

Then $D^*(x, y, y) = D^*(T_1x, T_2y, y)$

$$\le \alpha D^*(x, y, y)$$

This implies $(1 -\alpha) D^*(x, y, y) \le 0$

Since $x \ne y$ we have $D^*(x, y, y) > 0$ her $(1 -\alpha) < 0$.

This implies $\alpha > 1$ which contraction to $\alpha < ½$.

Thus T_1 and T_2 have a unique common fixed point.

Theorem 2

Let X be a complete D* - metric space and $T_1, T_2, T_3 : X \rightarrow X$ be any three maps such that $D^*(T_1x, T_2y, T_3z) \le \alpha D^*(x, y, z)$ for all $x, y, z \in X$ and $0 \le \alpha < 1$. Then T_1, T_2, T_3 have a unique common fixed point.

Proof

Let $x_0 \in X$ he any fixed arbitrary element Define a sequence $\{x_n\}$ in X as

$$x_{n+1} = T_1 x_n$$
$$x_{n+2} = T_2 x_{n+1}$$
$$x_{n+3} = T_3 x_{n+2} \qquad \text{for } n = 0, 1, 2, \ldots.$$

Let $d_n = D^*(x_n, x_{n+1}, x_{n+2})$

$$d_1 = D^*(x_1, x_2, x_3)$$
$$= D^*(T_1x_0, T_2x_1, T_3x_2)$$
$$\le \alpha D^*(x_0, x_1, x_2)$$
$$d_1 \le \alpha d_0$$
$$d_2 = D^*(x_2, x_3, x_4)$$
$$= D^*(T_2 x_1, T_3 x_2, T_1 x_3)$$
$$\le \alpha D^*(x_1, x_2, x_3)$$
$$\le \alpha d_1.$$
$$\le \alpha^2 do$$

Continuing in thus way he get $d_n \le \alpha^n d_0 \rightarrow 0$ as $n\rightarrow\infty$ (since $0 \le \alpha < 1$).

Now we shall prome that $\{x_n\}$ is a Cauchy sequence in X.

Let $d_n^* = D^*(x_n, x_n, x_{n+1})$

Then $d_{n+1}^* = D^*(x_{n+1}, x_{n+1}, x_{n+2})$

$$\le D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1})$$
$$\le d_{n+} d_n^*$$
$$d_{n+1}^* - d_n^* \le d_n. \le \alpha^n d_0 \rightarrow 0 \text{ as } n\rightarrow\infty \text{ (since } 0 \le \alpha < 1)$$
$$d_{n+1}^* \le d_n^* \quad \text{for all n}$$

Hence $\{ d_n^* \}$ is monotonically decreasasing sequence of positive real number and it converges to its glb. Let it be d .Then $d_n^* \rightarrow d$ as $n\rightarrow\infty$.

Now we shall prove that d = 0 .Suppose $d \ne 0$.

Now $d = \lim_{n_x \to \propto} d_{n+2}^*$

$\leq \lim_{n_x \to \propto} \{ d_{n+1} + d_{n+1}^* \}$

$\leq \lim_{n_x \to \propto} \{ \alpha\, d_n + d_{n+1}^* \}$

$< \lim_{n_x \to \propto} \{ d_n + d_{n+1}^* \}$

$= d$ .which is contraction .Thus $d = 0$.

Hence $D^*(x_n, x_n, x_m) \to 0$ as $m, n \to \propto$

Therefore $\{x_n\}$ is a D^* Cauchy sequence in X.

Since X is D^* complete $x_n \to x$ in X

Now we prove that x is fixed point of T_1

To prove that $T_1x = x$

Suppose $T_1x \neq x$

Then $D^*(T_1x, x, x) = \lim_{n_x \to \propto} D^*(T_1x, x_{n+2}, x_{n+3})$

$= D^*(T_1x, T_2x_{n+1}, T_3x_{n+2})$

$\leq \alpha \lim_{n \to \propto} D^*(x, x_{n+1}, x_{n+2})$

$\leq \alpha\ D^*(x, x, x)$

$= 0$

Thus $T_1x = x$.

Similarly we can prome that $T_2x = T_3x = x$.

Now we prove that x is a unique common fixed point of T_1, T_2, T_3

Suppose $x \neq y$ and $T_1x = T_2x = T_3x = x$ & $T_1y = T_2y = T_3y = y$

Then $D^*(x,y,y) = D^*(T_1x, T_2y, T_3y)$

$\leq \alpha\, D^*(x, y, y)$

This impulse $(1-\alpha)D^*(x, y, y) \leq 0$

Since $x \neq y$ we have $D^*(x, y, y) > 0$

This $((1-\alpha) < 0$

This impulse $\alpha > 1$ which in contradiction Hence T_1, T_2 &T_3 have a unique common fixed point

Theorem 3

Let X be a D^* complete metric space and $S,T\ X \to X$ be any two maps such that

$$D^*(STx, T\,x, y) \leq \alpha\, D^*(T\,x, x, y)$$

for all $x, y \in X$ and $0 \leq \alpha < ½$. Then S and T have a unique common fixed point

Proof

Let $x_0 \in X$ he any fixed arbitrary element. Define a sequence $\{x_n\}$ in X as

$x_{n+1} = Tx_n$

$x_{n+2} = Sx_{n+1}$ for n=0, 1, 2…….

Let $d_n = D^*(x_x, x_{n+1}, x_{n+1})$

$d_1 = D^*(x_1, x_2, x_2)$

$= D^*(Tx_0, STx_0, x_2)$

$\leq \alpha\, D^*(x_0, Tx_0, x_2)$

$= \alpha\, D^*(x_0, x_1, , x_2)$

$\leq \alpha\, D^*(x_0, x_1, x_1) + \alpha\, D^*(x_1, x_2, x_2)$

$= \alpha\, d_0 + \alpha\, d_1$

$(1-\alpha)\, d_1 \leq \alpha\, d_0$

$d_1 \leq \frac{\alpha}{1-\alpha} d_0$

$d_1 \leq \beta\, do$ where $\beta = \frac{\alpha}{1-\alpha} < 1$ (Since $0 \leq \alpha < ½$.).

