SOME COMMON FIXED POINT THEOREMS IN D*-METRIC SPACE

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Abstract

In this paper we establish some common Fixed Point Theorems for contraction and generalized contraction mappings in D*-metric space which is introduced by Shaban Sedghi, Nabi Shobe and Haiyun Zhou [10]. In what follows (X , D*) will denote D*-metric space, N, the set of all natural number and \( R^+ \), the set of all positive real number.

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1. Introduction

There have been an number of generalization in generalized metric space (or D-Metric space) initiated by Dhage [2] in 1992. He proved the existence of unique fixed point theorems of a self map satisfying a contractive conditions in complete and bounded D-Metric space. Dealing with D-Metric space, Ahmad etal. [1], Dhage [2, 3, 4] Rhoades [8], Singh and Sharma [9] and others made a significant contribution in fixed point theory of D-Metric space. Unfortunately almost all theorems in D-Metric space are not valid (See S.V.R Naidu and others [5-7]). Here our aim is to prove some common fixed point theorems using some generalized contractive conditions in D*- Metric space as a probable modification of the definition of D-Metric spaces introduced by Dhage [2].

Definition 1.1.

Let X be a non empty set. A generalized metric (or D*-metric) on X is a function \( D^* : X^3 \rightarrow [0, \infty) \) that satisfies the following conditions for each x, y, z, a \( \in X \).

1. \( D^*(x, y, z) \geq 0 \)
2. \( D^*(x, y, z) = 0 \) if and only if \( x = y = z \)
3. \( D^*(x, y, z) = D^*(\rho\{x, y, z\}) \) where \( \rho \) is permutation.
4. \( D^*(x, y, a) \leq D^*(x, y, z) + D^*(a, z, z) \).

The pair \((X, D^*)\) is called generalized metric (or D*-metric) space.

Example 1.2:

(a) \( D^*(x, y, z) = \max \{d(x, y), d(y, z), d(z, x)\} \),
(b) \( D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x) \).

Here, d is the ordinary metric on X.

(c) If \( X = R^p \) then we define
\( D^*(x, y, z) = (||x - y||^p + ||y - z||^p + ||z - x||^p)^{1/p} \) for every \( p \in R^+ \)

(d) If \( X = R \) then we define
\( D^*(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{otherwise}. \end{cases} \)

Remark 1.3.

In D*-metric space \( D^*(x, y, y) = D^*(x, x, y) \)

Definition 1.4.
A open ball in a \(D^*\) - metric space \(X\) with centre \(x\) and radius \(r\) is denoted by 
\[B_{D^*}(x, r) = \{ y \in X : D^*(x, y, y) < r \}\]

**Example 1.5.**

Let \(X = \mathbb{R}\) denote \(D^*(x, y, z) = |x-y| + |y-z| + |z-x|\) for all \(x, y, z \in \mathbb{R}\).
Thus 
\[B_{D^*}(0, 1) = \{ y \in \mathbb{R} / D^*(0, y, y) < 1 \} = \{ y \in \mathbb{R} / |0-y| + |y-y| + |y| < 1 \} = \{ y \in \mathbb{R} / |y| < \frac{1}{2} \} = \{ y \in \mathbb{R} / -\frac{1}{2} < y < \frac{1}{2} \} = \left(-\frac{1}{2}, \frac{1}{2}\right).\]

**Definition 1.6.**

Let \((X, D^*)\) be a \(D^*\) - metric space and \(A \subseteq X\).
1. If for every \(x \in A\), there exist \(r > 0\) such that \(B_{D^*}(x, r) \subseteq A\), then subset \(A\) is called open subset of \(X\).
2. Subset \(A\) of \(X\) is said to be \(D^*\) - bounded if there exist \(r > 0\) such that \(D^*(x, y, y) < r\) for all \(x, y \in A\).
3. A sequence \(\{x_n\}\) in \(X\) converges to \(x\) if and only if 
\[D^*(x_n, x, x) = D^*(x, x, x_n) \rightarrow 0 \text{ as } n \rightarrow \infty.\]
That is, for each \(\varepsilon > 0\) there exist \(n_0 \in \mathbb{N}\) such that for all \(n \geq n_0\) implies \(D^*(x_n, x, x) < \varepsilon\).
It is also noted that \(D^*(x_n, x, x) = D^*(x, x, x_n) < \varepsilon\) for all \(n \geq n_0\), for some \(n_0 \in \mathbb{N}\).
4. A sequence \(\{x_n\}\) in \(X\) is called a Cauchy sequence if for each \(\varepsilon > 0\), there exist \(n_0 \in \mathbb{N}\) such that 
\[D^*(x_n, x_m, x) \leq \varepsilon\] 
for each \(n, m \geq n_0\) The \(D^*\) - metric space \((X, D^*)\) is said to complete if every Cauchy sequence is convergent.

**Remark 1.7.**

(1) \(D^*\) is continuous function \(X^3\)
(2) If sequence \(\{x_n\}\) in \(X\) converges to \(x\), then \(x\) is unique.
(3) Any convergent sequence in \((X, D^*)\) is a Cauchy sequence.

**Definition 1.8.**

A point \(x\) in \(X\) is a fixed point of the map \(T : X \rightarrow X\) if \(Tx = x\).

**Definition 1.9.**

A point \(x\) in \(X\) is a common fixed point of the two maps \(T_1, T_2 : X \rightarrow X\) if \(T_1(x) = T_2(x) = x\).

**Theorem 1**

Let \(X\) be a \(D^*\) - complete metric space and \(T_1, T_2 : X \rightarrow X\) be any two maps such that
\[D^*(T_1x, T_2y, z) \leq \alpha D^*(x, y, z)\] 
for all \(x, y, z \in X\) and \(0 \leq \alpha \leq \frac{1}{2}\) Then \(T_1\) & \(T_2\) have a unique common fixed point.

