Int Jr. of Mathematical Sciences & Applications
Vol. 5, No. 2, (July-December, 2015)

ISSN No: 2230-9888
www.journalshub.com

Local Asymptotic Stability For Non-Linear Functional Integro-Differential Equations

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Abstract

In this paper, we proved the local asymptotic stability condition for a certain non-linear functional integro-differential equations. The present study, using the characterizations of measures of non-compactness we prove the theorem on the existence and local asymptotic stability of solutions for a quadratic functional integral equation via a fixed point theorem of Darbo. The investigations are used in the Banach space of real functions defined continuous and bounded on an unbounded interval. An example is indicated to demonstrate the natural realizations of abstract result presented in the paper.

Keywords: *Non-linear functional integral equation, measure of non-compactness, fixed point theorem, attractive solution etc.*

Mathematics subject classifications: *primary: 47410; secondary: 34A60.*

1 Introduction

In this paper, we deal with the stability of non-linear quadratic functional integro-differential equation in Banach algebras and discuss the existence as well as existence result for extermal solution wide, Lipschitz, Caratheodory and monotonic conditions. The main tool used in our considerations is the technique of measure of non-compactness and the fixed point theorem of Darbo [1]. The integral equation in question and has rather general form and contains as particular cases a lot of functional equations and non-linear integral equations of Voltera type.

In this paper the measure of non-compactness used to obtain the existence of solutions of the mentioned function at integral equation but also to characterize the solutions in terms of uniform global asymptotic attractivity. The assumptions imposed on the non-linearities in our main existence theorem admit several natural realizations with are illustrated by an example. The result several ones obtained earlier in a lot of papers concerning asymptotic stability of solutions of some functional integral equations. (of [3, 4, 5, 8]).

2 Notations, Definitions and Auxiliary facts

At the beginning of this section, we present some basic facts concerning the measures of non-compactness $[1,2]$ in Banach spaces.

Assume that $(E,\|\cdot\|)$ is an infinite dimensional Banach space with zero element θ Denote by $\overline{B}r(x)$ the closed ball centered at *x* and with radius *r*. Thus $\overline{B}r(\theta)$ is the closed ball centered at origin of radius *r*. If *X* is a subset of *E* then the symbols $\overline{X}$, Conv *X* stand for the closure and closed convex hull of *X*, respectively. Moreover, we denote by $\rho_{bd}(E)$ the family of all nonempty and bounded subset of *E* and by $\rho_{rcp}(E)$ its subfamily consisting of all relatively compact subsets of *E*.

The following definition of a measure of non-compactness appears in Banas and Goebel $[1]$.

Definition 2.1 :- A mapping $\mu : \rho_{bd}(E) \to R_+ = [0,\infty)$ is said to be a measure of non-compactness in *E* if it satisfies the following conditions:

1° The family $ker\,\mu = \{X \in \rho_{bd}(E) : \mu(X) = 0\}$ is nonempty and $ker\,\mu \subset \rho_{rcp}(E)$

2° $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$

3° $\mu(\overline{X}) = \mu(X)$

4° $\mu(ConvX) = \mu(X)$

5° $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ $for\ \lambda \in [0,1]$

6° If (X_n) is a sequence of closed sets from $\rho_{bd}(E)$ such that $X_{n+1} \subset X_n\ (n = 1,2,......)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection set $X_\infty = \cap_{n=1}^{\infty} X_n$ is nonempty.

The family $ker\,\mu$ described in 1° is said to be the kernel of the measure of non-compactness μ . Observe that the intersection set X_∞ from 6° is a member of the family $ker\,\mu$. In fact, since $\mu(X_\infty) \le \mu(X_n)$ for any *n*, we infer that $\mu(X_\infty) = 0$. This yields that $X_\infty \in ker\,\mu$. This simple observation will be essential in our further investigations.

Now we state a fixed point theorem of Darbo type which will be used in the sequel (see Banas [2, page 17]).

Theorem 2.1 :- Let Ω be a nonempty bounded closed and convex subset of the Banach space *E* and let $F : \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0,1]$ such that $\mu(FX) \le k\mu(X)$ for any nonempty subset *X* of Ω . Then *F* has a fixed point in the set Ω .

Remark 2.1 :- Let us denote by $Fix(F)$ the set of all fixed points of the operator *F* which belong to Ω. It can be shown $[1]$ that the set $Fix(F)$ belongs to the family $ker\,\mu$.

