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Intercourses with Bi-Pseudo-Integrals based on modified functions and measures by $\bar{g}-Transform$

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Abstract

A wide class of some elementary modified functions ($f_{\bar{g}}$) is investegated in treating, solving of pseudo-linear problems by showing the important role of sistem of pseudo-arithmetical operations $\{\overline{\oplus},\overline{\odot},\overline{\ominus},\overline{\oslash}\}$ and the extension of axiomatic concepts of pseudo-operations. The notion of modified pseudo-additive measure $\left(\overline{\oplus}-m_{\bar{g}}\right)$ by $\bar{g}-transform$ is followed and completed with the meaning of bi-pseudo-integral of modified function ($f_{\bar{g}}$) with respect to a $m_{\bar{g}}$. More links between different types of the bi-pseudo-integrals can be noted.
Key Words: *$\bar{g}-calculus$, $\bar{g}-Transform$, $\bar{g}-function$, modifed measure $\left(\overline{\oplus}-m_{\bar{g}}\right)$, bi- pseudo-integral*

1. Introduction

The concept of *pseudo-arithmetical operations* $\{\oplus,\odot\ominus,\oslash\}$ first was introduced on $[0,+\infty]$ interval and then to the whole extended real line $[-\infty,+\infty]$ [2], [3], [4], [7], [8], [10], [15], [16]. Following Mesiar and Rybárik [10], the binary operations $(\oplus,\odot)$ (pseudo-addition, pseudo-multiplication) are respectively the binary function $\oplus: [0,+\infty]^2 \to [0,+\infty]$ and $\odot$: $[0,+\infty]^2 \to [0,+\infty]$ that [8], [16], [18] fulfill the system of axioms ($\oplus$-A.1÷A.7) and ($\odot$-A.1÷ $A.7$). The system of the axioms ($\oplus,\odot$-A.1÷ $A.5$) was formulated by the system of the Axioms of Sugeno and Murofushi [5], [14]. Pseudo-arithmetical operations are useful tools in treating [2] of nonlinear problems and some elementary $\bar{g}-functions$ are derived as solutions of some functional equations using results of Aczél [1]. A further development of $g-calculus$ [3], [4] is obtained by applying these pseudo-arithmetical operations [2], [9] to $\bar{g}-functions$ and their $\bar{g}-derivatives$ [2], so $\bar{g}-functions$ are modified functions by $\bar{g}-transform$.

1.1. Concept of pseudo-arithmetical operations and their extension.

The pseudo-addition $\oplus$ be a binary operation on $[0,+\infty]$ with the properties (A1) ÷ (A7) which is either $\oplus = \vee$ (max) or the operation of the type $\oplus = \oplus_g$, (i.e., $\oplus = \oplus_g = \oplus_{g_{a,r}}$ is generated by any of the function $g = g_{a,r} - generator$, $g_{a,r}(x) = a \cdot x^r, a > 0$ and [1], [10], [18] defined by $x \oplus_{g_{a,r}} y = (x^r + y^r)^{1/r}$ for some $r \geq 1$).

The group of Axioms (A.1÷ $A.7$) show their common and important properties, such as associativity (A2) and continuity (A7). All operations $\oplus_{g_{a,r}}$- are Archimedean but the operation $\vee$ has not this property [15], [16], [18].

- The extensions $\overline{\oplus}_{\bar{g}_{a,r}}$ of the Archimedian $\oplus_{g_{a,r}}$ remain continuous and associative.
- $\oplus = \vee$ For the latter $\vee$ is considerated as it has no generator. The question is how $\vee$ could be extended on the interval $[-\infty, +\infty]$. It is considered that $\vee$ on the interval $[0, +\infty]$ is the limit of the operation $\oplus_{g_{a,r}}$, $r \geq 1$ so, it is suggested to extend $\vee$ on the interval $[-\infty, +\infty]$ in the same way as the limit of the extended operation $\overline{\oplus}_{\bar{g}_{a,r}}$. The extended operation $\overline{\vee}$ of the operation $\vee$ is neither associative nor continuous [14], [15], [16], [18].

Let a generator $\bar{g}: [-\infty, +\infty] \longrightarrow [-\infty, +\infty]$ be a continuous, monotone strictly increaing unbouded function of the pseudo-addition $\overline{\oplus}$ on the interval $[-\infty, +\infty]$ such that $\bar{g}(0) = = 0_{\oplus}$, $\bar{g}(1) = 1_{\odot}$, $\bar{g}(+\infty) = +\infty$, with the convention $0 \cdot (+\infty) = 0$ and some valued undefined (or an odd extension of a given generator g from $[0, +\infty]$ to $[-\infty, +\infty]$, briefly $\bar{g}(x) = sgn\, x \cdot g(|x|)$, $x \in [-\infty, \infty]$). Then, the sistem of pseudo-arithmetical operations $\{\overline{\oplus}, \overline{\odot}, \overline{\ominus}, \overline{\oslash}\} = \{\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}}, \overline{\ominus}_{\bar{g}}, \overline{\oslash}_{\bar{g}}\}$ generated by this function $\bar{g}$ is said to be a *consistent* sistem [8]. By these extension are taken also $\bar{\boldsymbol{g}} - calculus$ based on definition of $\boldsymbol{g} - calculus$ [3], [4], [6], [7], [12], [16], [18], [21]:

