(α, β) – Cut of intuitionistic fuzzy modules - II

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Abstract

For any intuitionistic fuzzy set A = {< x, μA(x), νA(x) > : x ∈ X} of a set X, the crisp subset Cα, β(A) = {x ∈ X : μA(x) ≥ α, νA(x) ≤ β} of X called the (α, β) – cut of A is studied by the author in [6] and [7] when A is intuitionistic fuzzy subgroup of a group G and when A is intuitionistic fuzzy submodule of an R-module M respectively. Basnet in [4] studied the (α, β)- cut set when A is intuitionistic fuzzy ideal of a ring R. This paper is a continuation of author’s earlier paper [7]. In this paper, some results concerning the sum and the product of two intuitionistic fuzzy submodules are obtained. A relationship between the (α, β)- cut set of sum and product of two intuitionistic fuzzy submodules with the (α, β)- cut sets of the two intuitionistic fuzzy submodules is established.

Keywords: Intuitionistic fuzzy submodule (IFSM), (α, β) – Cut set.

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1. Introduction:

The concept of intuitionistic fuzzy sets was introduced by Atanassov [1, 2] as a generalization of that of fuzzy sets and it is a very effective tool to study the case of vagueness. Further many researches applied this notion in various branches of mathematics especially in algebra and defined intuitionistic fuzzy subgroups, intuitionistic fuzzy subrings, and intuitionistic fuzzy sublattices, intuitionistic fuzzy submodules and so forth.

2. Preliminaries:

In this section we recall some definitions and results which will be used later

Definition (2.1)[1] Let X be a fixed non-empty set. An Intuitionistic fuzzy set (IFS) A of X is an object of the following form A = {< x, μA(x), νA(x) > : x ∈ X}, where μA : X → [0, 1] and νA : X → [0, 1] define the degree of membership and degree of non-membership of the element x ∈ X respectively and for any x ∈ X, we have 0 ≤ μA(x) + νA(x) ≤ 1.

Remark (2.2)(i): When μA(x) + νA(x) = 1, i.e., when νA(x) = 1 - μA(x) = μcA(x). Then A is called fuzzy set.
(ii) We write \( A = (\mu_A, \nu_A) \) to denote the IFS \( A = \{ < x, \mu_A(x), \nu_A(x) > : x \in X \} \).

**Definition (2.3)**[7] Let \( M \) be a modules over a ring \( R \). An IFS \( A = (\mu_A, \nu_A) \) of \( M \) is called intuitionistic fuzzy (left) submodule (IFSM) if

(i) \( \mu_A(0) = 1, \nu_A(0) = 0 \)

(ii) \( \mu_A(x + y) \geq \min \{ \mu_A(x), \mu_A(y) \} \) and \( \nu_A(x + y) \leq \max \{ \nu_A(x), \nu_A(y) \} \), \( \forall x, y \in M \)

(iii) \( \mu_A(rx) \geq \mu_A(x) \) and \( \nu_A(rx) \leq \nu_A(x) \), \( \forall x \in M, r \in R \)

If we replace the condition (iii) with \( \mu_A(rx) \geq \mu_A(x) \) and \( \nu_A(rx) \leq \nu_A(x) \), \( \forall x \in M, r \in R \), it is called intuitionistic fuzzy (right) module. When \( R \) is commutative ring, then these two modules coincides. From this onward, \( R \) will be a commutative ring with unity.

**Definition(2.4)**[5, 4] Let \( A = (\mu_A, \nu_A) \) and \( B = (\mu_B, \nu_B) \) be two intuitionistic fuzzy submodules of an \( R \)-module \( M \), then their sum \( A + B = (\mu_{A+B}, \nu_{A+B}) \) is defined as

\[
\mu_{A+B}(x) = \sup_{x = a+b} \{ \min \{ \mu_A(a), \mu_B(b) \} \} \quad \text{and} \quad \nu_{A+B}(x) = \inf_{x = a+b} \{ \max \{ \nu_A(a), \nu_B(b) \} \} , \text{for all } x \in M
\]

**Theorem(2.5)** Let \( A \) and \( B \) be two intuitionistic fuzzy submodules of an \( R \)-module \( M \). Then the sum \( A + B \) of \( A \) and \( B \) is also intuitionistic fuzzy submodule of \( M \).