Our further considerations will be placed in the Banach space $BC(R_+,R)$ consisting of all real functions $x=x(t)$ defined continuous and bounded on R_+. This space is equipped with the standard supremum norm

$$\|x\| = sup\{|x(t)| : t\in R_+\} \qquad (2.1)$$

for our purpose we will use the ball measure of non-compactness in $E=BC(R_+,R)$ defined by

$$\beta(A) = inf\left\{r>0 : A\subset \bigcup_{i=1}^{n}\beta(x_i,r)\ for\ x_i\in E\right\} \qquad (2.2)$$

for all bounded subsets *A* of $BC(R_+,R)$, where $\beta(x_i,r)=\{\, x\in X \mid \|x_i - x\| < r\}$.

The ball measure of non-compactness is also called Hausdorff measure of non-compactness since it has close connections with the Hausdorff metric in the Banach space *E*. We use a handy formula for ball or Hausdorff measure of non-compactness in $BC(R_+,R)$ discussed in Banas $[2]$. To derive this formula, let us fix a nonempty and bounded subset *X* of the space $BC(R_+,R)$ and a positive number *T*. For $x\in X$ and $\varepsilon\ge 0$ denote by $\omega^T(x,\varepsilon)$ the modulus of continuity of the function *x* on the interval $[0,T]$, i.e.

$$\omega^T(x,\varepsilon) = sup\{|x(t)-x(s)| : t,s\in[0,T], |t-s|\le\varepsilon\}$$

Next, let us put

$$\omega^T(X,\varepsilon) = sup\{\omega^T(x,\varepsilon) : x\in X\},$$

$$\omega_0^T(X) = \lim_{\varepsilon\to\infty} \omega^T(X,\varepsilon),$$

$$\omega_0(X) = \lim_{T\to\infty} \omega_0^T(X).$$

It is known that

$$\beta(A) = \frac{1}{2}\omega_0(A)$$

for any bounded subset *A* of $BC(R_+,R)$ (see Banas and Goebel [1] and the reference given therein).

Now, for a fixed number $t\in R_+$ let us denote

$$X(t) = \{x(t) : x\in X\}$$

and

$$\|X(t)\| = sup\{|x(t)| : x,y\in X\}$$

Finally, let us consider the function μ defined on the family $BC(R_+,R)$ by the formula

$$\mu(X)=\omega_0(X)+\limsup_{t\to\infty}\|X(t)\| \qquad (2.3)$$

It can be shown as in Banas $[2]$ that the function μ is a measure of non-compactness in the space $BC(R_+,R)$. The kernel $ker\,\mu$ of this measure consists nonempty and bounded subsets *X* of $BC(R_+,R)$ such that the function from *X* are locally equicontinous on R_+ and the thickness of the bundle formed by functions form *X* tends to zero at infinity. This particular characteristic of $ker\,\mu$ has been utilized in establishing the local attractivity of the solutions for quadratic integral equation.

In order to introduce further concepts used in the paper let us assume that Ω is a nonempty subset of the space $BC(R_+,R)$. Moreover, let *Q* be an operator defined on Ω with values in $BC(R_+,R)$.

Consider the operator equation of the form

$$x(t)=Q\,x(t),t\in R_+ \qquad (2.4)$$

Definition 2.2 :- We say that solutions of the equation (2.3) are locally attractive if there exists a ball $\overline{B}r(x_0)$ in the space $BC(R_+,R)$ such that for arbitrary solutions $x=x(t)$ and $y=y(t)$ of equation (2.3) belonging to $\overline{B}r(x_0)\cap\Omega$ we have that

$$\lim_{t\to\infty}\big(x(t)-y(t)\big)=0 \qquad (2.5)$$

In the case when the limit (2.4) is uniform with respect to the set $\overline{B}(x_0)\cap\Omega$ i.e. when for each $\varepsilon>0$ there exists $T>0$ such that

$$|x(t)-y(t)|\le\varepsilon \qquad (2.6)$$

for all $x,y\in\overline{B}(x_0)\cap\Omega$ being solutions of (2.4) and for $t\ge T$, we will say that solutions of equation (2.4) are uniformly locally attractive (or equivalently, that solutions of (2.4) are asymptotically stable).

Definition 2.3 :- A line $y=m_1\,t+m_2$, where m_1 and m_2 are real numbers, is called a attractor for the solution $x\in BC(R_+,R)$ to the equation (2.1) if $lim_{t\to\infty}\left[x(t)-(m_1\,t+m_2)\right]=0$. In this case the solution *x* to the equation (2.1) is also called to be asymptotic to the line $y=m_1\,t+m_2$ and the line is an asymptote for the solution *x* on R_+.