$$x \,\overline{\oplus}_{\bar{g}}\, y = x \,\overline{\oplus}_{\bar{g}_{a,r}}\, y = \bar{g}^{\,\mathbf{-1}}(\bar{g}(x) + \bar{g}(y)) = \bar{g}_{a,r}^{\;\mathbf{-1}}(\bar{g}_{a,r}(x) + \bar{g}_{a,r}(y))$$

$$x \,\overline{\odot}_{\bar{g}}\, y = x \,\overline{\odot}_{\bar{g}_{a,r}}\, y = \bar{g}^{\,\mathbf{-1}}(\bar{g}(x) \cdot \bar{g}(y)) = \bar{g}_{a,r}^{\;\mathbf{-1}}(\bar{g}_{a,r}(x) \cdot \bar{g}_{a,r}(y))$$

$$x \,\overline{\ominus}_{\bar{g}}\, y = x \,\overline{\ominus}_{\bar{g}_{a,r}}\, y = \bar{g}^{\,\mathbf{-1}}(\bar{g}(x) - \bar{g}(y)) = \bar{g}_{a,r}^{\;\mathbf{-1}}(\bar{g}_{a,r}(x) - \bar{g}_{a,r}(y))$$

$$x \,\overline{\oslash}_{\bar{g}}\, y = x \,\overline{\oslash}_{\bar{g}_{a,r}}\, y = \bar{g}^{\,\mathbf{-1}}(\bar{g}(x) / \bar{g}(y)) = \bar{g}_{a,r}^{\;\mathbf{-1}}(\bar{g}_{a,r}(x) / \bar{g}_{a,r}(y)).$$

So, for $\mathrm{x}, \mathrm{y} \in [-\infty, +\infty]$, let $\bar{\mathrm{g}}$ be a generator on $[-\infty, +\infty]$, with some valued undefined [10], [18], [21]:

$$\begin{pmatrix} -\infty \,\overline{\oplus}_{\bar{g}_{a,r}} - \infty \\ -\infty \,\overline{\oplus}_{\bar{g}_{a,r}} + \infty \\ +\infty \,\overline{\oplus}_{\bar{g}_{a,r}} + \infty \end{pmatrix}, \begin{pmatrix} 0 \,\overline{\odot}_{\bar{g}_{a,r}} - \infty \\ 0 \,\overline{\odot}_{\bar{g}_{a,r}} + \infty \\ 0 \,\overline{\odot}_{\bar{g}_{a,r}}\, 0 \end{pmatrix}, \begin{pmatrix} -\infty \overline{\ominus}_{\bar{g}_{a,r}} - \infty \\ +\infty \overline{\ominus}_{\bar{g}_{a,r}} + \infty \end{pmatrix}, \begin{pmatrix} -\infty \,\overline{\oslash}_{\bar{g}_{a,r}} - \infty & -\infty \,\overline{\oslash}_{\bar{g}_{a,r}}\, 0 & -\infty \,\overline{\oslash}_{\bar{g}_{a,r}} + \infty \\ -\infty \,\overline{\oslash}_{\bar{g}_{a,r}} + \infty & +\infty \,\overline{\oslash}_{\bar{g}_{a,r}}\, 0 & +\infty \,\overline{\oslash}_{\bar{g}_{a,r}} + \infty \end{pmatrix}.$$

we will select some most important generators $\bar{g}$ and $\bar{g} - calculus$ derived from them [4], [6], [10], [18], [21]:

$\{\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}},\overline{\ominus}_{\bar{g}},\overline{\oslash}_{\bar{g}}\}$	$\bar{g}_{1,1}(x)=x$	$\bar{g}_{a,1}(x)=a\cdot x$	$\bar{g}_{1,r}(x)=x^r$	$\bar{g}_{a,r}(x)=a\cdot x^r$
$x\,\overline{\oplus}_{\bar{g}_{a,r}}\,y$ $x\,\overline{\odot}_{\bar{g}_{a,r}}\,y$ $x\,\overline{\ominus}_{\bar{g}_{a,r}}\,y$ $x\,\overline{\oslash}_{\bar{g}_{a,r}}\,y$	$x+y$ $x\cdot y$ $x-y$ x/y	$x+y$ $\boldsymbol{a}\cdot(x\cdot y)$ $x-y$ $a^{-1}\cdot(x/y)$	$(x^r+y^r)^{1/r}$ $x\cdot y$ $(x^r-y^r)^{1/r}$ x/y	$(x^r+y^r)^{1/r}$ $\boldsymbol{a}^{1/r}\cdot(x\cdot y)$ $(x^r-y^r)^{1/r}$ $a^{-1/r}\cdot(x/y)$
Conditions $(\boldsymbol{a},\boldsymbol{r},\boldsymbol{x},\boldsymbol{y})$	$r=1,a=1$ $\overline{\oslash}_{\bar{g}_{1,1}};y\neq 0$	$r=1,a>0$ $\overline{\oslash}_{\bar{g}_{a,1}};y\neq 0$	$r\geq 1,a=1$ $\overline{\oslash}_{\bar{g}_{1,r}};y\neq 0$	$r\geq 1,a>0$ $\overline{\oslash}_{\bar{g}_{a,r}};y\neq 0$