**Proof.** Let \( x, y \in M \) be any two elements and let \( \min \{ \mu_{A+B}(x), \mu_{A+B}(y) \} = \alpha \) (say)

Let \( \varepsilon > 0 \) be given, then

\[
\alpha - \varepsilon < \mu_{A+B}(x) = \sup_{x = a+b} \{ \min \{ \mu_A(a), \mu_B(b) \} \} \quad \text{and} \quad \alpha - \varepsilon < \mu_{A+B}(y) = \sup_{y = c+d} \{ \min \{ \mu_A(c), \mu_B(d) \} \}
\]

so there exists a representation \( x = a + b, y = c + d \), where \( a, b, c, d \in M \) such that

\[
\alpha - \varepsilon < \min \{ \mu_A(a), \mu_B(b) \} \quad \text{and} \quad \alpha - \varepsilon < \min \{ \mu_A(c), \mu_B(d) \}
\]

\[
\Rightarrow \alpha - \varepsilon < \mu_A(a), \alpha - \varepsilon < \mu_B(b) \quad \text{and} \quad \alpha - \varepsilon < \mu_A(c), \alpha - \varepsilon < \mu_B(d)
\]

Thus, we get \( x + y = (a + b) + (c + d) = (a + c) + (b + d) \) such that

\[
\alpha - \varepsilon < \min \{ \mu_A(a + c), \mu_B(b + d) \}
\]

\[
\Rightarrow \alpha - \varepsilon < \sup_{x = (a+c)+(b+d)} \{ \min \{ \mu_A(a+c), \mu_B(b+d) \} \} = \mu_{A+B}(x + y)
\]

Since \( \varepsilon \) is arbitrary, it follows that \( \mu_{A+B}(x + y) \geq \alpha = \min \{ \mu_{A+B}(x), \mu_{A+B}(y) \} \).

Similarly, we can show that \( \nu_{A+B}(x + y) \leq \max \{ \nu_{A+B}(x), \nu_{A+B}(y) \} \).

Further, let \( \beta = \max \{ \mu_{A+B}(x), \mu_{A+B}(y) \} = \mu_{A+B}(x) \), and let \( \varepsilon > 0 \), then

\[
\beta - \varepsilon < \mu_{A+B}(x) = \sup_{x = a+b} \{ \min \{ \mu_A(a), \mu_B(b) \} \}, \text{so there exists a representation}
\]

\[
x = a + b \text{ such that } \beta - \varepsilon < \min \{ \mu_A(a), \mu_B(b) \} \Rightarrow \beta - \varepsilon < \mu_A(a), \beta - \varepsilon < \mu_B(b)
\]
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\[\Rightarrow \beta - \varepsilon < \mu_\alpha(a) \leq \mu_\alpha(ra), \quad \beta - \varepsilon < \mu_\beta(b) \leq \mu_\beta(tb), \quad \text{for any } r \in \mathbb{R}\]

\[\Rightarrow \beta - \varepsilon < \min\{\mu_\alpha(ra), \mu_\beta(tb)\}, \quad \text{for any } r \in \mathbb{R}\]

Now, \(rx = r(a + b) = ra + rb\) so that \(\beta - \varepsilon < \min\{\mu_\alpha(ra), \mu_\beta(rb)\}\)

\[\Rightarrow \beta - \varepsilon < \sup_{rx = r(a+b)} \left\{\min\{\mu_\alpha(ra), \mu_\beta(rb)\}\right\} = \mu_{A+B}(rx)\]

Since \(\varepsilon\) is arbitrary, it follows that

\[\mu_{A+B}(rx) \geq \beta = \mu_{A+B}(x).\]

Similarly, we can show that \(\nu_{A+B}(rx) \leq \nu_{A+B}(x)\)

Moreover, it is easy to check that \(\mu_{A+B}(0) = 0\) and \(\nu_{A+B}(0) = 0\).