Now we introduce the following definition useful in the sequel.

Definition 2.4 :- The solutions of the equation (2.1) are said to be locally asymptotically attractive if there exists a $x_0\in BC(R_+,R)$ and an $r>0$ such that for any two solutions $x=x(t)$ and $y=y(t)$ of the equation (2.1) belonging to $\overline{B}_r(x_0)\cap\Omega$ the condition (2.3) is satisfied and there is a line which is a common attractor to them on R_+. In the case when condition (2.3) is satisfied uniformly with respect to the set $\overline{B}r(x_0)\cap\Omega$, that is if for every $\varepsilon>0$ there exists $T>0$ such that the inequality (2.4) is satisfied for $t>T$ and for all $x,y\in\overline{B}r(x_0)\cap\Omega$ being solution of (2.1) having a line as a common attractor we will say that solutions of the equation (2.1) are uniformly locally asymptotically attractive on R_+.

Remark 2.2 :- Not that two solutions *x* and *y* of the equation (2.1) existing on R_+ are called asymptotically attractive if the condition (2.3) is satisfied and there is a line as a common attractor on R_+. Therefore, locally asymptotically attractive solutions are asymptotically attractive, but the converse may not be true. Similarly uniformly locally asymptotically attractive solutions are asymptotically attractive, but the converse may not be true. A asymptotically attractive solution for the operator equation (2.1) existing on R_+ is also called asymptotically stable on R_+.

Let us mention that the concept of attractive of solutions was introduced in Hu and Yan $[9]$ and Banas and Rzepka $[3]$ while the concept of asymptotic attractively is introduced in Dhage [7,9,10]

3 Statement of the problem integral Equation

In this section we will discuss the following non-linear function integro differential equation (in short FBVP)

$$x(t)=h(t)+\left[f\left(t,x\left(\mu(t)\right)\right)\right]\int_0^{d(t)} g\left(\left(t,x\left(\delta(s)\right),x'\left(\eta(s)\right)\right)ds \qquad (3.1)$$

for all $t\in R_+$ where

$$h:R_+\to R,\ f:R_+\times R\to R \text{ and } g:R_+\times R_+\times R\to R.$$

By a solution of FBVP (3.1) we mean a functions $x\in AC'\left(R_+,R\right)$ that satisfies the equation (3.1), where $AC'\left(R_+,R\right)$ is the space of all continuous real valued functions on R_+.

(B_1) The function $\mu,\alpha,\eta:R_+\to R_+$ are continuous and $lim_{t\to\infty}\mu(t)=\infty$.

(B_2) The function $f:R_+\times R\to R$ is continuous and there exists a bounded function $f:R_+\to R$ with bound L such that

$$\left|f(t,x)-f(t,y)\right|\le \ell(t)\left|x-y\right|$$

for $t\in R_+$ and for $x,y\in R.$

(B_3) The function $f:R_+\to R_+$ defined by $F(t)=\left|f(t,0)\right|$ is bounded on R_+ with $F_0=sup_{t\ge 0}F(t)$.

(D_1) The function $q:R\to R$ continuous and $lim_{t\to\infty}q(t)=0$.

(D_2) The function $g:R_+\times R_+ R\to R$ is continuous and there exist continuous function $p,q:R_+\to R_+$ such that

$$\left|g(t,s,x)\right|\le p(t)q(s),$$

for $t\, s \in R_+$. We assume that

$$lim_{t\to\infty}\ p(t)\int_0^{\alpha(t)} q(s)\,ds = 0$$

The main result of this paper.

Theorem 3.1 :- Assume that the hypotheses (B_1) through (B_3) and D_1 and D_2 hold. Furthermore, if $LN_2 < 1$, where $N_2 = sup_{t\geq 0}\ p(t)\int_0^{\alpha(t)} q(s)\ ds$.

Proof :- Set $D = AC(R_+, R)$ consider the operator Q defined on the Banach space P by the formula

$$Q\,x(t) = h(t) + f\left[(t), x(\mu(t))\right]\int_0^{\alpha(t)} g\big(t, x(\sigma(s)), x'(\eta(s))\big)ds \qquad (3.2)$$

for $t \in R.$ From our assumptions, for any function $x \in P,\ Qx$ is a real valued continuous function on R_+. Since the function

$$v(t) = lim_{t\to\infty}\ p(t),\int_0^{\alpha(t)} q(s)\,ds \qquad (3.3)$$

Is continuous and from the hypothesis (H_2). The number $N_2 = sup_{t\geq 0}\, v(t)$ exists. Define a closed ball $\overline{B}r(0)$ in P centered at the origin O of the radius equal to $r = \dfrac{N_1 + N_2}{1 - LN_2}$, $LN_2 < 1$.