$d_2 = D^*(x_2, x_3, x_3)$
$= D^*(, STx_0, Tx_2, x_3)$
$\leq \alpha D^*(Tx_0, x_2, x_3)$
$= \alpha D^*(x_1, x_2, x_3)$
$\leq \alpha D^*(x_1, x_2, x_2) + \alpha D^*(x_2, x_3, x_3)$
$= \alpha d_1 + \alpha d_2$
$(1-\alpha)d_2 \leq \alpha d_1$

$$d_2 \leq \frac{\alpha}{1-\alpha} d_1 = \beta d_1$$

Continuing in this way we get

$d_n \leq \beta d_{n-1}$ for all n>0.

$\leq \beta^n d_0 \to 0$ as $n\to\infty$ (since $\beta = \frac{\alpha}{1-\alpha} < 1$).

Now we shall prome that $\{x_n\}$ is a Cauchy sequence in X. Let m> n> no for some no $\in$ N.

$$D^*(x_n, x_n, x_m) \leq \sum_{k=n}^{m-1} D^*(x_k, x_{k+1}, x_{k+1})$$

$$= \sum_{k=n}^{m-1} d_k$$

$$\leq \beta^n \frac{1}{1-\beta} d_0 \to 0 \text{ as } n, m \to \infty$$

Therefore $\{x_n\}$ is D* Cauchy sequence in X Since X is D* complete $x_n \to x$ in X

Now we prove that Tx=x
Suppose $Tx \neq x$

$$D^*(Tx, x, x) = \lim_{n\to\infty} D^*(Tx, x_{n+2}, x)$$

$$= \lim_{n\to\infty} D^*(Tx, ST\, x_n, x)$$

$$\leq \alpha \lim_{n\to\infty} D^*(x, T\, x_n, x)$$

$$= \alpha \lim_{n\to\infty} D^*(x, x_{n+1}, x)$$

$= 0$

Therefore Tx=x .
Next to prom that Sx = x
$D^*(STx, Tx, x) \leq \alpha D^*(T x, x , x)$
$= 0$ (since $T x = x$)
Thus $ST x = Tx = x$
Hence Sx = x(since Tx= x)
Therefore x is common fixed point S & T
Suppose $x \neq y$ Such that $Sx = Tx = x$ and $Sy = Ty = y$

Then $D^*(x, x, y) = D^*(STx, T x, y)$
$\leq \alpha D^*(Tx, x, y)$
$= \alpha D^*(x, x, y)$
$(1-\alpha) D^*(x, x, y) \leq 0$

Thus $1 - \alpha < 0$ This implies $\alpha > 1$ which is contradiction.
Therefore x = y
Hence x is a unique common fixed point.

Theorem 4:

Let X be a D* complete metric space and R, S,T $X\to X$ be any three maps such that

$$D^*(RSTx , ST x ,Tx) \leq \alpha D^*(ST x, Tx, x\})$$

for all x, $\in$ X and $0 \leq \alpha < 1$. Then R, S and T have a unique common fixed point

Proof

Let $x_0 \in$ X he any fixed arbitrary element Define a sequence $\{x_n\}$ in X as

$x_{n+1} = T x_n$

$x_{n+2} = S\, x_{n+1}$

$x_{n+3} = R\, x_{n+2}$ for n = 0, 1, 2, ….

Let $d_n = D^*(x_n, x_{n+1}, x_{n+2})$

$d_1 = D^*(x_1, x_2, x_3)$

$= D^*(Tx_0, STx_0, RSTx_0)$

$\leq \alpha D^*(x_0, Tx_0, STx_0)$

$\leq \alpha D^*(x_0, x_1, x_2)$

$d_1 \leq \alpha d_0$

$d_2 = D^*(x_2, x_3, x_4)$

$= D^*(Tx_1, STx_1, RSTx_1)$

$\leq \alpha D^*(x_1, Tx_1, STx_1)$

$\leq \alpha D^*(x_1, x_2, x_3)$

$\leq \alpha d_1.$

$\leq \alpha^2 do$

Continuing in thus way we get

$d_n \leq \alpha^n d_0 \to 0$ as $n \to \infty$ (since $0 \leq \alpha < 1$).

Now we shall prome that $\{x_n\}$ is a Cauchy sequence in X.

Let $d_n^* = D^*(x_n, x_n, x_{n+1})$

Then $d_{n+1}^* = D^*(x_{n+1}, x_{n+1}, x_{n+2})$

$\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1})$

$\leq d_n + d_n^*$

$d_{n+1}^* - d_n^* \leq d_n. \leq \alpha^n d_0 \to 0$ as $n \to \infty$ (since $0 \leq \alpha < 1$)

$d_{n+1}^* \leq d_n^*$ for all n

Hence $\{ d_n^* \}$ is monotonically decreasasing sequence of positive real number and it converges to its glb. Let it be d .Then $d_n^* \to d$ as $n \to \infty$.

Now we shall prove that d = 0 .Suppose $d \neq 0$.

Now $d = \lim_{n_x \to \propto} d_{n+2}^*$

$\leq \lim_{n_x \to \propto} \{ d_{n+1} + d_{n+1}^* \}$

$\leq \lim_{n_x \to \propto} \{\alpha d_n + d_{n+1}^* \}$

$< \lim_{n_x \to \propto} \{ d_n + d_{n+1}^* \}$

=d .which is contraction .Thus d = 0.

Hence $D^*(x_n, x_n, x_m) \to 0$ as m, n $\rightarrow \propto$

Therefore $\{x_n\}$ is a D^* Cauchy sequence in X.

Since X is D^* complete the sequence $x_n \rightarrow x$ in X.

Now we prove that x is fixed point of T

To prove that Tx = x

Suppose $Tx \neq x$

Then $D^*(Tx, x, x) = \lim_{n_x \to \propto} D^*(Tx, x_{n+2}, x_{n+3})$

$= \lim_{n \to \propto} D^*(Tx, STx_n, RSTx_n)$

$\leq \alpha \lim_{n \to \propto} D^*(x, x_{n+1}, x_{n+2})$

$= 0$

Thus Tx = x.

Now we can prove that Sx = x.