Hence \(A + B\) is intuitionistic fuzzy submodule of \(M\).

**Definition (2.6) [4]** Let \(A = (\mu_A, \nu_A)\) and \(B = (\mu_B, \nu_B)\) be two intuitionistic fuzzy submodules of an \(R\)-module \(M\), then their product \(AB = (\mu_{AB}, \nu_{AB})\) is defined as

\[\mu_{AB}(x) = \sup_{x = \sum_{i \in A} a_i b_i} \left\{\min\{\mu_A(a_i), \mu_B(b_i)\}\right\}\]

and

\[\nu_{AB}(x) = \inf_{x = \sum_{i \in A} a_i b_i} \left\{\max\{\nu_A(a_i), \nu_B(b_i)\}\right\}, \quad \text{for all } x \in M\]

**Theorem (2.7)** Let \(A\) and \(B\) be two intuitionistic fuzzy submodules of an \(R\)-module \(M\). Then the product \(AB\) of \(A\) and \(B\) is also intuitionistic fuzzy submodule of \(M\).

**Proof.** Let \(x, y \in M\) be any two elements and let \(\min\{\mu_{AB}(x), \mu_{AB}(y)\} = \alpha\) (say)

Let \(\varepsilon > 0\) be given, then

\[\alpha - \varepsilon < \mu_{AB}(x) = \sup_{x = \sum_{i \in A} a_i b_i} \left\{\min\{\mu_A(a_i), \mu_B(b_i)\}\right\}\]

and

\[\alpha - \varepsilon < \mu_{AB}(y) = \sup_{y = \sum_{i \in B} a_i b_i} \left\{\min\{\mu_A(p_i), \mu_B(q_i)\}\right\}\]

Thus,

\[\Rightarrow \alpha - \varepsilon < \min\{\mu_A(a_i), \mu_B(b_i)\}\]

\[\Rightarrow \alpha - \varepsilon < \min\{\mu_A(a_i), \mu_B(b_i)\}, \quad \text{for all } i,\]

\[\Rightarrow \alpha - \varepsilon < \mu_A(a_i), \quad \alpha - \varepsilon < \mu_B(b_i), \quad \text{for all } i,\]

\[\Rightarrow \alpha - \varepsilon < \mu_A(a_i) + \mu_B(b_i)\]

Thus, we get \(x + y = \sum (a_i b_i + p_i q_i)\), where \(a_i, b_i, p_i, q_i \in M\) such that

\[\alpha - \varepsilon < \min\{\mu_A(a_i + p_i), \mu_B(b_i + q_i)\}, \quad \text{for all } i, \quad \Rightarrow \alpha - \varepsilon < \min\{\mu_A(a_i + p_i), \mu_B(b_i + q_i)\}\]

\[\Rightarrow \alpha - \varepsilon < \sup_{x+y = \sum (a_i b_i + p_i q_i)} \left\{\min\{\mu_A(a_i + p_i), \mu_B(b_i + q_i)\}\right\} = \mu_{AB}(x + y)\]

Since \(\varepsilon > 0\) is arbitrary, so we have \(\mu_{AB}(x + y) \geq \alpha = \min\{\mu_{AB}(x), \mu_{AB}(y)\}\)
Similarly, we can show that \( V_{AB}(x+y) \leq \max\{V_{AB}(x), V_{AB}(y)\} \)