If $a=1$, we have the follow form of the generator i.e., the *normed generator* $\bar{g}(x)=\bar{g}_{1,r}(x)=$ $=x^r$ and $\bar{g}(1)=\bar{g}_{1,r}(1)=1$. Also, easily we can control that ${\bar{g}_{a,r}}^{-1}(a)=1$, ${\bar{g}_{a,1}}^{-1}(a)=1$ and ${\bar{g}_{a,r}}^{-1}(1)=1$, ${\bar{g}_{a,1}}^{-1}(1)=1$ [8], [10], [16], [18], [21].
The pseudo-operator $\overline{\oplus}_{\bar{g}}$ that fulfill the system of axioms (A.1÷A.7), is defined as follow form:

$$\bar{g}(x)=\bar{\boldsymbol{g}}_{\boldsymbol{a},\boldsymbol{r}}(x)=\begin{cases} g(x)=\begin{cases} g_{a,r}(x), & if\ x\in[0,+\infty],\quad a>0,a\neq 1\\ g_{1,r}(x), & if\ x\in[0,+\infty]\ \ g=g_{1,r}-normed\end{cases}\\ -g(-x)=\begin{cases} g_{a,r}(x), & if\ x\in[0,+\infty],\quad a>0,a\neq 1\\ -g_{1,r}(-x), & if\ x\in[0,+\infty]\ \ g=g_{1,r}-normed\end{cases}\end{cases}$$

$$\overline{\oplus}=\begin{cases}\overline{\oplus}_{\vee}=\begin{cases}\vee & for\ x\in[0,+\infty]\\ \overline{\vee} & for\ x\in[-\infty,+\infty]\end{cases}\\ \overline{\oplus}_{\bar{g}}=\overline{\oplus}_{\bar{g}_{a,r}}=\begin{cases}\begin{cases}\oplus_{g_{a,r}} & for\ x\in[0,+\infty],\quad a>0,a\neq 1\\ \oplus_{g_{1,r}} & for\ x\in[0,+\infty],\ g=g_{1,r}-normed\end{cases}\\ \begin{cases}\overline{\oplus}_{\bar{g}_{a,r}} & for\quad x\in[-\infty,+\infty],a>0,a\neq 1\\ \overline{\oplus}_{\bar{g}_{1,r}} & for\ x\in[-\infty,+\infty],\bar{g}=\bar{g}_{1,r}-normed\end{cases}\end{cases}\end{cases}$$

- $\{\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}},\overline{\ominus}_{\bar{g}},\overline{\oslash}_{\bar{g}}\}=\{\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}},\overline{\ominus}_{\bar{g}_{a,r}},\overline{\oslash}_{\bar{g}_{a,r}}\}\xrightarrow{\bar{g}(x)=\bar{g}_{1,1}(x)=x}\{(+),(\cdot),(-),(/)\}$.

2. MODIFICATION BY $\bar{g}-Transform$

2.1. Modification of functions by $\bar{g}-Transform$

Definition 2.1.1. [2], [9] Let f be a function on $]a,b[\subseteq]-\infty,+\infty[$ and the function $\bar{g}$ be a generator of the consistent system of pseudo-arithmetical operations $\{\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}},\overline{\ominus}_{\bar{g}},\overline{\oslash}_{\bar{g}}\}$. The function $f_{\bar{g}}$ given by $f_{\bar{g}}(x) = \bar{g}^{-1}\left(f\left(\bar{g}(x)\right)\right)$ for every $x \in \left(\bar{g}^{-1}(a), \bar{g}^{-1}(b)\right)$ is said to be $\bar{g} - function$ corresponding to the function f.

Definition 2.1.2. [9] Let f be a function on $]a, b[\subseteq]-\infty, +\infty[$ and the function $\bar{g}$ be a generator of the consistent system of pseudo-arithmetical operations $\{\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}}, \overline{\ominus}_{\bar{g}}, \overline{\oslash}_{\bar{g}}\}$. The function $f_{\bar{g}}$ given by $f_{\bar{g}}(x, y) = \bar{g}^{-1}\left(f\left(\bar{g}(x), \bar{g}(y)\right)\right)$ for every $x, y \in \left(\bar{g}^{-1}(a), \bar{g}^{-1}(b)\right)$ is said to be $\bar{g} - function$ corresponding to the function f.

Definition 2.1.3. [2] A continuous function $f_{\bar{g}}$ such that is a solution of the functional equations $f_{\bar{g}}(x)\, \overline{\odot}\, f_{\bar{g}}(y) = f_{\bar{g}}(x\, \overline{\odot}\, \mathrm{y})$ and, where $r > 0$, $x \in]-\infty, +\infty[$ will be called $\bar{g} - power$ functions and denoted by $f_{\bar{g}-r,power}$. This function is given by $f_{\bar{g}-r,power}(x) = \bar{g}^{-1}\left(\left(\bar{g}(x)\right)^{r}\right)$, $r > 0$ (for $x < 0$ hold $\bar{g}(x) = -g(-x)$).