Then $D^*(x, Sx, x) = \lim_{n_x \to \propto} D^*(Tx, STx, x_{n+3})$

$$= \lim_{n \to \infty} D^*(Tx, STx, RSTx_n)$$
$$\leq \alpha \lim_{n \to \infty} D^*(x, Tx, x_{n+2})$$
$$= \alpha \lim_{n \to \infty} D^*(x, x, x_{n+2})$$
$$= 0$$

Thus Sx = x..

Finally meprove that Rx = x.

Then $D^*(x, x, Rx) = D^*(Tx, STx, RSTx)$
$$\leq \alpha D^*(x, Tx, STx)$$
$$= \alpha D^*(x, x, x)$$
$$= 0$$

Now we prove that x is a unique common fixed point of R,S T .

Suppose $x \neq y$ and $Tx=Sx=Rx=x$ & $Ty = Sy = R y= y$

Then $D^*(x,y,y) = D^*(Tx, STy, RST_Y)$
$$\leq \alpha D^*(x, y, y)$$

this impulse $(1-\alpha)D^*(x, y, y) \leq 0$

since $x \neq y$ we have $D^*(x, y, z) > 0$

This $((1-\alpha) < 0$

This impulse $\alpha > 1$ which in contradiction Hence R,S,T have a unique common fixed point

Theorem 2.2. Let X be a complete D* - metric space and $T_1, T_2, T_3 : X \rightarrow X$ be any three maps such that $D^*(T_1x, T_2y, T_3z) \leq a \{ D^*(x, y, z) + D^*(x, T_1x, T_2y) + D^*(y, T_2y, T_3z) \}$ for all $x, y, z \in X$ and $0 \leq a < 1/3$. Then T_1, T_2, T_3 have a unique common fixed point.

Proof:-

Let $x_0 \in X$ a fixed arbitrary element and define a sequence $\{x_n\}$ in X as

$x_{n+1} = T_1 x_n$

$x_{n+2} = T_2 x_{n+1}$

$x_{n+3} = T_3 x_{n+2}$ for n = 0, 1, 2, ….

Let $d_n = D^*(x_n, x_{n+1}, x_{n+2})$.

Then $d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3})$
$$= D^*(T_1 x_n, T_2 x_{n+1} T_3 x_{n+2})$$
$$\leq a \{D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, T_1 x_n, T_2 x_{n+1}) + D^*(x_{n+1}, T_2 x_{n+1} T_3 x_{n+2})\}$$
$$= a \{ D^*(x_n x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\}$$
$$= a \{2 D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\}$$
$$d_{n+1} \leq 2a\, d_n + a\, d_{n+1}$$
$$d_{n+1} \leq \{ 2a/(1-a)\} d_n$$
$$d_{n+1} \leq b\, d_n \quad . \quad \text{where } b = 2a/(1-a) < 1.$$

Hence $d_n \leq b^n d_0 \rightarrow 0$, as $n \rightarrow \infty$.

Now we shall prove that $\{x_n\}$ is a Cauchy sequence in X.

Let $d_n^* = D^*(x_n, x_n, x_{n+1})$

Then $d_{n+1}^* = D^*(x_{n+1}, x_{n+1}, x_{n+2})$
$$\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1})$$
$$\leq d_n + d_n^*$$
$$d_{n+1}^* - d_n^* \leq d_n. \leq \alpha^n d_0 \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (since } 0 \leq \alpha < 1)$$
$$d_{n+1}^* \leq d_n^* \quad \text{for all n}$$

Hence $\{ d_n^* \}$ is monotonically decreasasing sequence of positive real number and it converges to its glb. Let it be d .Then $d_n^* \rightarrow d$ as $n \rightarrow \infty$.

Now we shall prove that d = 0 .Suppose $d \neq 0$.

$$\text{Now } d = \lim_{n_x \to \infty} d_{n+2}^*$$
$$\leq \lim_{n_x \to \infty} \{ d_{n+1} + d_{n+1}^*\}$$
$$\leq \lim_{n_x \to \infty} \{b\, d_n + d_{n+1}^*\}$$

$$< \lim_{n_x \to \infty} \{ d_n + d_{n+1}^* \}$$

$=d$ .which is contraction .Thus $d = 0$.

Hence $D^*(x_n, x_n, x_m) \to 0$ as $m, n \to \infty$

Therefore $\{x_n\}$ is a D^* Cauchy sequence in X.

Since X is D^* complete $x_n \to x$ in X

Now we prove that x is fixed point of T_1

To prove that $T_1x = x$

Suppose $T_1x \neq x$, Then

$$D^*(T_1x, x, x) = \lim_{n_x \to \infty} D^*(T_1x, x_{n+2}, x_{n+3})$$

$$= D^*(T_1x, T_2x_{n+1}, T_3x_{n+2})$$

$$\leq a \lim_{n \to \infty} \{D^*(x, x_{n+1}, x_{n+2}) + D^*(x, T_1 x, T_2 x_{n+1}) + D^*(x_{n+1}, T_2 x_{n+1} T_3 x_{n+2})\}$$

$$\leq a \lim_{n \to \infty} \{D^*(x, x_{n+1}, x_{n+2}) + D^*(x, T_1 x, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\}$$

$$\leq a\ D^*(x, T_1 x, x)$$

$(1-a)\ D^*(x, T_1 x, x) \leq 0$ Hence $(1-a) < 0$

Therefore $a>1$,which is contradiction to $a<1$.

Thus $T_1x = x$.Similarly we can prome that $T_2x = T_3x = x$.

Now we prove that x is a unique common fixed point of T_1, T_2, T_3

Suppose $x \neq y$ and $T_1x=T_2x=T_3x=x$ & $T_1y=T_2y=T_3y=y$

Then $D^*(x,y,y) = D^*(T_1x, T_2y, T_3 y)$

$$\leq a \{D(x, y, y) + D^*(x, x, y) + D^*(y, y, y)\}$$

$$\leq 2a\ D^*(x, y, y)$$

$$< D^*(x, y, y),$$

which is contradiction.

Hence T_1, T_2 & T_3 have a unique common fixed

Theorem 2.3. Let X be a complete D^* - metric space and $T_1, T_2, T_3 : X \to X$ be any three maps such that D^*

$$(T_1x, T_2y, T_3z) \leq a_1 D^*(x, y, z) + a_2 \{D^*(x, T_1x, T_2y) + D^*(y, T_2y, T_3z)\} + a_3 \{D^*(x, y, T_2y) + D^*(y, z, T_3z)\}$$

for all $x, y, z \in X$, and $0 \leq a_1 + 2a_2 + 2a_3 < 1$. Then T_1, T_2, and T_3 have a unique common fixed point.