Further, let \( \beta = \max\{\mu_{AB}(x), \mu_{AB}(y)\} = \mu_{AB}(x) \), and let \( \varepsilon > 0 \), then

\[
\beta - \varepsilon < \mu_{AB}(x) = \sup_{x = \sum_{i} a_{ib_{i}}} \min\{\min\{\mu_{A}(a_{i}), \mu_{B}(b_{i})\}\} \quad \text{so there exists a representation}
\]

\[
x = \sum_{i \in \mathbb{R}} a_{ib_{i}} \quad \text{such that} \quad \beta - \varepsilon < \min\{\min\{\mu_{A}(a_{i}), \mu_{B}(b_{i})\}\}
\]

\[
\Rightarrow \beta - \varepsilon < \min\{\min\{\mu_{A}(r_{a_{i}}), \mu_{B}(b_{i})\}\} \quad \text{for all } i.
\]

Hence \( \beta - \varepsilon < \min\{\min\{\mu_{A}(r_{a_{i}}), \mu_{B}(b_{i})\}\} < \sup_{x = \sum_{i} (r_{a_{i}}, b_{i})} \min\{\min\{\mu_{A}(r_{a_{i}}), \mu_{B}(b_{i})\}\} = \mu_{AB}(rx) \)

As \( \varepsilon > 0 \) is arbitrary, so we have \( \mu_{AB}(rx) \geq \beta = \mu_{AB}(x) \)

Similarly, we can show that \( V_{AB}(rx) \leq V_{AB}(x) \)

Also, it can be easily checked that \( \mu_{AB}(0) = 1 \) and \( V_{AB}(0) = 0 \)

Hence \( AB \) is intuitionistic fuzzy submodule of \( M \).

3. \((\alpha, \beta)\) – Cut of Intuitionistic fuzzy set

Definition (3.1)[6] Let \( A \) be intuitionistic fuzzy set of a universe set \( X \). Then \((\alpha, \beta)\)-cut of \( A \) is a crisp subset \( C_{\alpha, \beta}(A) \) of the IFS \( A \) is given by

\[
C_{\alpha, \beta}(A) = \{ x : x \in X \text{ such that } \mu_{A}(x) \geq \alpha, \upsilon_{A}(x) \leq \beta \},
\]

where \( \alpha, \beta \in [0,1] \) with \( \alpha + \beta \leq 1 \).

Proposition (3.2)[6] If \( A \) and \( B \) be two IFS’s of a universe set \( X \), then following holds

(i) \( C_{\alpha, \beta}(A) \subseteq C_{\delta, \theta}(A) \) if \( \alpha \geq \delta \) and \( \beta \leq \theta \)

(ii) \( C_{1, \beta}(A) \subseteq C_{\alpha, \beta}(A) \subseteq C_{\alpha - 1, \beta}(A) \)

(iii) \( A \subseteq B \) implies \( C_{\alpha, \beta}(A) \subseteq C_{\alpha, \beta}(B) \)

(iv) \( C_{\alpha, \beta}(A \cap B) = C_{\alpha, \beta}(A) \cap C_{\alpha, \beta}(B) \)

(v) \( C_{\alpha, \beta}(A \cup B) \supseteq C_{\alpha, \beta}(A) \cup C_{\alpha, \beta}(B) \) equality hold if \( \alpha + \beta = 1 \)

(vi) \( C_{\alpha, \beta}(A_{1}) = \cap C_{\alpha, \beta}(A_{1}) \)

(vii) \( C_{0, 1}(A) = X \).

Theorem (3.3)[7] If \( A = (\mu_{A}, \upsilon_{A}) \) be IFS of an \( R \)-module \( M \), then \( A \) is a IFSM of \( M \) if and only if \( C_{\alpha, \beta}(A) \) is submodule of \( M \), for all \( \alpha, \beta \in [0,1] \) with \( \alpha + \beta \leq 1 \), where \( \mu_{A}(0) \geq \alpha \), \( \upsilon_{A}(0) \leq \beta \).