2.2. Modification by $\bar{g} - transform$ for $\bar{g} - calculus$ and the extensions of axiomatic concepts of pseudo-operations $\left(\overline{\ominus}_{\bar{g}}, \overline{\oslash}_{\bar{g}}\right)$

The pseudo-arithmetical operations generated by generator $\bar{g}$, as $\bar{g}$-modified function of the arithmetic operations $\{\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}}, \overline{\ominus}_{\bar{g}}, \overline{\oslash}_{\bar{g}}\}$ by $\bar{g} - transform$ [6], [9], [10] are given as:

$$\overline{\oplus}\,(x, y) = x\, \overline{\oplus}_{\bar{g}}\, y = \bar{g}^{-1}(\bar{g}(x) + \bar{g}(y)) = \bar{g}^{-1}\left((+)(\bar{g}(x), \bar{g}(y)\right) = \boldsymbol{f}_{\bar{g}-(+)}(x, y)$$

$$\overline{\odot}\,(x, y) = x\, \overline{\odot}_{\bar{g}}\, y = \bar{g}^{-1}(\bar{g}(x) \cdot \bar{g}(y)) = \bar{g}^{-1}\left((\cdot)(\bar{g}(x), \bar{g}(y)\right) = \boldsymbol{f}_{\bar{g}-(\cdot)}(x, y)$$

$$\overline{\ominus}\,(x, y) = x\, \overline{\ominus}_{\bar{g}}\, y = \bar{g}^{-1}(\bar{g}(x) - \bar{g}(y)) = \bar{g}^{-1}\left((-)(\bar{g}(x), \bar{g}(y)\right) = \boldsymbol{f}_{\bar{g}-(-)}(x, y)$$

$$\overline{\oslash}\,(x, y) = x\, \overline{\oslash}_{\bar{g}}\, y = \bar{g}^{-1}(\bar{g}(x) / \bar{g}(y)) = \bar{g}^{-1}\left((/)(\bar{g}(x), \bar{g}(y)\right) = \boldsymbol{f}_{\bar{g}-(/)}(x, y).$$

2.3. The extensions of pseudo-operations $\left(\overline{\ominus}_{\bar{g}}, \overline{\oslash}_{\bar{g}}\right)$ based on $\bar{g} - transform$

We are treating again the problems of the extensions of the axiomatic concepts of pseudo-operations $\left(\overline{\ominus}_{\bar{g}}, \overline{\oslash}_{\bar{g}}\right)$ [2], [6], [9] support by $\bar{g} - functions$ and the parameterized Linear and Pseudo-Linear Functional Equation with f and f_g solutions (respectively the classes I and IV by $\bar{g} - transform$ [9]):

$$x\, \overline{\ominus}_{\bar{g}}\, y = x\, \overline{\oplus}\,(-y) = \bar{g}^{-1}(\bar{g}(x) - \bar{g}(y)) = g\left(g^{-1}\left(\left(\bar{g}(x) + \left(-\bar{g}(y)\right)\right)\right)\right) =$$

$$= \bar{g}^{-1}\left(\bar{g}(x) + g\left(g^{-1}\left(-\bar{g}(y)\right)\right)\right) = \bar{g}^{-1}\left(\bar{g}(x) + \bar{g}\left(f_g(y)\right)\right) = x\, \overline{\oplus}_{\bar{g}}\, \boldsymbol{f}_{\bar{g}}(y).$$

$$x\, \overline{\ominus}_{\bar{g}}\, y = x\, \overline{\oplus}_{\bar{g}}\, f_{\bar{g}-\bar{g}^{-1}(-1),lin}(y) : \begin{cases} Class\ \mathbf{I} \\ f(y) = -y \\ f_{\bar{g}}(y) = (-y)_{\bar{g}} = \bar{g}^{-1}(-1)\, \overline{\odot}_{\bar{g}}\, y = \boldsymbol{f}_{\bar{g}-\bar{g}^{-1}(-1),lin}(y) \end{cases}$$

- $x\, \overline{\ominus}_{\bar{g}}\, y = x\, \overline{\oplus}_{\bar{g}}\, f_{\bar{g}-\bar{g}^{-1}(-1),lin}(y)$

$$x\,\overline{\oslash}_{\bar{g}}\,y=\bar{g}^{\,-\mathbf{1}}\left(\frac{\bar{g}(x}{\bar{g}(y)}\right)=\bar{g}^{\,-\mathbf{1}}\left(\bar{g}(x)\cdot\frac{1}{\bar{g}(y)}\right)=\bar{g}^{\,-\mathbf{1}}\left(\bar{g}(x)\cdot\bar{g}\left(\bar{g}^{\,-1}\left(\frac{1}{\bar{g}(y)}\right)\right)\right)=$$

$$=\bar{g}^{\,-\mathbf{1}}\left(\bar{g}(x)\cdot\bar{g}\left(f_{\bar{g}}(y)\right)\right)=x\,\overline{\odot}_{\bar{g}}\,f_{\bar{g}}(y)=x\,\overline{\odot}_{\bar{g}}\,f_{\bar{g}-(-1),power}(y).$$