**Proof**

Let \(x_0 \in X\) be any fixed arbitrary element Define a sequence \(\{x_n\}\) in \(X\) as.
\[x_{n+1} = T_1x_n\text{ and }x_{n+2} = T_2x_{n+1}\text{ for }n = 0, 1, 2, \ldots.\]
Let \(d_n = D^*(x_n, x_{n+1}, x_{n+2})\) for all \(n = 0, 1, 2, \ldots.\)

Now 
\[d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_n) \leq D^*(T_1x_n, T_2x_{n+1}, x_n)\]
\[\leq \alpha D^*(x_n, x_{n+1}, x_{n+2}) + \alpha D^*(x_{n+1}, x_{n+2}, x_n)\]
\[= \alpha d_n + \alpha d_{n+1}\]
\[= (1 - \alpha) d_{n+1} + \alpha d_n\]
\[d_{n+1} \leq \frac{\alpha}{1 - \alpha} d_n\]
\[d_{n+1} \leq k d_n \text{ for all } n = 0, 1, 2, \ldots, \text{ where } k = \frac{\alpha}{1 - \alpha} < 1 \text{ (Since } \alpha < \frac{1}{2}\)
\[d_n \leq k d_{n-1}\]
\[\leq k^s d_0 \rightarrow 0 \text{ as } n \rightarrow \infty\]
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Thus \( \lim_{n \to \infty} d_n = 0 \) Thus \( \lim_{n \to \infty} D^*(x_n, x_{n+1}, x_{n+2}) = 0 \)

Now we shall prove that \( \{x_n\} \) is a \( D^* \) - Cauchy sequence in \( X \).
Let \( m > n > n_0 \) for some \( n_0 \in \mathbb{N} \).
Now \( D^*(x_m, x_n, x_n) \leq D^*(x_m, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+2}) \)
\( \leq \sum_{k=n}^{\infty} D^*(x_k, x_{k+1}) \to 0 \) as \( m, n \to \infty \)

Thus \( \lim_{n,m \to \infty} D^*(x_n, x_n, x_m) = 0 \)

There fore \( \{x_n\} \) is \( D^* \) - Cauchy sequence in \( X \).
Since \( X \) is \( D^* \) - Complete \( x \) is a fixed point of \( T_1 \) suppr \( x \neq T_1 x \).
Then \( D^*(T_1 x, x, x) = \lim_{n \to \infty} D^*(T_1 x, x_{n+1}, x) \)
\( \leq \alpha \lim_{n \to \infty} D^*(x, x_{n+1}, x) \)
\( = 0 \).

There fore \( D^*(T_1 x, x, x) = 0 \). Therefore \( T_1 x = x \) Simillarly we can prove that \( T_2 x = x \).
Hence \( T_1 x = T_2 x = x \).Thus \( x \) is common fixed point of \( T_1 \) and \( T_2 \).

Uniqueness
Supper \( x \neq y \) such that \( T_1 y = T_2 y = y \)
Then \( D^*(x, y, y) = D^*(T_1 x, T_2 y, y) \)
\( \leq \alpha D^*(x, y, y) \)
This implies \((1-\alpha)D^*(x, y, y) \leq 0 \)
Since \( x \neq y \) we have \( D^*(x, y, y) > 0 \) her \((1-\alpha) < 0 \).
This implies \( \alpha > 1 \) which contraction to \( \alpha < \frac{1}{2} \).
Thus \( T_1 \) and \( T_2 \) have a unique common fixed point.

Theorem 2
Let \( X \) be a complete \( D^* \) - metric space and \( T_1, T_2, T_3 : X \to X \) be any three maps such that \( D^*(T_1 x, T_2 y, T_3 z) \leq \alpha D^*(x, y, z) \) for all \( x, y, z \in X \) and \( 0 \leq \alpha < 1 \). Then \( T_1, T_2, T_3 \) have a unique common fixed point.

Proof
Let \( x_0 \in X \) he any fixed arbitrary element Define a sequence \( \{x_n\} \) in \( X \) as
\[ x_{n+1} = T_1 x_n \]
\[ x_{n+2} = T_2 x_{n+1} \]
\[ x_{n+3} = T_3 x_{n+2} \]
for \( n = 0, 1, 2, \ldots \)
Let \( d_n = D^*(x_n, x_{n+1}, x_{n+2}) \)
\[ d_1 = D^*(x_1, x_2, x_3) \]
\[ = D^*(T_1 x_0, T_2 x_1, T_3 x_2) \]
\( \leq \alpha D^*(x_0, x_1, x_2) \)
\[ d_2 = D^*(x_2, x_3, x_4) \]
\[ = D^*(T_2 x_1, T_3 x_2, T_1 x_3) \]
\( \leq \alpha D^*(x_1, x_2, x_3) \)
\( \leq \alpha d_1 \),
\( \leq \alpha^2 d_0 \)
Continuing in thus way he get \( d_n \leq \alpha^n d_0 \to 0 \) as \( n \to \infty \) (since \( 0 \leq \alpha < 1 \)).

Now we shall prove that \( \{x_n\} \) is a Cauchy sequence in \( X \).
Let \( d_{n+1} = D^*(x_n, x_{n+1}, x_{n+2}) \)
Then \( d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3}) \)
\( \leq D^*(x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}) \)
\( \leq d_n + d_n \)
\( d_{n+1}^* - d_n^* \leq d_n \leq \alpha^n d_0 \to 0 \) as \( n \to \infty \) (since \( 0 \leq \alpha < 1 \))
\( d_{n+1}^* \leq d_n^* \) for all \( n \)
Hence \( \{d_n^* \} \) is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be \( d \).Then \( d_n^* \to d \) as \( n \to \infty \).
Now we shall prove that \( d = 0 \).Suppose \( d \neq 0 \).

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Now \( d = \lim_{n \to \infty} d_{n+2} \)
\[ \leq \lim_{n \to \infty} \{ d_{n+1} + d_{n+1}^* \} \]
\[ \leq \lim_{n \to \infty} \{ \alpha d_n + d_{n+1}^* \} \]
\[ < \lim_{n \to \infty} \{ d_n + d_{n+1}^* \} \]
\[ = d, \text{ which is contraction. Thus } d = 0. \]

Hence \( D^*(x_n, x_n, x_m) \to 0 \) as \( m, n \to \infty \)
Therefore \( \{x_n\} \) is a \( D^* \) Cauchy sequence in \( X \).
Since \( X \) is \( D^* \) complete \( x_n \to x \) in \( X \).