Let $x \in \overline{B}r(0)$ be arbitrarily fixed. Then by hypotheses $(A_1)-(B_2)$ and $(H_1),(H_2)$ we obtain.

$$\begin{aligned}
\left|Q\,x(t)\right| &\leq \left|h(t)\right| + \left|f\left(t, x(\delta(t))\right)\right|\left(\int_0^{\alpha(t)}\left|g\left(t, x(\sigma(s), x'\eta(s)\right)ds\right.\right) \\
&\leq \left|q(t)\right| + \left[\left|f\left(t, x(\sigma(t))\right) - f(t,0)\right| + \left|f(t,o)\right|\right]\left(p(t)\int_0^{\alpha(t)} q(s)\,ds\right) \\
&\leq q(t) + \left[l(t)\left|x(\sigma(t))\right| + F(t)\right]v(t) \\
&\leq \left|q(t)\right| + \left[L\left|x(\sigma(t)\right| + F_0\right]N_2 \\
&\leq N_1 + L\,N_2\left\|x\right\| + F_0\,N_2 \\
&= \frac{N_1 + F_0\,N_2}{1 - L\,N_2}
\end{aligned}$$

for all $t \in R_+$. Taking the supremum over t, we obtain the result.

$$\left\|Q\,x\right\| \leq \frac{N_1 + F_0\,N_2}{1 - LN_2} \qquad (3.4)$$

for all $x \in \overline{B}r(0)$.

Now Q defines a mapping $Q:\overline{B}r(0)\to\overline{B}r(0)$. To prove the operator Q is continuous on the ball $\overline{B}r(0)$. Let us fix arbitrarily $\in>0$ and take $x,y\in\overline{B}r(0)$ such that $\|x-y\|\le\varepsilon.$ Then by hypotheses $(A_1)-(A_2)$ and $(B_1)-(B_2)$ we get:

$$|Qx(t)-Qy(t)|\le\left|f\left(t,x(\mu(t)),\int_0^{\alpha(t)}g\left(t,x(\sigma(s)),x'(\eta(s))\right)ds\right.\right.$$

$$\left.-f\left(t,y(\mu(t)),\int_0^{\alpha(t)}g\left(t,x(\sigma(s)),x'(\eta(s))\right)ds\right|$$

$$\le\left|f\left(t,x(\mu(t)),\int_0^{\alpha(t)}g\left(t,x(\sigma(s)),x'(\eta(s))\right)ds\right.\right.$$

$$\left.-f\left(t,y(\mu(t)),\int_0^{\alpha(t)}g\left(t,x(\sigma(s)),x'(\eta(s))\right)ds\right|$$

$$+\left|f\left(t,y(\alpha(t)),\int_0^{\alpha(t)}g\left(t,x(\sigma(s)),x'(\eta(s))\right)ds\right|$$

$$\le\left|f\left(t,x(\mu(t))\right)-f\left(t,y(\mu(t))\right)\right|$$

$$\int_0^{\alpha(t)}g\left(t,x(\sigma(s)),x'(\eta(s))\right)ds$$

$$-f\left(t,y(\mu(t)),\int_0^{\alpha(t)}g\left(t,x(\sigma(s)),x'(\eta(s))\right)ds\right.$$

$$+\left[\left|f\left(t,y(\mu(t))\right)-f(t,0)\right|+\left|f(t,0)\right|\right]$$

$$\int_0^{\alpha(r)}\left[\left|g\left(t,x(\sigma(s)),x'(\eta(s))\right)\right|+\left|g\left(t,x(\sigma(s)),x'(\eta(s))\right)ds\right)\right.$$

$$\le\ell(t)\left[\left|x(\mu(t))-y(\mu(t))\right|\right]\left(p(t)\int_0^{\alpha(t)}q(s)ds\right)$$

$$+2\left[\ell(t)\left|y(\mu(t))+F_0\right|\right]\left(p(t)\int_0^{\alpha(t)}q(s)ds\right)$$

$$\le L\|x-y\|v(t)+2L\|y\|v(t)$$

$$\le LN_2\in+2Lrv(t)$$

Hence from hypothesis (D_2), there exists $T>0$ such that $v(t)=\dfrac{\varepsilon}{2Lr}$ for $t\ge T$. Thus for $t\ge T$ from the estimate (3.3) we get