- $x\,\overline{\oslash}_{\bar{g}}\,y=x\,\overline{\odot}_{\bar{g}}\,f_{\bar{g}-(-1),power}(y)$

$$x\,\overline{\oslash}_{\bar{g}}\,y=x\,\overline{\odot}_{\bar{g}}\,f_{\bar{g}-(-1),power}(y):\begin{cases} Class\ \mathbf{IV} \\ f(y)=\ 1/y \\ f_{\bar{g}}(y)=\ (1/y)_{\bar{g}}=g^{\,-1}\left(\left(g(x)\right)^{-1}\right)=\boldsymbol{f}_{\bar{\boldsymbol{g}}-(-\mathbf{1}),\boldsymbol{power}}(y).\end{cases}$$

3. SOME SPECIAL BI-PSEUDO-INTEGRALS

3.1. Again for the first Bi-Pseudo-Integral

The integral of Marinov$\acute{a}$ presents one of the first attemps of defining an integral that would be an extention of the Lebesgue integral and would included integration with respect to some non addtitive –measure. With respect to a $\oplus-measure\ m$, by Marinov$\acute{a}$ [18] was defined the bi-pseudo-integral $\left(\boldsymbol{M}-\int_{\boldsymbol{X}}^{(\oplus,\odot)}\right)$ on X with respect to a $\oplus$- measure as an integral based on non-additive set functions. Marinov$\acute{a}'s$ integral is based on special types of a pseudo-operations $(\oplus_g,\odot_g)=(\oplus_{g_{a,r}},\odot_{g_{a,r}})$ or $\oplus=\vee=\max$ on $[0,+\infty]$ that fulfill the system of axioms (A.1÷A.7) and on $\oplus-measure$ [8], [16]. There are shown connections between $(\oplus_{g_{a,r}},\odot_{g_{a,r}})-integral$ and the Lebesgue integral of non-negative function so, in the case of $(\oplus_{g_{1,r}},\odot_{g_{1,r}})=(\oplus_{g_{1,r}}\ ,\ \cdot\ )$, Marinov$\acute{a}'s$ integral is the *Lebesgue* integral [8], [16], [18]. In the case of ($\oplus=\vee=\max$), the $\left(\boldsymbol{M}-\int_{\boldsymbol{X}}^{(\vee,\odot)}\right)$ leads to the *Shilkret's* integral [20]. For ($\oplus\neq\vee=\max$) is discover the connection between the $\left(\boldsymbol{M}-\int_{\boldsymbol{X}}^{(\oplus_{g_{a,r}},\odot_{g_{a,r}})}\right)$ and the *Lebesgue* integral, but *Kolesárová* in [16] has explained the reasons for another definition of the $(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})-integral$ of real functions which is more appropriate than the definition that was given in [18]. Different properties of integrals with respect to $\overline{\oplus}_{\bar{g}}$- measure and $\vee$-measure for f - *RMF* are caused by the essential difference between $\oplus_{g_{a,r}}$ and $\overline{\vee}$ [8], [14],[16], [18], [20].

Definition 3.1.1: For a *simple non-negative measurable* function (in short form s - S*NNMF*) defined on X, $s(x)=\sum_{i=1}^{n}\propto_i\cdot\mathbb{I}_{A_i}$ where sets $A_i\in\mathcal{A},A_i\neq A_j,\ A_i\cap A_j=\emptyset,$ for $i\neq j,\ i,j=$ $=\overline{1,n},\ 0<\propto_i<+\infty$, then $\left(\boldsymbol{M}-\int_{\boldsymbol{X}}^{(\oplus_{g_{a,r}},\ \odot_{g_{a,r}})}\right)(m,s\ -SNNMF)=$

$= \int_{X}^{\left(\oplus_{g_{a,r}},\ \odot_{g_{a,r}}\right)} s \odot_{g_{a,r}} dm = \oplus_{i=1}^{\infty} \propto_i \odot_{g_{a,r}} m(A_i)$ [8], [16], [18].

Definition 3.1.2: For a *non-negative measurable function* (*f-NNMF*), $f: X \to [0, +\infty]$, then

$\left(\boldsymbol{M} - \int_{X}^{\left(\oplus_{g_{a,r}},\ \odot_{g_{a,r}}\right)}\right)(m,\ f - NNMF) = \int_{X}^{\left(\oplus_{g_{a,r}},\ \odot_{g_{a,r}}\right)} f \odot_{g_{a,r}} dm =$

$= \sup\left\{\int_{X}^{\left(\oplus_{g_{a,r}},\ \odot_{g_{a,r}}\right)} s \odot_{g_{a,r}} dm:\ s \leq f,\ s - \ is\ a\ SNNMF\right\}$ and say that f is integrable if $\left(\boldsymbol{M} - \int_{X}^{\left(\oplus_{g_{a,r}},\ \odot_{g_{a,r}}\right)}\right)(m,\ f - NNMF) = \int_{X}^{\left(\oplus_{g_{a,r}},\ \odot_{g_{a,r}}\right)} f \odot_{g_{a,r}} dm < +\infty.$