Now we prove that \( x \) is fixed point of \( T_1 \)
To prove that \( T_1x = x \)
Suppose \( T_1x \neq x \)
Then \( D^*(T_1x, x, y) = \lim_{n \to \infty} D^*(T_1x, x_{n+1}, x_{n+2}) \)
\[ \leq \alpha \lim_{n \to \infty} D^*(x, x_{n+1}, x_{n+2}) \]
\[ \leq \alpha D^*(x, x, x) = 0 \]
Thus \( T_1x = x \).
Similarly we can prove that \( T_2x = T_3x = x \).
Now we prove that \( x \) is a unique common fixed point of \( T_1, T_2, T_3 \)
Suppose \( x \neq y \) and \( T_1x = T_2x = T_3x = x \) & \( T_1y = T_2y = T_3y = y \)
Then \( D^*(x, y, y) = D^*(T_1x, T_2y, T_3y) \leq \alpha D^*(x, y, y) \)
This impulse \( (1-\alpha)D^*(x, y, y) \leq 0 \)
Since \( x \neq y \) we have \( D^*(x, y, y) > 0 \)
This \( (1-\alpha) < 0 \)
This impulse \( \alpha > 1 \) which in contradiction Hence \( T_1, T_2 \& T_3 \) have a unique common fixed point

**Theorem 3**
Let \( X \) be a \( D^* \) complete metric space and \( S, T X \to X \) be any two maps such that
\[ D^*(STx, T x, y) \leq \alpha D^*(T x, x, y) \]
for all \( x, y \in X \) and \( 0 \leq \alpha < \frac{1}{2} \). Then \( S \) and \( T \) have a unique common fixed point

**Proof**
Let \( x_0 \in X \) be any fixed arbitrary element. Define a sequence \( \{x_n\} \) in \( X \) as
\( x_{n+1} = Tx_n \)
\( x_{n+2} = Sx_{n+1} \) for \( n = 0, 1, 2, \ldots \)
Let \( d_n = D^*(x_n, x_{n+1}, x_{n+1}) \)
\( d_1 = D^*(x_1, x_2, x_2) = D^*(Tx_0, STx_0, x_2) \leq \alpha D^*(x_0, x_0, x_2) = \alpha D^*(x_0, x_1, x_2) \leq \alpha D^*(x_0, x_1, x_1) + \alpha D^*(x_1, x_2) = \alpha d_0 + \alpha d_1 \)
\((1-\alpha) d_1 \leq \alpha d_0 \)
\( d_1 \leq \frac{\alpha}{1-\alpha} d_0 \)
\( d_1 \leq \beta d_0 \) where \( \beta = \frac{\alpha}{1-\alpha} < 1 \) (Since \( 0 \leq \alpha < \frac{1}{2} \).)
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\[ d_2 = D^*(x_2, x_1, x_3) \]
\[ = D^*(STx_0, Tx_2, x_3) \]
\[ \leq \alpha D^*(Tx_0, x_2, x_3) \]
\[ = \alpha D^*(x_1, x_2, x_3) \]
\[ \leq \alpha D^*(x_1, x_2, x_3) + \alpha D^*(x_2, x_3, x_3) \]
\[ = \alpha d_1 + \alpha d_2 \]
\[ (1-\alpha)d_2 \leq \alpha d_1 \]

Continuing in this way we get
\[ d_n \leq \beta d_{n-1} \text{ for all } n>0. \]
\[ \leq \beta^n d_0 \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (since } \beta = \frac{\alpha}{1-\alpha} < 1). \]

Now we shall prove that \( \{x_n\} \) is a Cauchy sequence in X. Let \( m > n \) for some \( n \in \mathbb{N} \).
\[ D^*(x_n, x_m, x_m) \leq \sum_{k=n}^{m-1} D^*(x_k, x_{k+1}, x_{k+1}) \]
\[ = \sum_{k=n}^{m-1} d_k \]
\[ \leq \beta^n \frac{1}{1-\beta} d_0 \rightarrow 0 \text{ as } n, m \rightarrow \infty \]

Therefore \( \{x_n\} \) is \( D^* \) Cauchy sequence in X Since X is \( D^* \) complete \( x_n \rightarrow x \) in X

Now we prove that \( Tx = x \)
Suppose \( Tx \neq x \)
\[ D^*(Tx, x, x) = \lim_{n \rightarrow \infty} D^*(Tx, x_{n+2}, x) \]
\[ = \lim_{n \rightarrow \infty} D^*(Tx, STx_n, x) \]
\[ \leq \alpha \lim_{n \rightarrow \infty} D^*(x, Tx_n, x) \]
\[ = \alpha \lim_{n \rightarrow \infty} D^*(x, x_{n+1}, x) \]
\[ = 0 \]

Therefore \( Tx = x \).
Next to prove that \( Sx = x \)
\[ D^*(STx, Tx, x) \leq \alpha D^*(T x, x, x) \]
\[ = 0 \text{ (since } T x = x) \]
Thus \( STx = Tx = x \) Hence \( Sx = x \text{ (since } Tx = x) \)
Therefore \( x \) is common fixed point \( S \) & \( T \)
Suppose \( x \neq y \) Such that \( Sx = Tx = x \) and \( Sy = Ty = y \)
Then \[ D^*(x, y) = D^*(STx, T x, y) \]
\[ \leq \alpha D^*(Tx, y) \]
\[ = \alpha D^*(x, y) \]
\[ (1-\alpha) D^*(x, y) \leq 0 \]
Thus \( 1 - \alpha < 0 \) This implies \( \alpha > 1 \) which is contradiction.
Therefore \( x = y \)
Hence \( x \) is a unique common fixed point.