Theorem (3.4) Let \( A \) and \( B \) be two IFSM of an \( R \)-module \( M \), then

\[
C_{\alpha, \beta}(A + B) \subseteq C_{\alpha, \beta}(A) \cap C_{\alpha, \beta}(B) \quad \text{and the equality holds if } \alpha + \beta = 1
\]

Proof. Let \( x = y + z \in C_{\alpha, \beta}(A) + C_{\alpha, \beta}(B) \), where \( y \in C_{\alpha, \beta}(A) \) and \( z \in C_{\alpha, \beta}(B) \)

\[
\Rightarrow \mu_{A}(y) \geq \alpha, \upsilon_{A}(y) \leq \beta \quad \text{and} \quad \mu_{B}(z) \geq \alpha, \upsilon_{B}(z) \leq \beta
\]
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\[
\begin{align*}
\Rightarrow & \quad \min \{ \mu_A(y), \mu_B(z) \} \geq \alpha \quad \text{and} \quad \max \{ \nu_A(y), \nu_B(z) \} \leq \beta \\
\Rightarrow & \quad \sup_{x+y+z} \{ \min \{ \mu_A(y), \mu_B(z) \} \} \geq \alpha \quad \text{and} \quad \inf_{x+y+z} \{ \max \{ \nu_A(y), \nu_B(z) \} \} \leq \beta \\
\Rightarrow & \quad \mu_{A+B}(x) \geq \alpha \quad \text{and} \quad \nu_{A+B}(x) \leq \beta \quad \text{i.e.} \quad x \in C_{\alpha,\beta}(A + B)
\end{align*}
\]

Thus $C_{\alpha,\beta}(A) + C_{\alpha,\beta}(B) \subseteq C_{\alpha,\beta}(A + B)$. For the other part, let $\alpha + \beta = 1$ and $x \in C_{\alpha,\beta}(A + B)$. Then

\[
\begin{align*}
& \mu_{A+B}(x) \geq \alpha \quad \text{and} \quad \nu_{A+B}(x) \leq \beta \quad \text{i.e.} \quad x \in C_{\alpha,\beta}(A + B) \\
& \Rightarrow \quad \mu_{A+B}(x) \geq \alpha \quad \text{and} \quad \nu_{A+B}(x) \leq \beta \quad \text{i.e.} \quad \sup \{ \min \{ \mu_A(y), \mu_B(z) \} \} \geq \alpha \\
& \Rightarrow \quad \min \{ \mu_A(a), \mu_B(b) \} \geq \alpha \quad \text{for some} \quad x = a + b \\
& \Rightarrow \quad \mu_A(a) \geq \alpha \quad \text{and} \quad \mu_B(b) \geq \alpha \\
& \Rightarrow \quad \nu_A(a) \leq 1 - \mu_A(a) \leq 1 - \alpha = \beta \quad \text{and} \quad \nu_B(b) \leq 1 - \mu_B(b) \leq 1 - \alpha = \beta \\
& \Rightarrow \quad a \in C_{\alpha,\beta}(A) \quad \text{and} \quad b \in C_{\alpha,\beta}(B) \quad \text{and so} \quad x = a + b \in C_{\alpha,\beta}(A) + C_{\alpha,\beta}(B)
\end{align*}
\]

Hence $C_{\alpha,\beta}(A + B) = C_{\alpha,\beta}(A) + C_{\alpha,\beta}(B)$.

**Remark (3.5)** In the above theorem the equality does not hold as can be seen from the following example:

**Example (3.6)** Consider the $\mathbb{Z}$-module $\mathbb{Z}_6$, where $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ and define

\[
\begin{align*}
& \mu_A(0) = 1, \quad \mu_A(4) = 0.6 = \mu_A(2), \quad \mu_A(1) = \mu_A(3) = \mu_A(5) = 0 \\
& \nu_A(0) = 0, \quad \nu_A(4) = 0.3 = \nu_A(2), \quad \nu_A(1) = \nu_A(3) = \nu_A(5) = 1
\end{align*}
\]

and $\mu_B(0) = 1, \quad \mu_B(4) = \mu_B(2) = \mu_B(1) = \mu_B(5) = 0, \quad \mu_B(3) = 0.5 \\
\nu_B(0) = 0, \quad \nu_B(4) = \nu_B(2) = \nu_B(1) = \nu_B(5) = 1, \quad \nu_B(3) = 0.3$