Proposition 3.1.3: Let ***s*** be a *simple non-negative measurable* function (short form s - *SNNMF*) defined on X, let m be a $\oplus -measure$ on $(X, \mathcal{A})$, $(\oplus \neq \vee)$ and let $g = g_{1,r}$be a normed generator of the operation $\oplus_{g_{a,r}}$. Then$\left(\boldsymbol{M} - \int_{X}^{\left(\oplus_{g_{a,r}},\ \odot_{g_{a,r}}\right)}\right)(m, s - SNNMF) =$
$= g_{a,r}{}^{-1}\left(\frac{1}{a}\right) \odot_{g_{a,r}} g_{a,r}{}^{-1}\left(\int_{X}^{(+,\ \cdot\ )}(g_{a,r} \circ s) \cdot d(g_{a,r} \circ m)\right)$ where the right-hand side is the Lebesgue integral $\left(\boldsymbol{L} - \int_{X}^{(+,\ \cdot)}\right)(g_{a,r} \circ m, g_{a,r} \circ s - SNNMF)$ [8], [11], [16], [18].

G*eneralized definition* by *Kolesárová* on [16] for $(\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}})$ −integral, in case of *f* – *RMF* with respect to a $\overline{\oplus}_{\bar{g}}$- measure m where $\bar{g}_{a,r}|_{[0,+\infty]} = g_{a,r}, \overline{\oplus}_{\bar{g}_{a,r}}|_{[0,\infty]} = \oplus_{g_{a,r}}$ are given as bellow.

Definition 3.1.4: [11], [16] For *a real measurable* function (in short form *f* - *RMF*), $(\oplus \neq \vee)$, $f: X \to (-\infty, +\infty)$, if at least one of the functions $f^+,\ f^-$ is integrable, then

$$\left(\boldsymbol{M}, \boldsymbol{K} - \int_{X}^{(\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}})}\right)(m,\ f - RMF) = \int_{X}^{(\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}})} f^+ \overline{\odot}_{\bar{g}}\ dm\ \overline{\ominus}_{\bar{g}} \int_{X}^{(\overline{\oplus},\ \cdot)} f^- \overline{\odot}_{\bar{g}}\ dm =$$

$$= \left(\boldsymbol{M}, \boldsymbol{K} - \int_{X}^{(\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}})}\right)(m,\ f^+)\ \overline{\ominus}_{\bar{g}} \left(\boldsymbol{M}, \boldsymbol{K} - \int_{X}^{(\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}})}\right)(m,\ f^-)$$

A function f is called integrable iff

$$-\infty < \left(\boldsymbol{M}, \boldsymbol{K} - \int_{X}^{(\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}})}\right)(m,\ f - RMF) = \int_{X}^{(\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}})} f\ \overline{\odot}_{\bar{g}}\ dm < +\infty.$$

Proposition 3.1.5: [16] Let $(X, \mathcal{A},\ m)$ be a $\oplus$- measure space. The integral of a real measurable *f*- *RMF* with respect to a $\oplus$- measure m, in case $\oplus \neq \vee$ (when $\int_{X}^{(\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}})}$is defined) is given by:

$$\left(\boldsymbol{M}, \boldsymbol{K} - \int_{X}^{(\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}})}\right)(m,\ f - RMF) = \left(\boldsymbol{M}, \boldsymbol{K} - \int_{X}^{(\overline{\oplus}_{\bar{g}_{a,r}}, \overline{\odot}_{\bar{g}_{a,r}})}\right)(m,\ f - RMF) =$$

$$= \bar{g}_{a,r}{}^{-1}\left(\bar{g}_{a,r}{}^{-1}\left(\frac{1}{a}\right) \odot_{g_{a,r}} \left(\boldsymbol{L} - \int_{X}^{(+,\ \cdot)}\right)(\bar{g}_{a,r} \circ m, \bar{g}_{a,r} \circ f - RMF)\right)$$

and the integral in the right-hand side $\left(\boldsymbol{L} - \int_{X}^{(+,\ \cdot)}\right)(\bar{g}_{a,r} \circ m, \bar{g}_{a,r} \circ f - RMF)$ is *Lebesgue integral*, also $\bar{g}_{a,r} \circ m$ is the *Lebesgue measure*.

If $\bar{g}$ is the *normed generator*$\bar{g} = \bar{g}_{1,r} - normed\ generator$ hold [8], [16], [18]:

$$\left(\boldsymbol{M},\boldsymbol{K}-\int_{\boldsymbol{X}}^{\left(\overline{\oplus}_{\bar{g}_{1,r}},\overline{\odot}_{\bar{g}_{1,r}}\right)}\right)(m,\ f-RMF)=\left(\boldsymbol{M},\boldsymbol{K}-\int_{\boldsymbol{X}}^{\left(\overline{\oplus}_{\bar{g}_{1,r}},\ \cdot\right)}\right)(m,\ f-RMF)=$$
$$=\int_{\boldsymbol{X}}^{\left(\overline{\oplus}_{\bar{g}_{1,r}},\cdot\right)}f\cdot dm=\bar{g}_{1,r}{}^{-1}\left(\left(\boldsymbol{L}-\int_{\boldsymbol{X}}^{(+,\ \cdot)}\right)\left(\bar{g}_{1,r}\circ m,\bar{g}_{1,r}\circ f-RMF\right)\right).$$

4. MODIFIED PSEUDO-ADDITIVE MEASURE BY $\bar{g}-Transform$

Let X be a non-emptyset and let $\mathcal{A}$ be a σ –algebra of subsets of X [3], [5], [6], [8], [13], [15], [18].