**Theorem 4:**
Let \( X \) be a \( D^* \) complete metric space and \( R, S, T \in X \) be any three maps such that
\[ D^*(RSTx, STx, Tx) \leq \alpha D^*(STx, Tx, x) \]
for all \( x \in X \) and \( 0 \leq \alpha < 1 \). Then \( R, S \) and \( T \) have a unique common fixed point

**Proof**
Let \( x_0 \in X \) be any fixed arbitrary element Define a sequence \( \{x_n\} \) in \( X \) as
\[ x_{n+1} = T x_n \]
Let \( d_0 = D^*(x_0, x_{n+1}, x_{n+2}) \)
\[
\begin{align*}
d_1 &= D^*(x_1, x_2, x_3) \\
&= D^*(Tx_0, STx_0, RSTx_0) \\
&\leq \alpha D^*(x_0, Tx_0, STx_0) \\
&\leq \alpha D^*(x_0, x_1, x_2) \\
\end{align*}
\]
\[
\begin{align*}
d_0 &= D^*(x_2, x_3, x_4) \\
&= D^*(Tx_1, STx_1, RSTx_1) \\
&\leq \alpha D^*(x_1, Tx_1, STx_1) \\
&\leq \alpha D^*(x_1, x_2, x_3) \\
&\leq \alpha d_1, \\
&\leq \alpha^2 d_0.
\end{align*}
\]
Continuing in this way we get
\[
\begin{align*}
d_n &\leq \alpha d_{n-1} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (since } 0 \leq \alpha < 1). \\
\end{align*}
\]
Now we shall prove that \( \{x_0\} \) is a Cauchy sequence in X.
Let \( d_n^* = D^*(x_n, x_n, x_{n+1}) \)
\[
\begin{align*}
d_n^* &= D^*(x_n, x_{n+1}, x_{n+2}) \\
&\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}) \\
&\leq d_n + d_n^* \\
&\leq \alpha d_{n-1} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (since } 0 \leq \alpha < 1) \\
\end{align*}
\]
Hence \( \{d_n^*\} \) is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be d. Then \( d_n^* \rightarrow d \) as \( n \rightarrow \infty \).
Now we shall prove that \( d = 0 \). Suppose \( d \neq 0 \).
Now \( d = \lim_{n \rightarrow \infty} d_{n+2}^* \)
\[
\begin{align*}
&\leq \lim_{n \rightarrow \infty} d_{n+1}^* + d_{n+2}^* \\
&\leq \lim_{n \rightarrow \infty} \alpha d_{n+1}^* + d_{n+2}^* \\
&\leq \lim_{n \rightarrow \infty} \alpha d_n + d_{n+1}^* \\
&= d, \text{ which is contraction. Thus } d = 0.
\end{align*}
\]
Hence \( D^*(x_n, x_n, x_m) \rightarrow 0 \) as \( m, n \rightarrow \infty \)
Therefore \( \{x_n\} \) is a \( D^* \) Cauchy sequence in X.
Since X is \( D^* \) complete the sequence \( x_n \rightarrow x \) in X.
Now we prove that \( x \) is fixed point of \( T \)
To prove that \( Tx = x \)
Suppose \( Tx \neq x \)
Then \( D^*(Tx, x, x) = \lim_{n \rightarrow \infty} D^*(Tx, x_{n+2}, x_{n+3}) \)
\[
\begin{align*}
&= \lim_{n \rightarrow \infty} D^*(Tx, STx_0, RSTx_0) \\
&\leq \alpha \lim_{n \rightarrow \infty} D^*(x, x_{n+1}, x_{n+2}) \\
&= 0
\end{align*}
\]
Thus \( Tx = x \).
Now we can prove that \( Sx = x \).
Then \( D^*(x, Sx, x) = \lim_{n \rightarrow \infty} D^*(Tx, STx_0, x_{n+3}) \)
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\[
\lim_{n \to \infty} D^*(Tx, STx, RSTx) = \bigtriangleup
\]

\[
\leq \alpha \lim_{n \to \infty} D^*(x, Tx, x_{n+2}) = \bigtriangleup
\]

\[
= \alpha \lim_{n \to \infty} D^*(x, x, x_{n+2}) = 0
\]

Thus \( Sx = x \).

Finally we prove that \( Rx = x \).

Then \( D^*(x, x, Rx) = D^*(Tx, STx, RSTx) \)

\[
\leq \alpha D^*(x, Tx, STx) = \bigtriangleup
\]

\[
= \alpha D^*(x, x, x) = 0
\]

Thus we prove that \( x \) is a unique common fixed point of \( R, S, T \).

Let \( x_0 \in X \) a fixed arbitrary element and define a sequence \( \{x_n\} \) in \( X \) as

\[
x_{n+1} = T_1 x_n
\]

\[
x_{n+2} = T_2 x_{n+1}
\]

\[
x_{n+3} = T_3 x_{n+2} \quad \text{for } n = 0, 1, 2, \ldots
\]

Let \( d_n = D^*(x_n, x_{n+1}, x_{n+2}) \).

Then \( d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3}) \)

\[
\leq a \left\{ D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, T_1 x_{n+1}, T_2 x_{n+1}) \right\}
\]

\[
= a \left\{ 2 D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3}) \right\}
\]

\[
d_{n+1} \leq 2a d_n + a d_{n+1} - d_n \leq 2a d_n + \alpha d_{n+1} \leq [2a/(1-\alpha)] d_n
\]

\[
d_{n+1} \leq b d_n \quad \text{where } b = 2a/(1-\alpha) < 1.
\]

Hence \( d_n \leq b^n d_0 \to 0 \) as \( n \to \infty \).

Now we shall prove that \( \{x_n\} \) is a Cauchy sequence in \( X \).

Let \( d_n^* = D^*(x_n, x_{n+1}) \).

Then \( d_{n+1}^* = D^*(x_{n+1}, x_{n+2}, x_{n+3}) \)

\[
\leq a \left\{ D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1}) \right\}
\]

\[
= a \left\{ 2 D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+2}) \right\}
\]

\[
d_{n+1}^* \leq b d_n^* \quad \text{for all } n
\]

Hence \( \{d_n^*\} \) is monotonically decreasing sequence of positive real numbers and it converges to its glb. Let it be \( d \).

Now \( d = \lim_{n \to \infty} d_n^* \to d \) as \( n \to \infty \).

Now we shall prove that \( d = 0 \). Suppose \( d \neq 0 \).

\[
\lim_{n \to \infty} d_{n+1}^* \leq \lim_{n \to \infty} \left\{ d_{n+1} + d_{n+1}^* \right\}
\]

\[
\leq \lim_{n \to \infty} \left\{ b d_n + d_n^* \right\}
\]
To prove that $T_1 x = x$, we first prove that $x$ is a unique common fixed point of $T_1$.