Proof.

Let $x_0 \in X$ a fixed arbitrary element and define a sequence $\{x_n\}$ in X as

$x_{n+1} = T_1 x_n$

$x_{n+2} = T_2 x_{n+1}$

$x_{n+3} = T_3 x_{n+2}$ for $n = 0, 1, 2, \ldots$.

Let $d_n = D^*(x_n, x_{n+1}, x_{n+2})$.

Then $d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3})$

$$= D^*(T_1 x_n, T_2 x_{n+1} T_3 x_{n+2})$$

$$\leq a_1 D^*(x_n, x_{n+1}, x_{n+2}) + a_2 (\{D^*(x_n, T_1x_n, T_2 x_{n+1}) + D^*(x_{n+1}, T_2 x_{n+1}, T_3 x_{n+2})\}$$

$$+ a_3 \{D^*(x_n, x_{n-+1}, T_2 x_{n+1}) + D^*(x_{n+1}, x_{n+2}, T_3 x_{n+2})\}$$

$$= a_1 D^*(x_n, x_{n+1}, x_{n+2}) + a_2 \{D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n-+1}, x_{n+2}, x_{n+3})\}$$

$$+ a_3 \{D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\}$$

$$\leq (a_1 + a_2 + a_3) D^*(x_n, x_{n+1}, x_{n+2}) + (a_2 + a_3) D^*(x_{n+1}, x_{n+2}, x_{n+3})$$

$$\leq (a_1 + a_2 + a_3) d_n + (a_2 + a_3) d_{n+1}$$

$$(1 - a_2 - a_3) d_{n+1} \leq (a_1+a_2+a_3) d_n$$

$$d_{n+1} \leq \{(a_1+a_2+a_3) / (1 - a_2 - a_3)\} d_n$$

$d_{n+1} \leq a\ d_n$, for all n where $a = \{(a_1+a_2+a_3) / (1-a_2 - a_3)\} < 1$.

Hence $d_n \leq a^n d_0 \to 0$ as $n \to \infty$

Now we prove that $\{x_n\}$ is D^* - Cauchy sequence in X .

Let $d_n^* = D^*(x_n, x_n, x_{n+1})$

Then $d_{n+1}^* = D^*(x_{n+1}, x_{n+1}, x_{n+2})$

$\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1})$

$\leq d_n + d_n^*$

$d_{n+1}^* - d_n^* \leq d_n. \leq \alpha^n d_0 \to 0$ as $n \to \infty$ (since $0 \leq \alpha < 1$)

$d_{n+1}^* \leq d_n^*$ for all n

Hence $\{ d_n^* \}$ is monotonically decreasasing sequence of positive real number and it converges to its glb. Let it be d .Then $d_n^* \to d$ as $n \to \infty$.

Now we shall prove that d = 0 .Suppose $d \neq 0$.

$$\text{Now } d = \lim_{n_x \to \infty} d_{n+2}^*$$

$$\leq \lim_{n_x \to \infty} \{ d_{n+1} + d_{n+1}^* \}$$

$$\leq \lim_{n_x \to \infty} \{ b\, d_n + d_{n+1}^* \}$$

$$< \lim_{n_x \to \infty} \{ d_n + d_{n+1}^* \}$$

=d .which is contraction .Thus d = 0.

For m > n , we have

$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m)$

$\leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \ldots + D^*(x_{m-1}, x_{m-1}, x_m)$

$\to 0$ as n, m $\to \infty$. Hence $D^*(x_n, x_n, x_m) \to 0$ as m, n $\to \infty$

Therefore $\{x_n\}$ is a D^* Cauchy sequence in X.

Since X is D^* complete $x_n \to x$ in X

Now we prove that x is fixed point of T_1

To prove that $T_1x = x$

Suppose $T_1x \neq x$, Then

$$D^*(T_1x, x, x) = \lim_{n \to \infty} D^*(T_1x, x_{n+2}, x_{n+3})$$

$$= \lim_{n \to \infty} D^*(T_1x, T_2x_{n+1}, T_3x_{n+2})$$

$$\leq \lim_{n \to \infty} \{ a_1 D^*(x , x_{n+1}, x_{n+2}) + a_2 \{D^*(x , T_1x , T_2 x_{n+1}) + D^*(x_{n+1}, T_2 x_{n+1}, T_3 x_{n+2})\}$$

$$+ a_3 \{D^*(x , x_{n-+1}, T_2 x_{n+1}) + D^*(x_{n+1}, x_{n+2}, T_3 x_{n+2})\}$$

$$= \lim_{n \to \infty} \{ a_1 D^*(x , x_{n+1}, x_{n+2}) + a_2 \{D^*(x , T_1x , x_{n+2}) + D^*(x_{n-+1}, x_{n+2}, x_{n+3})\}$$

$$+ a_3 \{ D^*(x , x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\} \}$$

$\leq a_2 D^*(x, T_1 x , x)$

$< D^*(x, T_1 x , x)$,which is contradiction . Thus $T_1x = x$..

Similarly we can prome that $T_2x = T_3x = x$.

Now we prove that x is a unique common fixed point of T_1, T_2, T_3

Suppose $x \neq y$ and $T_1x=T_2x=T_3x=x$ & $T_1y=T_2y=T_3y= y$

Then $D^*(x,y,y) = D^*(T_1x, T_2y, T_3 y)$

$\leq a_1 D^*(x, y, y) + a_2 \{D^*(x, T_1x, T_2y) + D^*(y, T_2y, T_3y)\}$

$+ a_3 \{D^*(x, y, T_2y) + D^*(y, y, T_3y)\}$

$= a_1 D^*(x, y, y) + a_2 \{D^*(x, x, y) + D^*(y, y, y)\}$

$+ a_3\{D^*(x, y, y) + D^*(y, y, y)\}$

$= (a_1 + a_2 + a_3) D^*(x, y, y)$

$< D^*(x, y, y)$, which is contradiction.