$$|Qx(t)-Qy(t)|\le(LN_2+1)\varepsilon \tag{3.5}$$

Further, let us assume that $t\in[0,T]$. Then evaluating similarly as above we get:

$$\left|Q\,x(t)-Q\,y(t)\right|\le\left|f\left(t,x\left(\mu(t)\right)-f\left(t,y\left(\mu(t)\right)\right|\right|\int_0^{\alpha(t)}g\left(t,x\left(\sigma(s)\right),x'\left(\eta(s)\right)\right)ds$$

$$+\left[\left|f\left(t,y\left(\mu(t)\right)-f(t,0)\right|\right]\right|\left(\int_0^{\alpha(t)}\left|g\left(t,x\left(\sigma(s)\right),x'\left(\eta(s)\right)\right)-g\left(t,y\left(\sigma(s)\right),y\left(\eta(s)\right)\right)\right|ds\right)$$
$$\le LK_2\left|x\left(\mu(t)\right)-y\left(\mu(t)\right)\right|$$
$$+\left(Lr+F_0\right)\left(\int_0^{\alpha_T}\left[\left|g\left(t,x\left(\sigma(s)\right),x'\left(\eta(s)\right)\right)-g\left(t,y\left(\sigma(s)\right),y'\left(\eta(s)\right)\right)\right|\right]ds\right)$$
$$\le LK_2\in+\alpha_T\,\omega_r^T\left(g,\varepsilon\right) \tag{3.6}$$

Where

$$\alpha_T=sup\left\{\alpha(t),t\in[0,T]\right\}$$

and

$$\omega_r^T\left(g,\in\right)=sup\left\{\left|g\left(t,x,x'\right)g\left(t,y,y'\right)\right|:\right.$$
$$\left.t,s\in[0,T],\le\in[0,\alpha_T],x,y\in[-r,r],\left|x-y\right|\le t\right\} \tag{3.7}$$

On the view of continuity of α we have that $\alpha_T<\infty$. Moreover, from the uniform continuity of the function $g(t,x,x')$ on the set $[0,T]\times[0,\alpha_T]\times[-r,r]$ we derive that $\omega_r^T(g,\in)\to 0$ as $\in\to 0$. Now using the inequalities (3.5), (3.6) and the above established facts we conclude that the operator θ maps continuously the ball $\overline{B}r(0)$ into itself.

Further for non-empty subset X of the ball $\overline{B}r(0)$. Let for arbitrarily $T>0$ and $\in>0$. Let us choose $x\in X$ and $t_1,t_2\in[0,T]$ with $\left|t_2-t_1\right|\le\varepsilon$. Without loss of generality we may assume that $t_1<t_2$. Then, taking into account our assumptions, we get

$$\left|Q\,x(t_1)-Q\,x(t_2)\right|\le\left|h(t_1)-h(t_2)\right|$$
$$+\left|f\left(t_1,x\left(\mu(t_1)\right)\right)\int_0^{\alpha(t_1)}g\left(t_1,x\left(\sigma(s)\right),x'\left(\eta(s)\right)\right)ds\right.$$
$$\left.-\left|f\left(t_2,x\left(\mu(t_2)\right)\right)\int_0^{\alpha(t_2)}g\left(t_2,x\left(\sigma(s)\right),x'\left(\eta(s)\right)\right)ds\right|\right.$$
$$\le\left|q(t_1)-q(t_2)\right|+\left|f\left(t_1,x\left(\mu(t_1)\right)\right)\int_0^{\alpha(t_1)}g\left(t_1,x\left(\sigma(s)\right),x'\left(\eta(s)\right)\right)ds-f\left(t_2,x\left(\mu(t_2)\right)\right).\right.$$

$$\left.\int_0^{\alpha(t_1)}g\left(t_1,x\left(\sigma(s)\right),x'\left(\eta(s)\right)\right)ds\right|+\left|f\left(t_2,x\left(\mu(t_2)\right)\right)\int_0^{\alpha(t_1)}g\left(t_1,x\left(\sigma(s)\right),x'\left(\eta(s)\right)\right)ds-\right.$$

$$\left.f\left(t_2,x\left(\mu(t_2)\right)\right)\int_0^{\alpha(t_2)}g\left(t_2,x\left(\sigma(s)\right),x'\left(\eta(s)\right)\right)ds\right|-f\left(t_2,x\left(\mu(t_2)\right)\right)$$

$$\leq \left|h(t_1)-h(t_2)\right|+\left|f\left(t_1,x(\mu(t_1))-f\left(t_2,x(\mu(t_2))\right)\right|$$