Let $X$ be any three maps such that $D^*(x, y, z) = D^*(y, z, x)$ for all $x, y, z \in X$.

**Theorem 2.3.** Let $X$ be a complete $D^*$-metric space and $T_1, T_2, T_3 : X \to X$ be any three maps such that $D^*(T_1 x, T_2 y, T_3 z) \leq a_1 D^*(x, y, z) + a_2 \{ D^*(x, T_1 x, T_2 y) + D^*(y, T_2 y, T_3 z) \}$

$$+ a_3 \{ D^*(x, y, T_3 z) + D^*(y, z, T_2 y) \}$$

for all $x, y, z \in X$, and $0 \leq a_1 + 2a_2 + 2a_3 < 1$. Then $T_1, T_2$, and $T_3$ have a unique common fixed point.

**Proof.**

Let $x_0 \in X$ be a fixed arbitrary element and define a sequence $\{ x_n \}$ in $X$ as

- $x_n = T_1 x_{n-1}$,
- $x_{n+1} = T_2 x_n$, and $x_{n+2} = T_3 x_{n+1}$ for $n = 0, 1, 2, \ldots$.

Let $d_n = D^*(x_n, x_{n+1}, x_{n+2})$.

Then $d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3})$

$$= D^*(T_1 x_n, T_2 x_{n+1}, T_3 x_{n+2})$$

$$\leq a_1 D^*(x_{n+1}, x_{n+2}, x_{n+3}) + a_2 \{ D^*(x_n, T_1 x_n, T_2 x_{n+1}) + D^*(x_{n+1}, T_2 x_{n+1}, T_3 x_{n+2}) \}$$

$$+ a_3 \{ D^*(x_{n+1}, x_{n+2}, x_{n+3}) + D^*(x_n, x_{n+1}, x_{n+2}) \}$$

$$\leq a_1 + 2a_2 + 2a_3 \} D^*(x_n, x_{n+1}, x_{n+2}) + (a_2 + a_3) \} D^*(x_{n+1}, x_{n+2}, x_{n+3})$$

$$\leq a_1 + a_2 + a_3 \} d_n + (a_2 + a_3) \} d_{n+1}$$

$$(1 - a_2 - a_3) \} d_{n+1} \leq (a_1 + a_2 + a_3) \} d_n$$

$$d_{n+1} \leq \frac{(a_1 + a_2 + a_3) \} d_n}{(1 - a_2 - a_3)}.$$
THEORY 2.5. Let X be a complete D* - metric space and T1, T2, T3 : X \to X be any three maps such that D*(T1x, T2y, T3z) \leq \max{D*(x, y, z), D*(x, T1x, T2y), D*(y, T1x, T2y), D*(z, T1x, T2y), D*(y, z, T3z)} for all x, y, z \in X, and 0 \leq a < 1. Then T1, T2, and T3 have a unique common fixed point.

Proof.
Let x0 \in X a fixed arbitrary element and define a sequence {xn} in X as
Let \( d_n = \max \{ D^*(x_n, x_{n+1}, x_{n+2}) \} \)
Then \( d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3}) \)
\[ \leq a \max \{ D^*(x_n, x_{n+1}, x_{n+2}) \} \]
\[ = a \max \{ D^*(x_n, x_{n+1}, x_{n+2}) \} \]
\[ = a \max \{ D^*(x_n, x_{n+1}, x_{n+2}) \} \]
\[ \leq a \max \{ D^*(x_n, x_{n+1}, x_{n+2}) \} \]
\[ \leq a \max \{ d_n, d_{n+1} \} \]
\[ d_{n+1} \leq d_n \text{ for all } n \]
Hence \( d_n \leq a^n d_0 \rightarrow 0 \text{ as } n \rightarrow \infty \)
Now we prove that \( \{x_n\} \) is \( D^* \) - Cauchy sequence in \( X \).
Let \( d_{n+1}^* = D^*(x_{n+1}, x_{n}, x_{n+1}) \)
Then
\[ d_{n+1}^* = D^*(x_{n+1}, x_{n+2}, x_{n+3}) \]
\[ \leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1}) \]
\[ \leq d_n + d_n^* \]
\[ d_{n+1}^* - d_n^* \leq d_n \leq a^\alpha d_{n+1} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (since } 0 \leq \alpha < 1) \]
\[ d_{n+1}^* \leq d_n^* \text{ for all } n \]
Hence \( \{ d_n^* \} \) is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be \( d \). Then \( d_n^* \rightarrow d \text{ as } n \rightarrow \infty \).
Now we shall prove that \( d = 0 \). Suppose \( d \neq 0 \).
Now \( d = \lim_{n \rightarrow \infty} \frac{d_{n+2}^*}{d_n} \)
\[ \leq \lim_{n \rightarrow \infty} \frac{\{ d_{n+1} + d_{n+1}^* \}}{d_n} \]
\[ \leq \lim_{n \rightarrow \infty} \frac{\{ d_n + d_n^* \}}{d_n} \]
\[ \leq \lim_{n \rightarrow \infty} \frac{d_n + d_n^*}{d_n} \]
\[ = d \]
Now we prove that \( \{x_n\} \) is \( D^* \) - Cauchy sequence in \( X \).
For \( m > n \) we have,
\[ D^*(x_n, x_{n+1}, x_{n+2}) \leq D^*(x_n, x_{n+1}, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+1}) + \ldots + D^*(x_{n+m-1}, x_{n-1}, x_{n+m}) \]
\[ = 0 \text{ as } m, n \rightarrow \infty \]
Thus \( \{x_n\} \) is a \( D^* \) Cauchy sequence in \( X \) and \( X \) is \( D^* \) - complete \( x_n \rightarrow x \) in \( X \).
Now we shall prove that \( T_x = x \)
\[ D^*(T_1x, x, x) = \lim_{n \rightarrow \infty} D^*(T_1x, x_{n+2}, x_{n+3}) \]
\[ = \lim_{n \rightarrow \infty} D^*(T_1x, T_2x_{n+1}, T_3x_{n+2}) \]
\[ \leq a \lim_{n \rightarrow \infty} \max \{ D^*(x, x_{n+1}, x_{n+2}) \} \]
\[ \leq a \lim_{n \rightarrow \infty} \max \{ D^*(x, x_{n+1}, x_{n+2}) \} \]
\[ \leq a \max \{ D^*(x, x_{n+1}, x_{n+2}) \} \]
\[ < D^*(T_1x, x, x) \]
Which is a contradiction.
Thus \( T_1x = x \).
Similarly we can prove that \( T_2x = T_3x = x \).
Now we prove that \( x \) is a unique common fixed point of \( T_1, T_2, T_3 \)
Suppose \( x \neq y \) and \( T_1x = T_2x = T_3x = x \) & \( T_1y = T_2y = T_3y = y \)
Then \( D^*(x,y,y) = D^*(T_1x, T_2y, T_3y) \)
To prove that \( T_1 \) has a unique common fixed point, let \( T_1, T_2 \) be any three maps such that
\[
\text{SOME COMMON FIXED POINT THEOREMS...}
\]