Clearly, A and B are IFSM's of $\mathbb{Z}$-module $\mathbb{Z}_6$. Now $A + B = (\mu_{A+B}, \nu_{A+B})$, where

\[
\begin{align*}
& \mu_{A+B}(x) = \sup_{x=a+b} \{ \min \{ \mu_A(a), \mu_B(b) \} \} \quad \text{and} \quad \nu_{A+B}(x) = \inf_{x=a+b} \{ \max \{ \nu_A(a), \nu_B(b) \} \} \quad \text{for all} \quad x \in \mathbb{Z}_6
\end{align*}
\]

Therefore, we get
\[ \mu_{A \ast B}(0) = \text{Sup} \{ \min \{ \mu_\alpha(0), \mu_\beta(0) \}, \{ \min \{ \mu_\alpha(1), \mu_\beta(5) \}, \{ \min \{ \mu_\alpha(2), \mu_\beta(4) \}, \} \} \]
\[
= \text{Sup} \{ \min \{1,1\}, \min \{0,0\}, \min \{0.6,0\}, \min \{0.0,0.5\}, \min \{0.6,0\}, \min \{0,0\} \}
\]
\[
= \text{Sup} \{ 1, 0, 0, 0, 0, 0 \}
\]
\[
\mu_{A \ast B}(l) = \text{Sup} \{ \min \{ \mu_\alpha(0), \mu_\beta(l) \}, \{ \min \{ \mu_\alpha(1), \mu_\beta(0) \}, \{ \min \{ \mu_\alpha(2), \mu_\beta(5) \}, \} \} \}
\]
\[
= \text{Sup} \{ \min \{1,0\}, \min \{0.1\}, \min \{0.6,0\}, \min \{0,0\}, \min \{0.6,0.5\}, \min \{0,0\} \}
\]
\[
= \text{Sup} \{ 0, 0, 0, 0, 0, 0 \}
\]

Similarly, we can find \( \mu_{A \ast B}(2) = 0.6 \), \( \mu_{A \ast B}(3) = 0.5 \), \( \mu_{A \ast B}(4) = 0.6 \), \( \mu_{A \ast B}(5) = 0.5 \)

Also, \( \nu_{A \ast B}(0) = \text{Inf} \{ \max \{ \nu_\alpha(0), \nu_\beta(0) \}, \max \{ \nu_\alpha(1), \nu_\beta(5) \}, \max \{ \nu_\alpha(2), \nu_\beta(4) \}, \} \}
\[
= \text{Inf} \{ \max \{0,0\}, \max \{1,1\}, \max \{0.3,1\}, \max \{1,0.3\}, \max \{0.3,1\}, \max \{1,1\} \}
\]
\[
= \text{Inf} \{ 0, 1, 1, 1, 1, 1 \} = 0
\]

Similarly, we can find \( \nu_{A \ast B}(1) = 0.3 \), \( \nu_{A \ast B}(2) = 0.6 \), \( \nu_{A \ast B}(3) = 0.3 \), \( \nu_{A \ast B}(4) = 0.3 \), \( \nu_{A \ast B}(5) = 0.3 \)

Now, \( C_{0.5,0.3}(A+B) = \{0,1,3,4,5\} \). Also, \( C_{0.5,0.3}(A) + C_{0.5,0.3}(B) = \{0,4,2\} + \{1,3\} = \{1,3,5\} \)

Clearly, \( C_{0.5,0.3}(A) + C_{0.5,0.3}(B) \subseteq C_{0.5,0.3}(A+B) \).