$$\int_0^{\alpha(t_1)}\left|g\left(t_1,x(\sigma(s)),x'(\eta(s))\right)ds+\left|f\left(t_2,x(\mu(t_2))\right)\right|\right.$$

$$\left.\int_0^{\alpha(t_1)}g\left(t_1,x(\sigma(s)),x'(\eta(s))\right)ds,\int_0^{\alpha(t_2)}g\left(t_2,x(\sigma(s)),x'(\eta(s))\right)ds\right|$$

$$\leq\left|h(t_1)-h(t_2)\right|+\left|f\left(t_1,x(\mu(t_1))-f\left(t_2,x(\mu(t_1))\right)\right|v(t_1)\right.$$

$$+\left|f\left(t_2,x(\mu(t_1))\right)-f\left(t_2,x(\mu(t_2))\right)\right|v(t_1)$$

$$+\left|f\left(t_2,x(\mu(t_1))\right)-f\left(t_2-x(\mu(t_2))\right)\right|v(t_1)$$

$$+\left(Lr+F_0\right|\int_0^{\alpha(t_1)}g\left(t_1,x(\sigma(s)),x'(\eta(s))\right)ds-\int_0^{\alpha(t_2)}g\left(t_2,x(\sigma(s)),x'(\eta(s))\right)ds\Big|$$

$$\leq\left|h(t_1)-h(t_2)\right|+N_2\left|f\left(t_1,x(\mu(t_1))\right)-f\left(t_2,x(\mu(t_2))\right)\right|+LN_2\left|x(\mu(t_1))-x(\mu(t_2))\right|$$

$$+\left(Lr+F_0\right)\left|\int_0^{\alpha(t_1)}g\left(t_1,x(\sigma(s)),x'(\eta(s))\right)ds-\int_0^{\alpha(t_2)}g\left(t_2,x(\sigma(s)),x'(\eta(s))\right)ds\right|\Big| \quad (3.8)$$

Again

$$\left|\int_0^{\alpha(t_1)}g\left(t_1,x(\sigma(s)),x'(\eta(s))\right)ds-\int_0^{\alpha(t_2)}g\left(t_2,x(\sigma(s)),x'(\eta(s))\right)ds\right|$$

$$\leq\left|\int_0^{\alpha(t_1)}g\left(t_1,x(\sigma(s)),x'(\eta(s))\right)ds-\int_0^{\alpha(t_1)}g\left(t_2,x(\sigma(s)),x'(\eta(s))\right)ds\right|$$

$$+\left|\int_0^{\alpha(t_1)}g\left(t_2,x(\sigma(s)),x'(\eta(s))\right)ds-\int_0^{\alpha(t_2)}g\left(t_2,x(\sigma(s)),x'(\eta(s))\right)ds\right|$$

$$\leq\int_0^{\alpha(t_1)}\left|g\left(t_1,x(\sigma(s)),x'(\eta(s))\right)-g\left(t_2,x(\sigma(s)),x'(\eta(s))\right)\right|ds$$

$$+\int_{\alpha(t_2)}^{\alpha(t_1)}\left|g\left(t_2,x(\sigma(s)),x'(\eta(s))\right)\right|ds$$

$$\leq\int_0^{\alpha_T}\left|g\left(t_1,x(\sigma(s)),x'(\eta(s))\right)-g\left(t_2,x(\sigma(s)),x'(\eta(s))\right)\right|ds+\left|v(t_1)-v(t_2)\right| \quad (3.9)$$

Now combining (3.8) and (3.9) we get,

$$\left|Qx(t_2)-Qx(t_1)\right|\leq\left|h(t_1)-h(t_2)\right|+N_2\left|f\left(t,x(\mu(t_1))-f\left(t_2,x(\mu(t_2))\right)\right)\right|$$

$$+LM_2\left|x(\mu(t_1))-x(\mu(t_2))\right|v(t_1)$$

$$+\left(Lr+F_0\right)\int_0^{\alpha_T}\left|g\left(t_1,x\left(\sigma(s)\right),x'\left(\eta(s)\right)\right)-g\left(t_2,x\left(\sigma(s)\right),x'\left(\eta(s)\right)\right)\right|ds$$

$$+\left(Lr+F_0\right)\left|v(t_1)-v(t_2)\right|$$

$$\leq\omega^T\left(h,\in\right)+LN_2\omega^T\left(x,\omega^T\left(\mu(\in)\right)\right)+N_2\,\omega_r^T\left(f,\in\right)$$

$$+\left(Lr+F_0\right)\int_0^{\alpha_T}\omega_r^T\left(g,\in\right)ds+\left(Lr+F_0\right)\omega^T\left(v,\in\right) \qquad (3.10)$$