\[
\leq a \max \{ D^*(x, y, y), \ D^*(x, T_1x, y), D^*(y, T_2y, y), D^*(y, T_2y, T_2y) \}
\]

\[
= a \max \{ D^*(x, y, y), D^*(x, y, y), D^*(x, y, y), D^*(y, y, y) \}
\]

which is contradiction. Hence \( T_1, T_2 \) have a unique common fixed point.

**Theorem 2.6.**

Let \( X \) be a complete \( D^* \) - metric space and \( T_1, T_2, T_3 : X \rightarrow X \) be any three maps such that
\[
D^* (T_1x, T_2x, T_3x) \leq a_1 D^* (x, y, z) + a_2 \max \{ D^*(x, T_1x, T_2x), D^*(y, T_2y, T_3y) \}
\]

for all \( x, y, z \in X \) and \( 0 \leq a_1 + 2a_2 < 1 \). Then \( T \) has a unique fixed point.

**Proof.**

Let \( x_0 \in X \) be an arbitrary element and define a sequence \( \{x_n\} \) in \( X \) as
\[
x_{n+1} = T_1x_n
\]

\[
x_{n+2} = T_2x_{n+1}
\]

\[
x_{n+3} = T_3x_{n+2}
\]

for \( n = 0, 1, 2, \ldots \).

Let \( d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3}) \).

Then
\[
d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3})
\]

\[
= a_1 D^*(x_n, T_1x_n, T_2x_n, T_3x_n) + a_2 \max \{ D^*(x_n, T_1x_n, T_2x_n), D^*(x_n, T_2x_n, T_3x_n) \}
\]

\[
= a_1 \{ D^*(x_{n+1}, x_{n+2}) + a_2 \max \{ D^*(x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}) \} \}
\]

\[
\leq 2a_1 d_n + a_2 d_{n+1}
\]

\[
d_{n+1} \leq b d_n \quad \text{where } b = \{ 2a_1 \} + 1.
\]

Hence \( d_n \leq b d_0 \rightarrow 0 \) as \( n \rightarrow \infty \).

Now we shall prove that \( \{x_n\} \) is a Cauchy sequence in \( X \).

Let
\[
d_n = D^*(x_n, x_{n+1}, x_{n+1})
\]

Then
\[
d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3})
\]

\[
\leq D^*(x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+1})
\]

\[
\leq d_{n+1} + d_n
\]

\[
d_{n+1} \leq d_{n+1} \alpha d_n \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{since } 0 \leq \alpha < 1.
\]

Hence \( \{d_n\} \) is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be \( d \) . Then \( d_n \rightarrow d \) as \( n \rightarrow \infty \).

Now we shall prove that \( d = 0 \) . Suppose \( d \neq 0 \).

Now \( d = \lim_{n \rightarrow \infty} d_{n+1} \) and
\[
\leq \lim_{n \rightarrow \infty} \{ d_{n+1}, d_n \}
\]

\[
\leq \lim_{n \rightarrow \infty} \{ b d_n, d_{n+1} \}
\]

\[
\leq \lim_{n \rightarrow \infty} \{ d_n, d_{n+1} \}
\]

\( = d \) . Hence \( d = 0 \) .

Therefore \( \{x_n\} \) is a \( D^* \) Cauchy sequence in \( X \).

Now we prove that \( x \) is a fixed point of \( T_1 \).

To prove that \( T_1x = x \) let \( T_1x = x \).

Suppose \( T_1x \neq x \) . Then
\[
D^*(T_1x, x, x) = \lim_{n \rightarrow \infty} D^*(T_1x, x_{n+1}, T_2x_{n+2})
\]

\[
= \lim_{n \rightarrow \infty} D^*(T_1x, T_2x_{n+1}, T_3x_{n+2})
\]

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Corollary 2.8.

Now we prove that $x$ is a unique common fixed point of $T$. Suppose $x \neq y$ and $T_x = T_y = x \neq y$.

Then $D^*(x,y) = D^*(x,x)$. Then $d \leq a_1 D^*(x,y)$. Hence $T$ has a unique fixed point.

Remark 2.7.

Let $x_0$, $x_1$, $x_2$, $x_3$ be any three maps such that $D^*(x_0, x_1, x_2, x_3)$ is a unique common fixed point.

Theorem 2.11.

Let $X$ be a complete $D^*$-space and $T_1$, $T_2$, $T_3 : X \to X$ be any three maps such that $D^*(T_1, T_2, T_3)$ is a unique common fixed point.

Proof.

Let $x_0 \in X$ be an arbitrary element and define a sequence $\{x_n\}$ in $X$ as

\[
x_n = T_1 x_{n-1} = T_2 x_{n-2} = T_3 x_{n-3}, \quad n = 0, 1, 2, \ldots
\]

Then $d_n = D^*(x_n, x_{n+1}, x_{n+2})$.

Then $d_n = D^*(x_n, x_{n+1}, x_{n+2})$.

\[
\lim_{n \to \infty} d_n = \lim_{n \to \infty} |d_1 D^*(x_n, x_{n+1}, x_{n+2}) + d_2 \max \{D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\} | \leq \lim_{n \to \infty} a_1 D^*(x_n, x_{n+1}, x_{n+2}) + a_2 \max \{D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\} \leq a_2 D^*(x, T_x, x)
\]

Similarly we can prove that $T_3 x = x$.