A ***set function*** $m:\mathcal{A}\longrightarrow[0,+\infty]$ will be called a $\overline{\oplus}$- ***measure,*** $(\overline{\oplus}-m)$ if the conditions $\left(\text{C1.}\, m\,(\emptyset)=0\right)$ and $\left(\text{C2.}\ m\left(\bigcup_{i=1}^{\infty}A_i\ \right)=\oplus_{i=1}^{\infty}m\,(A_i)\right)$ hold for any sequence $(A_i)_{n\in\mathbb{N}}$ of pairwise disjoints sets from $\mathcal{A}$.

So, shall write $\oplus_{i=1}^{n}a_i\ =a_1\ \overline{\oplus}\ a_2\ \overline{\oplus}\ \ ...\overline{\oplus}\ a_n$ and $\oplus_{i=1}^{\infty}a_i\ =sup_n\,(\oplus_{i=1}^{n}a_i)$. If $\overline{\oplus}$ is an idempotent operation ($\overline{\oplus}-\boldsymbol{ID}$), then the disjointness of sets and condition (1) can be omitted.

For $\left(\overline{\oplus}_{\bar{g}}-m\right)-measure$ by definition 4.1.1, based on the definition 2.2 for $\bar{g}-function$ [2], [3], [5], [14], [16], [17] and on the sistem of the pseudo-arithmetical operations generated by generator $\bar{g}$, can modify the measure by $\bar{g}-transform$ in the following form:

Definition 4.1: [9] Let m be a set function $:\mathcal{A}\longrightarrow[0,+\infty]$ and the function $\bar{g}$ be a generator of the consistent system of pseudo-arithmetical operations $\{\overline{\oplus},\overline{\odot},\overline{\ominus},\overline{\oslash}\}$. The function $m_{\bar{g}}$ given by $m_{\bar{g}}(A)=\bar{g}^{-1}\left(m\left(\bar{g}(A)\right)\right)$ for every $A\in\{\bar{g}^{-1}(A_1),\dots,\bar{g}^{-1}(A_n)\}$ is said to be $\bar{g}-\left(\overline{\oplus}_{\bar{g}}-m\right)measure$ function $\left(m_{\bar{g}}\right)$ corresponding to the set function m.

By $\bar{g}-calculus$, $\bar{g}-Transform$ and definition of $(\overline{\oplus}_{\bar{g}}-m)-measure$ we get [2], [4], [6], [9]:

$$m(A)\,\overline{\oplus}_{\boldsymbol{\bar{g}}}\,m(B)=\bar{g}^{-\mathbf{1}}\left(\bar{g}\left(m(A)\right)+\bar{g}\left(m(B)\right)\right)=\bar{g}^{-1}\left((+)\left(\bar{g}\left(m(A)\right),\bar{g}\left(m(A)\right)\right)\right)=$$
$$=\boldsymbol{f}_{\boldsymbol{\bar{g}}-(+)}\left(m(A),m(B)\right).$$

Moreover, we can write by the definition of $(\overline{\oplus}_{\bar{g}}-m)-measure$ and $m_{\bar{g}}-calculus$ [9]:

- $m_{\bar{g}}(A\cup B)=\bar{g}^{-1}\left(m\left(\bar{g}(A\cup B)\right)\right)=\bar{g}^{-1}\left(m\left(\bar{g}(A)\cup\bar{g}(B)\right)\right)$
- $m\left(\bar{g}(A)\cup\bar{g}(B)\right)=m\left(\bar{g}(A)\right)\overline{\oplus}_{\bar{g}}\,m\left(\bar{g}(B)\right)=\bar{g}\left(m_{\bar{g}}(A)\right)\overline{\oplus}_{\bar{g}}\,\bar{g}\left(m_{\bar{g}}(A)\right)$
- $\bar{g}\left(m_{\bar{g}}(A\cup B)\right)=\bar{g}\left(m_{\bar{g}}(A)\right)\overline{\oplus}_{\bar{g}}\,\bar{g}\left(m_{\bar{g}}(A)\right)$
- $m_{\bar{g}}(A\cup B)=\bar{g}^{-1}\left(\bar{g}\left(m_{\bar{g}}(A)\right)\overline{\oplus}_{\bar{g}}\,\bar{g}\left(m_{\bar{g}}(A)\right)\right)=f_{\bar{g}-(\overline{\oplus}\bar{g})}\left(m_{\bar{g}}(A),m_g(A)\right)$
- $m\left(\bar{g}(A)\right)\overline{\oplus}_{\bar{g}}\,m\left(\bar{g}(A)\right)=m\left(\bar{g}(A\cup B)\right)orm\left(\bar{g}(A)\right)\overline{\oplus}_{\bar{g}}\,m\left(\bar{g}(A)\right)=m\left(\bar{g}(A)\cup\bar{g}(B)\right)$
- $m_{\bar{g}}(A\cup B)=f_{\bar{g}-m}(A\cup B)=f_{\bar{g}-(\overline{\oplus}\bar{g})}\left(m_{\bar{g}}(A),m_{\bar{g}}(A)\right)$.