Where we have denoted

$$\omega^T\left(h,\in\right)=sup\left\{\left|h(t_2)-h(t_1)\right|:t_1,t_2\in[0,T],\left|t_2-t_1\right|<\varepsilon\right\},$$

$$\omega^T\left(\mu,\in\right)=sup\left\{\left|\mu(t_2)-\mu(t_1)\right|:t_1,t_2\in[0,T],\left|t_2-t_1\right|\leq\varepsilon\right\},$$

$$\omega^T\left(v,\in\right)=sup\left\{\left|v(t_2)-v(t_1)\right|:t_1,t_2\in[0,T],\left|t_2-t_1\right|\leq\varepsilon\right\},$$

$$\omega_r^T\left(f,\in\right)=sup\left\{\left|f(t_2,x)-f(t_1,x)\right|:t_1,t_2\in[0,T],\left|t_2-t_1\right|\leq\varepsilon,x\in[-r,r]\right\},$$

$$\omega_r^T\left(g,\in\right)=sup\left\{\left|g\left(t_2,x\left(\sigma(s)\right),x'\left(\eta(s)\right)\right)-g\left(t_1,x\left(\sigma(s)\right),x'\left(\eta(s)\right)\right)\right|:t_1,t_2\in[0,T],\right.$$
$$\left.\left|t_2-t_1\right|<\varepsilon,s\in[0,\alpha_T],x\in[-r,r]\right\}$$

From the above estimate we derive the following inequality

$$\omega^T\left(Q(X),\in\right)\leq\omega^T\left(h,\in\right)+LN_2\,\omega^T\left(X,\omega^T\left(\mu,\in\right)\right)$$
$$+N_2\,\omega_r^T\left(f,\in\right)+\left(Lr+F_0\right)$$
$$\int_0^{\alpha_T}\omega_r^T\left(g,\in\right)ds+\left(Lr+F_0\right)\omega^T\left(v,\in\right) \qquad (3.11)$$

Observe that $\omega_r^T\left(f,\in\right)\to 0$ and $\omega_r^T\left(g,\in\right)\to 0$ as $\in\to 0$, which is a simple consequence of the uniform continuity of the function $f\,\&\,g$ on the sets $[0,T]\times[-r,r]$ and $[0,T]\times[0,\alpha,T]\times[-r,r]$, respectively. Moreover, from the uniform continuity of h,μ,v on $[0,T]$, it follows that $\omega^T\left(h,\in\right)\to 0$, $\omega^T\left(\mu,\in\right)\to 0,$ $\omega^T\left(v,\in\right)\to 0$ as $\in\to 0$. Thus using the established facts with the estimate, (3.10) we get

$$\omega_r^T\left(Q(X)\right)\leq L\,N_2\,\omega_0^T\left(X\right).$$

Consequently are obtain

$$\omega_0\left(Q(X)\right)\leq L\,N_2\,\omega_0\left(X\right) \qquad (3.12)$$

Similarly for any $x\in X$, one has

$$|Qx(t)| \le |h(t)| + \left[f\left(t,x\left(\mu(t)\right)\right)\right] \left(\int_0^{\alpha(t)} \left| g\left(t,x\left(\sigma(s)\right),x'\left(\eta(s)\right)\right)\right| ds \right)$$

$$\le |h(t)| + \left[\left| f\left(t,x\left(\mu(t)\right) - f(t,0)\right| + \left| f(t,0)\right| \right]$$

$$\left(p(t) \int_0^{\alpha(t)} q(s)\, ds \right)$$

$$\le |h(t)| + \left[\ell(t) \left| x\left(\mu(S)\right| + F_0 \right] v(t)$$

$$\le |h(t)| + L\,N_2 \left| x\left(\mu(t)\right)\right| + F_0(t)$$

for all $t \in R_+$.