Hence $T_1$, $T_2$, $T_3$ have a unique common fixed point.

Corollary 2.8.

Let $(X, D^*)$ be a complete $D^*$-metric space and $T : X \to X$ be a map such that $D^*(x, y, z) \leq a D^*(x, y, z)$ for all $x, y, z \in X$ and $0 \leq a < 1$. Then $T$ has a unique fixed point. The above theorem is known as Banach contraction Type Theorem in $D^*$-metric space.

Remark 2.9.

If we put $a_1 = 0$ and $a_2 = a$ in the above theorem 1. We get the following theorem as corollary 2.10.

Corollary 2.10.

Let $(X, D^*)$ be a complete $D^*$-metric space and $T : X \to X$ be a map such that $D^*(x, y, z) \leq a_1 D^*(x, y, z) + a_2 \max \{D^*(x, y, z), D^*(x, z, T_y), D^*(y, z, T_z)\}$ for all $x, y, z \in X$ and $0 \leq a < 1$. Then $T$ has a unique fixed point.

Theorem 2.11.

Let $X$ be a complete $D^*$-metric space and $T_1$, $T_2$, $T_3 : X \to X$ be any three maps such that $D^*(T_1, T_2, T_3) \leq a_1 D^*(x, y, z) + a_2 \max \{D^*(x, y, z), D^*(x, z, T_y), D^*(y, z, T_z)\}$

\[
+ a_3 \max \{D^*(x, y, T_y) + D^*(y, z, T_z)\}
\]

for all $x, y, z \in X$, and $0 \leq a_1 + 2a_2 + 3a_3 < 1$. Then $T_1$, $T_2$, and $T_3$ have a unique common fixed point.

Proof.

Let $x_0 \in X$ be an arbitrary element and define a sequence $\{x_n\}$ in $X$ as

\[
x_n = T_1 x_{n-1} = T_2 x_{n-2} = T_3 x_{n-3}, \quad n = 0, 1, 2, \ldots
\]

Then $d_n = D^*(x_n, x_{n+1}, x_{n+2})$.

Then $d_n = D^*(x_n, x_{n+1}, x_{n+2})$.

\[
\lim_{n \to \infty} d_n = \lim_{n \to \infty} |d_1 D^*(x_n, x_{n+1}, x_{n+2}) + d_2 \max \{D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\} | \leq \lim_{n \to \infty} a_1 D^*(x_n, x_{n+1}, x_{n+2}) + a_2 \max \{D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\} \leq a_2 D^*(x, T_x, x).
\]

Similarly we can prove that $T_3 x = x$.

Hence $T_1$, $T_2$, $T_3$ have a unique common fixed point.
To prove that \( T_1 = H \), hence \( T < D^*(x, y, y) \), which is contradiction. Thus \( d = 0 \).

Now we shall prove that \( d = 0 \). Suppose \( d \neq 0 \).

Now \( d = \lim_{n \to \infty} d_{n+1}^* \)

\[
\begin{align*}
d_{n+1}^* - d_n^* & \leq d_n = \alpha \cdot d_{n-1} \to 0 \text{ as } n \to \infty \text{ (since } 0 \leq \alpha < 1) \\
d_n^* & \leq d_m^* \text{ for all } n
\end{align*}
\]

Hence \( \{ d_n^* \} \) is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be \( d \). Then \( d_n^* \to d \) as \( n \to \infty \).

Now we shall prove that \( d = 0 \). Suppose \( d \neq 0 \).

For \( m > n \), we have

\[
\begin{align*}
D^*(x_n, x_{n+1}) & \leq D^*(x_n, x_m) + D^*(x_m, x_{n+1}) \\
& \leq D^*(x_n, x_m) + D^*(x_m, x_{n+2}) + \ldots + D^*(x_{m+1}, x_{n+1}) \\
& \to 0 \text{ as } n, m \to \infty . \text{ Hence } D^*(x_n, x_m) \to 0 \text{ as } m \to \infty 
\end{align*}
\]

Therefore \( \{ x_n \} \) is a \( D^* \) Cauchy sequence in \( X \).

Since \( X \) is \( D^* \) complete \( x_n \to x \) in \( X \)

Now we prove that \( x \) is fixed point of \( T_1 \).

To prove that \( T_1 x = x \).

Suppose \( T_1 x \neq x \), Then

\[
\begin{align*}
D^*(T_1 x, x) &= \lim_{n \to \infty} D^*(T_1 x, x_{n+1}) \\
& = \lim_{n \to \infty} D^*(T_1 x, T_2 x_{n+1}, T_3 x_{n+2}) \\
& \leq \lim_{n \to \infty} \{ a_1 D^*(x, x_{n+1}, x_{n+2}) + a_2 \max \{ D^*(x, T_1 x, T_2 x_{n+1}), D^*(x_{n+1}, T_2 x_{n+1}, T_3 x_{n+2}) \} + a_3 \max \{ D^*(x, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, T_3 x_{n+2}) \} \} \\
& \leq a_2 \max \{ D^*(x, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, T_3 x_{n+2}) \} \\
& < D^*(x, T_1 x, x), \text{ which is contradiction. Thus } T_1 x = x .
\end{align*}
\]

Similarly we can prove that \( T_2 x = T_3 x = x \).

Now we prove that \( x \) is a unique common fixed point of \( T_1, T_2, T_3 \).

Suppose \( x \neq y \) and \( T_1 x = T_2 x = T_3 x = x \) and \( T_1 y = T_2 y = T_3 y = y \).

Then

\[
\begin{align*}
D^*(x, y) &= D^*(T_1 x, T_2 y, T_3 y) \\
& \leq a_1 D^*(x, y, y) + a_2 \max \{ D^*(x, T_1 x, T_2 y), D^*(y, T_1 y) \} + a_3 \max \{ D^*(x, y, T_1 y) \} \\
& = a_1 D^*(x, y, y) + a_2 \max \{ D^*(x, y, y) \} + a_3 \max \{ D^*(x, y, y) \} \\
& = (a_1 + a_2 + a_3) D^*(x, y, y) \\
& < D^*(x, y, y), \text{ which is contradiction.}
\end{align*}
\]

Hence \( T_1, T_2, T_3 \) have a unique common fixed point.