Hence T_1, T_2 &T_3 have a unique common fixed

Theorem 2.5. Let X be a complete D^* - metric space and T_1, T_2, $T_3 : X \to X$ be any three maps such that $D^*(T_1x, T_2y, T_3z) \leq a \max\{ D^*(x, y, z) , \{D^*(x, T_1,x, T_2y) , D^*(y , T_2y, T_3z) D^*(x, y, T_2y) , D^*(y , z , T_3z)\}$ for all $x, y, z \in X$, and $0 \leq a < 1$. Then T_1, T_2, and T_3 have a unique common fixed point.

Proof.

Let $x_0 \in X$ a fixed arbitrary element and define a sequence $\{x_n\}$ in X as

$x_{n+1} = T_1 x_n$

$x_{n+2} = T_2 x_{n+1}$

$x_{n+3} = T_3 x_{n+2}$ for n = 0, 1, 2, ….

Let $d_n = D^*(x_n, x_{n+1}, x_{n+2})$.

Then $d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3})$

$= D^*(T_1 x_n, T_2 x_{n+1} T_3 x_{n+2})$

$\leq a \max \{D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_n, T_1 x_n, T_2 x_{n+1}), D^*(x_{n+1}, T_2 x_{n+1}, T_3 x_{n+2}),$
$D^*(x_n, x_{n-+1}, T_2 x_{n+1}), D^*(x_{n+1}, x_{n+2}, T_3 x_{n+2})\}$

$= a \max \{ D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n-+1}, x_{n+2}, x_{n+3})$
$D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\}$

$\leq a \max \{ D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})$

$\leq a \max \{d_n, d_{n+1}\}$

$d_{n+1} \leq a\, d_n$ for all n

Hence $d_n. \leq a^n d_0 \to 0$ as $n \to \infty$

Now we prove that $\{x_n\}$ is D* - Cauchy sequence in X .

Let $d_n^* = D^*(x_n, x_n, x_{n+1})$

Then $d_{n+1}^* = D^*(x_{n+1}, x_{n+1}, x_{n+2})$

$\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1})$

$\leq d_n + d_n^*$

$d_{n+1}^* - d_n^* \leq d_n. \leq \alpha^n d_0 \to 0$ as $n \to \infty$ (since $0 \leq \alpha < 1$)

$d_{n+1}^* \leq d_n^*$ for all n

Hence $\{ d_n^* \}$ is monotonically decreasasing sequence of positive real number and it converges to its glb. Let it be d .Then $d_n^* \to d$ as $n \to \infty$.

Now we shall prove that d = 0 .Suppose $d \neq 0$.

Now $d = \lim_{n_x \to \infty} d_{n+2}^*$

$\leq \lim_{n_x \to \infty} \{ d_{n+1} + d_{n+1}^*\}$

$\leq \lim_{n_x \to \infty} \{a d_n + d_{n+1}^*\}$

$< \lim_{n_x \to \infty} \{d_n + d_{n+1}^*\}$

$= d$

Now we prove that $\{x_n\}$ is D* - Cauchy sequence in X.

For m > n we have,

$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \ldots + D^*(x_{m-1}, x_{m-1}, x_m)$

$\to 0$ as m. $n \to \infty$

Thus $\{x_n\}$ is a D* Cauchy sequence in X and X is D* - complete $x_n \to x$ in X .

Now we shall prove that $T_1 x = x$

$D^*(T_1 x, x, x) = \lim_{n\to\infty} D^*(T_1 x, x_{n+2}, x_{n+3})$

$= \lim_{n\to\infty} D^*(T_1 x, T_2 x_{n+1}, T_3 x_{n+2})$

$\leq a \lim_{n\to\infty} \max \{D^*(x, x_{n+1}, x_{n+2}), D^*(x, T_1 x, T_2 x_{n+1}), D^*(x_{n+1}, T_2 x_{n+1}, T_3 x_{n+2}),$
$D^*(x, x_{n-+1}, T_2 x_{n+1}), D^*(x_{n+1}, x_{n+2}, T_3 x_{n+2})\}$

$= a \lim_{n\to\infty} \max \{ D^*(x, x_{n+1}, x_{n+2}), D^*(x, T_1 x, x_{n+2}), D^*(x_{n-+1}, x_{n+2}, x_{n+3})$
$D^*(x, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\}$

$\leq a \max \{ D^*(x, T_1 x, x), 0\}$

$< D^*(Tx, x, x),$

Which is a contradiction.

Thus $T_1 x = x$..

Similarly we can prome that $T_2 x = T_3 x = x$.

Now we prove that x is a unique common fixed point of T_1, T_2, T_3

Suppose $x \neq y$ and $T_1 x = T_2 x = T_3 x = x$ & $T_1 y = T_2 y = T_3 y = y$

Then $D^*(x,y,y) = D^*(T_1 x, T_2 y, T_3 y)$

$\leq$ a max { $D^*(x, y, y)$, {$D^*(x, T_1x, T_2y)$, $D^*(y, T_2y, T_3y)$, $D^*(x, y, T_2y)$, $D^*(y, y, T_3y)$}

= a max { $D^*(x, y, y)$, $D^*(x, x, y)$, $D^*(x, y, y)$, $D^*(y, y, y)$} = a $D^*(x, y, y)$

< $D^*(x, y, y)$,

which is contradiction.

Hence T_1, T_2 &T_3 have a unique common fixed

Theorem 2.6.

Let X be a complete D* - metric space and $T_1, T_2, T_3 : X \rightarrow X$ be any three maps such that $D^*(T_1x, T_2y, T_3z) \leq a_1 D^*(x, y, z) + a_2 \max\{D^*(x, T_1x, T_2y), D^*(y, T_2y, T_3z)\}$

for all x, y, z, $\in$ X and $0 \leq a_1 + 2a_2 < 1$. Then T has a unique fixed point.

Proof:-

Let $x_0 \in X$ a fixed arbitrary element and define a sequence $\{x_n\}$ in X as

$x_{n+1} = T_1 x_n$

$x_{n+2} = T_2 x_{n+1}$

$x_{n+3} = T_3 x_{n+2}$ for n = 0, 1, 2, ….

Let $d_n = D^*(x_n, x_{n+1}, x_{n+2})$.