**Theorem (3.7)** Let A and B be two IFSM of an R-module M, then

\[ C_{\alpha, \beta}(A)C_{\alpha, \beta}(B) \subseteq C_{\alpha, \beta}(AB) \]

and the equality holds if \( \alpha + \beta = 1 \)

**Proof.** Now \( AB = (\mu_{AB}, \nu_{AB}) \), where

\[ \mu_{AB}(x) = \text{Sup} \{ \min \{ \mu_\alpha(a_i), \mu_\beta(b_i) \} \} \quad \text{and} \quad \nu_{AB}(x) = \text{Inf} \{ \max \{ \nu_\alpha(a_i), \nu_\beta(b_i) \} \} \]

for all \( x \in M \)

Let \( x = \sum_{i \in I} a_i b_i \in C_{\alpha,\beta}(A)C_{\alpha,\beta}(B) \), where \( a_i \in C_{\alpha,\beta}(A) \) and \( b_i \in C_{\alpha,\beta}(B) \), for all \( i \).

\[ \Rightarrow \mu_\alpha(a_i) \geq \alpha, \nu_\alpha(a_i) \leq \beta \quad \text{and} \quad \mu_\beta(b_i) \geq \alpha, \nu_\beta(b_i) \leq \beta \], for all \( i \)

\[ \Rightarrow \min \{ \mu_\alpha(a_i), \mu_\beta(b_i) \} \geq \alpha, \max \{ \nu_\alpha(a_i), \nu_\beta(b_i) \} \leq \beta \], for all \( i \)

\[ \Rightarrow \min \{ \min \{ \mu_\alpha(a_i), \mu_\beta(b_i) \} \} \geq \alpha \quad \text{and} \quad \max \{ \max \{ \nu_\alpha(a_i), \nu_\beta(b_i) \} \} \leq \beta \]

\[ \Rightarrow \text{Sup} \{ \min \{ \mu_\alpha(a_i), \mu_\beta(b_i) \} \} \geq \alpha \quad \text{and} \quad \text{Inf} \{ \max \{ \nu_\alpha(a_i), \nu_\beta(b_i) \} \} \leq \beta \]

\[ \Rightarrow \mu_{AB}(x) \geq \alpha \quad \text{and} \quad \nu_{AB}(x) \leq \beta \quad \text{i.e.} \quad x \in C_{\alpha,\beta}(AB) \]

Thus, we get \( C_{\alpha,\beta}(A)C_{\alpha,\beta}(B) \subseteq C_{\alpha,\beta}(AB) \).

For the second part, let \( \alpha + \beta = 1 \) and \( x \in C_{\alpha,\beta}(AB) \). Then
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$$\mu_{AB}(x) \geq \alpha \quad \text{and} \quad \nu_{AB}(x) \leq \beta.$$  

Now, $$\mu_{AB}(x) \geq \alpha \Rightarrow \sup_{x=\sum_{i=1}^{n}a_i} \min \{\min \{\mu_A(a_i), \mu_B(b_i)\}\} \geq \alpha$$

$$\Rightarrow \min \{\min \{\mu_A(a_i), \mu_B(b_i)\}\} \geq \alpha \quad \text{for some} \quad x = \sum_{i=1}^{n}a_i b_i$$

$$\Rightarrow \min \{\mu_A(a_i), \mu_B(b_i)\} \geq \alpha \quad \text{for all} \quad i \Rightarrow \mu_A(a_i) \geq \alpha \quad \text{and} \quad \mu_B(b_i) \geq \alpha \quad \text{for all} \quad i$$

$$\Rightarrow \nu_A(a_i) \leq 1 - \mu_A(a_i) \leq 1 - \alpha = \beta \quad \text{and} \quad \nu_B(b_i) \leq 1 - \mu_B(b_i) \leq 1 - \alpha = \beta \quad \text{for all} \quad i$$

$$\Rightarrow a_i \in C_{a, b}(A) \quad \text{and} \quad b_i \in C_{a, b}(B), \quad \text{for all} \quad i$$

$$\Rightarrow x = \sum_{i=1}^{n}a_i b_i \in C_{a, b}(A)C_{a, b}(B)$$

Hence $$C_{a, b}(AB) \subseteq C_{a, b}(A)C_{a, b}(B)$$ and so the equality follows.

References