**Theorem 2.12.** Let \( X \) be a complete \( D^* \) metric space and \( T_1, T_2, T_3 : X \to X \) be any three maps such that

\[
D^*(T_1 x, T_2 y, T_3 z) \leq \max \{ D^*(x, y, z), 1/2 \{ D^*(x, T_1 x, T_2 y) + D^*(y, T_2 y, T_3 z) \} \} \text{ for all } x, y, z \in X , \text{ and } 0 \leq a < 1/3.
\]

Then \( T_1, T_2, \) and \( T_3 \) have a unique common fixed point.

**Proof.**

Let \( x_0 \in X \) a fixed arbitrary element and define a sequence \( \{ x_n \} \) in \( X \) as

\[
\begin{align*}
x_{n+1} &= T_1 x_n \\
x_{n+2} &= T_2 x_{n+1} \\
x_{n+3} &= T_3 x_{n+2} \text{ for } n = 0, 1, 2, \ldots
\end{align*}
\]

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Let \( d_n = D^*(x_n, x_{n+1}, x_{n+2}) \).
Then \( d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3}) \)
\[\leq \max \{ D^*(x_n, x_{n+1}, x_{n+2}), \frac{1}{2} \{ D^*(x_n, T_1 x_n, T_2 x_{n+1}, T_3 x_{n+2}) \} \} \]
\[= \max \{ D^*(x_n, x_{n+1}, x_{n+2}), \frac{1}{2} \{ D^*(x_n, T_1 x_n, T_2 x_{n+1}, T_3 x_{n+2}) \} \} \]
\[\leq \frac{1}{2} \left\{ \frac{1}{2} \left\{ D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3}) \right\} \right\} \]
\[\leq \frac{1}{2} \left\{ D^*(x_n, x_{n+1}, x_{n+2}) \right\} \]
\[\leq a d_n, \quad \frac{1}{2} (d_n, d_{n+1}, d_{n+2}) \]
\[d_{n+1} \leq a(3d_n + 1/2 d_{n+1}) \]
\[d_{n+1} \leq a(3d_n) \quad \text{for all } n \]
Hence \( d_n \rightarrow 0 \) as \( n \rightarrow \infty \).
Now we prove that \( \{x_n\} \) is \( D^* \)-Cauchy sequence in \( X \).
Let \( d_n = D^*(x_n, x_{n+1}, x_{n+2}) \)
Then \( d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3}) \)
\[\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3}) \]
\[\leq d_n + d_n \]
\[d_{n+1} - d_n \leq \alpha d_n \rightarrow 0 \quad (\text{since } 0 \leq \alpha < 1) \]
\[d_{n+1} \leq d_n \quad \text{for all } n \]
Hence \( \{d_n\} \) is a monotonically decreasing sequence of positive real numbers and it converges to its glb. Let it be \( d \).
Then \( d_n \rightarrow d \) as \( n \rightarrow \infty \).
Now we shall prove that \( d = 0 \).
Suppose \( d \neq 0 \).
Now \( d = \lim_{n \rightarrow \infty} d_{n+1} \)
\[\leq \lim_{n \rightarrow \infty} \{d_{n+1} + d_{n+2}\} \]
\[\leq \lim_{n \rightarrow \infty} \{d_n + d_{n+1}\} \]
\[= d \]
Now we prove that \( \{x_n\} \) is \( D^* \)-Cauchy sequence in \( X \).
For \( m > n \) we have,
\[D^*(x_n, x_m) \leq D^*(x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+2}) + \ldots + D^*(x_{m-1}, x_m) \]
\[\rightarrow 0 \quad \text{as } m, n \rightarrow \infty \]
Thus \( \{x_n\} \) is a \( D^* \)-Cauchy sequence in \( X \) and \( X \) is \( D^* \)-complete \( x \rightarrow x \) in \( X \).
Now we shall prove that \( T_1 x = x \)
\[D^*(T_1 x, x, x) = \lim_{n \rightarrow \infty} D^*(T_1 x, x_{n+2}, x_{n+3}) \]
\[= \lim_{n \rightarrow \infty} D^*(T_1 x, T_2 x_{n+1}, T_3 x_{n+2}) \]
\[\leq a \lim_{n \rightarrow \infty} \max \{D^*(x, x_{n+1}, x_{n+2}), \frac{1}{2} \{D^*(x, T_1 x, T_2 x_{n+1}, T_3 x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3}) \}\} \]
\[\leq a \max \{D^*(x, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3}) \} \]
\[< D^*(T_1 x, x, x) \], Which is a contradiction.
Thus \( T_1 x = x \).
Similarly we can prove that \( T_2 x = T_3 x = x \).
Now we prove that \( x \) is a unique common fixed point of \( T_1, T_2, T_3 \).
Suppose \( x \neq y \) and \( T_1 x = T_2 x = T_3 x = x \) & \( T_1 y = T_2 y = T_3 y = y \).
Then \( D^*(x, y, y) = D^*(T_1 x, T_2 y, T_3 y) \)
\[\leq \max \{D^*(x, y, y), \frac{1}{2} \{D^*(x, T_1 x, T_2 y, T_3 y) + D^*(y, T_1 y, T_2 y, T_3 y) \} \}\]
SOME COMMON FIXED POINT THEOREMS...

\[ a \max \{ D^*(x, y, y), 1/2 D^*(x, x, y), 1/2 D^*(x, y, y) \} = a \ D^*(x, y, y) \]

which is contradiction.

Hence \( T_1, T_2 \& T_3 \) have a unique common fixed point.

**Theorem 2.6.**

Let \( X \) be a complete \( D^* \)-metric space and \( T_1, T_2, T_3 : X \rightarrow X \) be any three maps such that

\[ D^*(T_1x, T_2y, T_3z) \leq a_1 D^*(x, y, z) + a_2 \max \{ D^*(x, T_1x , T_2y), D^*(y, T_2y , T_3z ) \} + a_3 \max \{ D^*(x, y, T_3z ), D^*(y, z, T_3z ) \} \]

for all \( x, y, z \in X \) and \( 0 \leq a_1 + 2a_2 + 2a_3 < 1 \). Then \( T \) has a unique fixed point.

**References**