Then $d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3})$

$= D^*(T_1 x_n, T_2 x_{n+1} T_3 x_{n+2})$

$\leq a_1 D^*(x_n, x_{n+1}, x_{n+2}) + a_2 \max\{D^*(x_n, T_1 x_n, T_2 x_{n+1}), D^*(x_{n+1}, T_2 x_{n+1} T_3 x_{n+2})\}$

$= a_1 \{D^*(x_n x_{n+1}, x_{n+2}) + a_2 \max\{D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\}$

$= 2 a_1 D^*(x_n, x_{n+1}, x_{n+2}) + a_2 D^*(x_{n+1}, x_{n+2}, x_{n+3})\}$

$d_{n+1} \leq 2a_1 d_n + a_2 d_{n+1}$

$d_{n+1} \leq \{2a_1 \backslash (1- a_2)\} d_n$

$d_{n+1} \leq b\, d_n$. where $b = \{2a_1 \backslash (1- a_2)\} < 1$.

Hence $d_n \leq b^n d_0 \rightarrow 0$, as $n \rightarrow \infty$.

Now we shall prove that $\{x_n\}$ is a Cauchy sequence in X.

Let $d_n^* = D^*(x_n, x_n, x_{n+1})$

Then $d_{n+1}^* = D^*(x_{n+1}, x_{n+1}, x_{n+2})$

$\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1})$

$\leq d_n + d_n^*$

$d_{n+1}^* - d_n^* \leq d_n. \leq \alpha^n d_0 \rightarrow 0$ as $n \rightarrow \infty$ (since $0 \leq \alpha < 1$)

$d_{n+1}^* \leq d_n^*$ for all n

Hence $\{d_n^*\}$ is monotonically decreasasing sequence of positive real number and it converges to its glb. Let it be d .Then $d_n^* \rightarrow d$ as $n \rightarrow \infty$.

Now we shall prove that d = 0 .Suppose $d \neq 0$.

$$\text{Now } d = \lim_{n_x \rightarrow \infty} d_{n+2}^*$$

$$\leq \lim_{n_x \rightarrow \infty} \{d_{n+1} + d_{n+1}^*\}$$

$$\leq \lim_{n_x \rightarrow \infty} \{b\, d_n + d_{n+1}^*\}$$

$$< \lim_{n_x \rightarrow \infty} \{d_n + d_{n+1}^*\}$$

=d .which is contraction .Thus d = 0.

Hence $D^*(x_n, x_n, x_m) \rightarrow 0$ as m, n $\rightarrow \infty$

Therefore $\{x_n\}$ is a D* Cauchy sequence in X.

Since X is D* complete $x_n \rightarrow x$ in X

Now we prove that x is fixed point of T_1

To prove that $T_1x = x$

Suppose $T_1x \neq x$, Then

$$D^*(T_1x, x, x) = \lim_{n_x \rightarrow \infty} D^*(T_1x, x_{n+2}, x_{n+3})$$

$$= \lim_{n \rightarrow \infty} D^*(T_1x, T_2x_{n+1}, T_3x_{n+2})$$

$\leq \lim_{n \to \infty} \{ a_1 D^*(x, x_{n+1}, x_{n+2}) + a_2 \max \{D^*(x ,T_1 x ,T_2 x_{n+1}) , D^*(x_{n+1} ,T_2 x_{n+1} T_3 x_{n+2})\} \}$ $\leq \lim_{n \to \infty} \{ a_1 D^*(x, x_{n+1}, x_{n+2}) + a_2 \max \{ D^*(x, T_1 x ,, x_{n+2}) , D^*(x_{n+1},x_{n+2}, x_{n+3})\}$ $\leq a_2 \quad D^*(x, T_1 x , x)$
$< D^*(x, T_1 x , x)$,which is contradiction Thus $T_1x = x$.
Similarly we can prove that $T_2x = T_3x = x$.
Now we prove that x is a unique common fixed point of T_1, T_2, T_3
Suppose $x \neq y$ and $T_1x=T_2x=T_3x=x$ & $T_1y=T_2y=T_3y=y$
Then $D^*(x,y,y) = D^*(T_1x, T_2y, T_3 y)$
$\leq a_1 \{D(x, y, y) + a_2 \max \{D^*(x, x, y) , D^*(y, y, y) \}$
$\leq (a_{1+} a_2) D^*(x, y, y)$
$< D^* (x, y, y)$, which is contradiction.
Hence T_1, T_2 &T_3 have a unique common fixed

Remark 2.7.
If we put $a_2 = 0$ and $a_1 = a$ in the above theorem we get the following Theorem as corollary.

Corollary 2.8.
Let (X, D*) be a complete D* - metric space and T: X → X be a map such that $D^*(Tx, Ty, Tz) \leq a D^*(x, y, z)$ for all $x, y, z \in X$ and $0 \leq a < 1$. Then T has a unique fixed point. The above Theorem is know as Banach contraction Type Theorem in D* - metric space.

Remark 2.9.
If we put $a_1 = 0$ and $a_2 = a$ in the above theorem 1. We get the following theorem as corollary 2.10.

Corollary 2.10.
Let (X, D*) be a complete D* - metric space and T : X →X be a map such that $D^*(Tx, Ty, Tz) \leq a \max \{D^*(x, Tx, Ty), D^*(y, Ty, Tz)\}$ for all $x, y, z \in X$ and $0 \leq a < \frac{1}{2}$. Then T has a unique fixed point.

Theorem 2.11.
Let X be a complete D* - metric space and T_1, T_2, T_3 : X → X be any three maps such that $D^* (T_1x, T_2y, T_3z) \leq a_1 D^*(x, y, z) + a_2 \max\{D^*(x, T_1,x, T_2y) , D^* (y , T_2y, T_3z)\}$
$+ a_3 \max \{D^*(x, y, T_2y) + D^* (y , z , T_3z)\}$

for all $x, y, z \in X$, and $0 \leq a_1 +2a_2 +2 a_3 < 1$. Then T_1, T_2, and T_3 have a unique common fixed point.