Therefore , from the above inequality, it follows that

$$\left\| Q\,X(t)\right\| \le |h(t)| + LN_2 \left\| X\left(\mu(t)\right)\right\| + F_0 v(t)$$

Supremumovert,

$$lim_{t\to\infty}\, sup \left\| Q\,X(t)\right\| \le lim_{t\to\infty}\, sup\, h(t) + LN_2\, lim_{t\to\infty}\, sup \left\| X\left(\mu(t)\right)\right\| + F_0\, lim_{t\to\infty}\, sup\, v(t)$$

$$\le LN_2\, lim_{t\to\infty}\, sup \left\| X\left(\mu(t)\right)\right\|$$

$$\le LK_2\, lim_{t\to\infty}\, sup \left\| X(t)\right\|$$

Hence

$$\gamma(Q\,X) = \omega_0(Q\,X) + lim_{t\to\infty}\, sup \left\| Q\left(\lambda(t)\right)\right\|$$

$$\le LN_2\, \omega_0(X) + LN_2\, lim_{t\to\infty}\, sup \left\| X(t)\right\|$$

$$\le LN_2 \left[\omega_0(X)\, t\, lim_{t\to\infty}\, sup \left\| X(t)\right\| \right]$$

$$\le LN_2\, \gamma(X)$$

Where $LN_2 < 1$. This shows that Q is a set-contraction on $\overline{B}r(0)$ with the contraction constant $K = LN_2$.

We apply theorem (2.1) to the operator $Q\,x = x$ and deduce that the operator *Q* has a fixed point x in the ball $\overline{B}r(0)$. As x is a solution of the functional integro-differential equation (3.1). Moreover, taking into account that the image of the space *X* under the operator *Q* is contained in the ball $\overline{B}r(0)$ we infer that the set $Fix(Q)$ of all fixed points of *Q* is contained in $\overline{B}r(0)$. Obviously the set *Fix(Q)* contains all solutions of the equation (3.1). On the other hand, from Remark 2.3 we conclude that the set $Fix(Q)$ belongs to the family $ker\,\gamma$. Now taking into account of the description of sets belonging to $ker\,\gamma$ (given in section 2) we deduce that all solutions of the equation (3.1) are uniformly locally asymptotically stable on R_+ and the common attractor is the line $x(t) = 0$. This completes the proof.

4 An Example

Consider the non-linear $QFIE$ of the form

$$x(t) = \frac{1}{t+2} + \left[\frac{1}{2}\cos x(2t)\right]\left(\int_0^{t/4} \frac{e^{-t}x(s^2)}{1+\left|x(s^2)\right|}\,ds\right) \qquad (4.1)$$

for all $t \in R$. Comparing $QFIE$ (4.1) with (3.1), we obtain

$$\mu(t) = 2t,\ \alpha(t) = \frac{t}{4},\ h(t) = \frac{t}{t+1},\ \sigma(t) = t^2$$

for all $t \in R_+$ and

$$f(t,x) = \frac{1}{2}\cos x \text{ and } g(t,x,x') = \frac{e^{-t}x}{1+|x|}$$

for all $t,s \in R_+$ & $x \in R$ we shall show that all the above function satisfy the condition of theorem 3.1.

Clearly, the function α, μ, η are continuous and map the half real line R_+ into it self with $lim_{t\to\infty}\alpha(t) = lim_{t\to\infty}\frac{t}{4} = \infty$. Next *h* is continuous and

$$\lim_{t\to\infty} h(t) = 0 \ \ and \ \ N_1 = \sup_{t\geq 0}\left|h(t)\right| = \sup_{t\geq 0}\frac{1}{t+1} = 1$$

Further on, the function f is continuous on $R_+ + R$ and satisfies (A_2) with $L = \frac{1}{2}$. T see this, let $x, y \in R$ then

$$\left|f(t,x) - f(t,y)\right| \leq \frac{1}{2}\left|\cos x - \cos y\right| \leq \frac{1}{2}\left|x,y\right| \text{ for all } t \in R_+ .$$

Finally, the function g is continuous on $R_+ \times R_+ \times R$ and

$$g(t,x,x')\Big| = \left|\frac{xe^{-t}}{1+|x|}\right| \leq e^{-t} = p(t)\cdot q(s)$$

for all $t,s \in R_+$ and $x \in R$. Moreover,

$$\lim_{t\to\infty} p(t)\int_0^{\alpha(t)} b(s)\,ds = \lim_{t\to\infty} e^{-t}\int_0^{t/4} ds = 0 \ and \ N_2 = \sup_{t\geq 0} e^{-t}\int_0^{t/4} ds \leq 1$$

As $LK_2 = \frac{1}{4} < 1$. We apply theorem 3.1 to yield that the $QFIE$ (3.1) has a solution and all solution are uniformly locally asymptotically stable on R_+.

ACKNOWLEDGEMENTS

The authors are thankful to the referee abd B. C. Dhage for their valuable suggestions. The research of S. S. Bellale was supported by UGC, New Delhi (MRP-Major, R.N.41-750, 2012)

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