Proof.
Let $x_0 \in X$ a fixed arbitrary element and define a sequence $\{x_n\}$ in X as
$x_{n+1}= T_1 x_n$
$x_{n+2}= T_2 x_{n+1}$
$x_{n+3}= T_3 x_{n+2}$ for n = 0, 1, 2, ….
Let $d_n= D^* (x_n, x_{n+1}, x_{n+2})$.
Then $d_{n+1} = D^* (x_{n+1}, x_{n+2}, x_{n+3})$
$= D^* (T_1 x_n ,T_2 x_{n+1} T_3 x_{n+2})$
$\leq a_1D^*(x_n , x_{n+1}, x_{n+2}) + a_2 \max \{D^*(x_n , T_1x_n , T_2 x_{n+1}) , D^* (x_{n+1}, T_2 x_{n+1}, T_3 x_{n+2})\}$
$+ a_3 \max\{D^*(x_n , x_{n-+1}, T_2 x_{n+1}) , D^*(x_{n+1}, x_{n+2}, T_3 x_{n+2})\}$
$= a_1 D^*(x_n , x_{n+1}, x_{n+2}) + a_2 \max\{D^*(x_n , x_{n+1}, x_{n+2}) , D^*(x_{n-+1}, x_{n+2}, x_{n+3})\}$
$+ a_3 \max\{ D^*(x_n , x_{n+1}, x_{n+2}) , D^*(x_{n+1}, x_{n+2}, x_{n+3})\}$
$\leq (a_1 + a_2 + a_3) D^*(x_n , x_{n+1}, x_{n+2}) + (a_2 + a_3) D^*(x_{n+1}, x_{n+2}, x_{n+3})$
$\leq (a_1 + a_2 + a_3) d_n + (a_2 + a_3) d_{n+1}$
$(1 - a_2 - a_3) d_{n+1} \leq (a_1+a_2+a_3) d_n$
$d_{n+1} \leq \{(a_1+a_2+a_3) / (1 - a_2 - a_3)\} d_n$
$d_{n+1} \leq a d_n$, for all n where $a = \{ (a_1+a_2+a_3) / (1-a_2 - a_3) \} < 1$.
Hence $d_n . \leq a^n d_0 \to 0$ as $n \to \infty$
Now we prove that $\{x_n\}$ is D* - Cauchy sequence in X .
Let $d_n^* = D^* (x_n, x_n, x_{n+1})$
Then $d_{n+1}^* = D^* (x_{n+1}, x_{n+1}, x_{n+2})$
$\leq D^* (x_n, x_{n+1}, x_{n+2}) + D^* (x_n, x_{n+1}, x_{n+1})$
$\leq d_{n+} d_n^*$

$d_{n+1}^* - d_n^* \leq d_n. \leq \alpha^n d_0 \to 0$ as $n \to \infty$ (since $0 \leq \alpha < 1$)

$d_{n+1}^* \leq d_n^*$ for all n

Hence $\{ d_n^* \}$ is monotonically decreasasing sequence of positive real number and it converges to its glb. Let it be d .Then $d_n^* \to d$ as $n \to \infty$.

Now we shall prove that d = 0 .Suppose $d \neq 0$.

$$\text{Now } d = \lim_{n_x \to \infty} d_{n+2}^*$$

$$\leq \lim_{n_x \to \infty} \{ d_{n+1} + d_{n+1}^* \}$$

$$\leq \lim_{n_x \to \infty} \{ b\, d_n + d_{n+1}^* \}$$

$$< \lim_{n_x \to \infty} \{ d_n + d_{n+1}^* \}$$

=d .which is contraction .Thus d = 0.

For m > n , we have

$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m)$

$\leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \ldots + D^*(x_{m-1}, x_{m-1}, x_m)$

$\to 0$ as n, m $\to \infty$. Hence $D^*(x_n, x_n, x_m) \to 0$ as m, n $\to \infty$

Therefore $\{x_n\}$ is a D* Cauchy sequence in X.

Since X is D* complete $x_n \to x$ in X

Now we prove that x is fixed point of T_1

To prove that $T_1x = x$

Suppose $T_1x \neq x$, Then

$$D^*(T_1x, x, x) = \lim_{n \to \infty} D^*(T_1x, x_{n+2}, x_{n+3})$$

$$= \lim_{n \to \infty} D^*(T_1x, T_2x_{n+1}, T_3x_{n+2})$$

$$\leq \lim_{n \to \infty} \{ a_1 D^*(x , x_{n+1}, x_{n+2}) + a_2 \max\{D^*(x , T_1x , T_2 x_{n+1}) , D^* (x_{n+1}, T_2 x_{n+1}, T_3 x_{n+2})\}$$

$$+ a_3 \max \{ D^*(x , x_{n-+1}, T_2 x_{n+1}) , D^*(x_{n+1}, x_{n+2}, T_3 x_{n+2})$$

$$= \lim_{n \to \infty} \{ a_1 D^*(x , x_{n+1}, x_{n+2}) + a_2 \max \{D^*(x , T_1x , x_{n+2}) , D^*(x_{n-+1}, x_{n+2}, x_{n+3})\}$$

$$+ a_3 \max \{ D^*(x , x_{n+1}, x_{n+2}) , D^*(x_{n+1}, x_{n+2}, x_{n+3})\} \}$$

$$\leq a_2 D^*(x, T_1 x , x)$$

$$< D^*(x, T_1 x , x)$$,which is contradiction . Thus $T_1x = x$..

Similarly we can prome that $T_2x = T_3x = x$.

Now we prove that x is a unique common fixed point of T_1, T_2, T_3

Suppose $x \neq y$ and $T_1x=T_2x=T_3x=x$ & $T_1y=T_2y=T_3y= y$

Then $D^*(x,y,y) = D^*(T_1x, T_2y, T_3 y)$

$$\leq a_1 D^*(x, y, y) + a_2 \max\{D^*(x, T_1x, T_2y) , D^*(y, T_2y, T_3y)\}$$

$$+ a_3 \max \{D^*(x, y, T_2y) , D^* (y, y, T_3y)\}$$

$$= a_1 D^*(x, y, y) + a_2 \max \{D^*(x, x, y) , D^*(y, y, y)\}$$

$$+ \max a_3\{D^*(x, y, y) , D^* (y, y, y)\}$$

$$= (a_1 + a_2 + a_3) D^* (x, y, y)$$

$$< D^* (x, y, y),$$ which is contradiction.

Hence T_1, T_2 &T_3 have a unique common fixed

Theorem 2.12. Let X be a complete D* - metric space and T_1, T_2, T_3 : X → X be any three maps such that $D^* (T_1x, T_2y, T_3z) \leq a \max\{D^*(x, y, z) , 1/2\{D^*(x, T_1,x, T_2y) + D^* (y , T_2y, T_3z) \}, 1/2 \{D^*(x, y, T_2y) + D^* (y , z , T_3z)\}\}$ for all x, y, z ∈ X, and $0 \leq a < 1/3$. Then T_1, T_2, and T_3 have a unique common fixed point.

Proof.

Let $x_0 \in X$ a fixed arbitrary element and define a sequence $\{x_n\}$ in X as

$x_{n+1} = T_1 x_n$

$x_{n+2} = T_2 x_{n+1}$

$x_{n+3} = T_3 x_{n+2}$ for n = 0, 1, 2, ….

Let $d_n = D^*(x_n, x_{n+1}, x_{n+2})$.

Then $d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3})$

$= D^*(T_1 x_n, T_2 x_{n+1} T_3 x_{n+2})$

$\leq a \max \{D^*(x_n, x_{n+1}, x_{n+2}), 1/2\{D^*(x_n, T_1x_n, T_2 x_{n+1}) + D^*(x_{n+1}, T_2 x_{n+1}, T_3 x_{n+2})\},$

$1/2\{D^*(x_n, x_{n-+1}, T_2 x_{n+1}) + D^*(x_{n+1}, x_{n+2}, T_3 x_{n+2})\}\}$

$= a \max\{D^*(x_n, x_{n+1}, x_{n+2}), 1/2\{D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n-+1}, x_{n+2}, x_{n+3})\}$

$1/2\{D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\}\}$

$\leq a \max\{D^*(x_n, x_{n+1}, x_{n+2}), 1/2\{D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\}\}$

$\leq a \max\{d_n, 1/2(d_n + d_{n+1})\}$

$d_{n+1} \leq a(3/2\, d_n + 1/2\, d_{n+1})$

$d_{n+1} \leq a(3d_n)$ for all n

Hence $d_n. \leq a^n d_0 \to 0$ as $n \to \infty$

Now we prove that $\{x_n\}$ is D* - Cauchy sequence in X .

Let $d_n^* = D^*(x_n, x_n, x_{n+1})$

Then $d_{n+1}^* = D^*(x_{n+1}, x_{n+1}, x_{n+2})$

$\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1})$

$\leq d_n + d_n^*$

$d_{n+1}^* - d_n^* \leq d_n. \leq \alpha^n d_0 \to 0$ as $n \to \infty$ (since $0 \leq \alpha < 1$)

$d_{n+1}^* \leq d_n^*$ for all n

Hence $\{d_n^*\}$ is monotonically decreasasing sequence of positive real number and it converges to its glb. Let it be d .Then $d_n^* \to d$ as $n \to \infty$.

Now we shall prove that d = 0 .Suppose $d \neq 0$.

Now $d = \lim_{n_x \to \infty} d_{n+2}^*$

$\leq \lim_{n_x \to \infty} \{d_{n+1} + d_{n+1}^*\}$

$\leq \lim_{n_x \to \infty} \{ad_n + d_{n+1}^*\}$

$< \lim_{n_x \to \infty} \{d_n + d_{n+1}^*\}$

$= d$

Now we prove that $\{x_n\}$ is D* - Cauchy sequence in X.

For m > n we have,

$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \ldots + D^*(x_{m-1}, x_{m-1}, x_m)$

$\to 0$ as m. n $\to \infty$

Thus $\{x_n\}$ is a D* Cauchy sequence in X and X is D* - complete $x_n \to x$ in X .

Now we shall prove that $T_1x = x$

$D^*(T_1x, x, x) = \lim_{n\to\infty} D^*(T_1x, x_{n+2}, x_{n+3})$

$= \lim_{n\to\infty} D^*(T_1x, T_2 x_{n+1}, T_3 x_{n+2})$

$\leq a \lim_{n\to\infty} \max\{D^*(x, x_{n+1}, x_{n+2}), ½\{D^*(x, T_1x, T_2 x_{n+1}) + D^*(x_{n+1}, T_2 x_{n+1}, T_3 x_{n+2})\},$

$½\{D^*(x, x_{n-+1}, T_2 x_{n+1}) + D^*(x_{n+1}, x_{n+2}, T_3 x_{n+2})\}\}$

$= a \lim_{n\to\infty} \max\{D^*(x, x_{n+1}, x_{n+2}), ½\{D^*(x, T_1x, x_{n+2}) + D^*(x_{n-+1}, x_{n+2}, x_{n+3})\}$

$½\{D^*(x, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\}\}$

$\leq a \max\{D^*(x, T_1x, x), 0\}$

$< D^*(T_1x, x, x)$, Which is a contradiction.

Thus $T_1x = x$..

Similarly we can prome that $T_2x = T_3x = x$.

Now we prove that x is a unique common fixed point of T_1, T_2, T_3

Suppose $x \neq y$ and $T_1x=T_2x=T_3x=x$ & $T_1y=T_2y=T_3y= y$

Then $D^*(x,y,y) = D^*(T_1x, T_2y, T_3 y)$

$\leq a \max\{D^*(x, y, y), ½\{D^*(x, T_1x, T_2y)+D^*(y, T_2y, T_3y)\},$

$1/2\{D^*(x, y, T_2y) + D^*(y, y, T_3y)\}\}$

$= a \max \{ D^*(x, y, y) , 1/2 D^*(x, x, y), 1/2D^*(x, y, y)\} = a D^* (x, y, y)$

$< D^* (x, y, y)$,

which is contradiction.

Hence T_1, T_2 &T_3 have a unique common fixed

Theorem 2.6.

Let X be a complete D* - metric space and $T_1, T_2, T_3 : X \rightarrow X$ be any three maps such that $D^* (T_1x, T_2y, T_3z) \leq a_1 D^*(x, y, z) + a_2 \max \{D^*(x, T_1x , T_2y), D^*(y, T_2y , T_3z)\}$

$+ a_3 \max \{D^*(x, y, T_2y), D^*(y, z, T_3z)\}$

for all $x, y, z \in X$ and $0 \leq a_1 + 2a_2 + 2a_3 < 1$. Then T has a unique fixed point.

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