5. BI-PSEUDO-INTEGRAL WITH RESPECT TO A MODIFIED MEASURE $m_{\overline{g}}$

5.1. Modification of $s-simple\ non-negative$ *function* and measure *m* by $g-transform$

Modification of $s-simple\ non-negative$ *function* (*SNNF*):

$$s=\sum_{i=1}^{n}\propto_i\cdot\mathbb{I}_{A_i}\xrightarrow{(g-TR)}f_{g-(s)}=g^{-1}\big(s(g(A_i))\big)=g^{-1}\big(\textstyle\sum_{i=1}^{n}g(\propto_i)\cdot\ \mathbb{I}_{g(A_i)}\big)\ =$$

$$=g^{-1}\left(\textstyle\sum_{i=1}^{n}g(\propto_i)\cdot g\ \left(g^{-1}\big(\mathbb{I}_{g(A_i)}\big)\right)\right)=g^{-1}\left(\textstyle\sum_{i=1_i}^{n}g(\propto_i)\cdot g\ \left(g^{-1}\big(\mathbb{I}_{g(A_i)}\big)\right)\right)=$$

$$=g^{-1}\left(\textstyle\sum_{i=1}^{n}g\left(g^{-1}\left(g(\propto_i)\cdot g\ \left(g^{-1}\big(\mathbb{I}_{g(A_i)}\big)\right)\right)\right)\right)=$$

$$=g^{-1}\left(\textstyle\sum_{i=1}^{n}g\left(g(\propto_i)\odot_g\left(g^{-1}\big(\mathbb{I}_{g(A_i)}\big)\right)\right)\right)=\bigoplus_{i=1}^{n}g(\propto_i)\odot_g\left(g^{-1}\big(\mathbb{I}_{g(A_i)}\big)\right).$$

So, s_g is a pseudo –linear combination of $g(\propto_i)$.

- $s=\sum_{i=1}^{n}\propto_i\cdot\mathbb{I}_{A_i}\xrightarrow{(g-TR)}s_g=f_{g-(s)}=\ \bigoplus_{i=1}^{n}g(\propto_i)\odot_g\left(f_{g-(\mathbb{I}_A)}(A_i)\right)$ [9].

Modification of measure *m* by $g-Transform$:

$$m(s)=\sum_{i=1}^{n}\propto_i\cdot\ m(A_i)\xrightarrow{(g-TR)}g^{-1}\left(\textstyle\sum_{i=1}^{n}g(\propto_i)\cdot\ m\big(g(A_i)\big)\right)\ =$$

$$=g^{-1}\left(\textstyle\sum_{i=1}^{n}g(\propto_i)\cdot g\ \left(g^{-1}\left(m\big(g(A_i)\big)\right)\right)\right)=g^{-1}\left(\textstyle\sum_{i=1_i}^{n}g(\propto_i)\cdot g\ \left(m_g(A_i)\right)\right)=$$

$$=g^{-1}\left(\textstyle\sum_{i=1}^{n}g\left(g^{-1}\left(g(\propto_i)\cdot g\ \left(m_g(A_i)\right)\right)\right)\right)=g^{-1}\left(\textstyle\sum_{i=1}^{n}g\left(g(\propto_i)\odot_g m_g(A_i)\right)\right)=$$

$$=\bigoplus_{i=1}^{n}g(\propto_i)\odot_g m_g(A_i).$$

- $m(s)=\sum_{i=1}^{n}\propto_i\cdot\ m(A_i)\xrightarrow{(g-TR)}m_g\big(s_g\big)=\ \bigoplus_{i=1}^{n}g(\propto_i)\odot_g m_g(A_i)$ [9].

Definition 5.1.1: For a *simple non-negative measurable* function (short form s_g – $SNNMF_g$) defined on X , $s(x)=\sum_{i=1}^{n}\propto_i\cdot\mathbb{I}_{A_i}$ where sets $A_i\in\mathcal{A},A_i\neq A_j,\ A_i\cap A_j=\emptyset,$ for $i\neq j,\ i,j=$ $=\overline{1,n},$ $0<\propto_i<+\infty$, then $\left(\boldsymbol{BP}_{(g-TR)}-\int_{g^{-1}(X)}^{(\oplus_g,\odot_g)}\right)\big(m_g,\ s_g\text{ – }SNNMF_g\big)=$

$$=\int_{g^{-1}(X)}^{(\oplus_g,\odot_g)}s_g\odot_g dm_g=\bigoplus_{i=1}^{n}g(\propto_i)\odot_g m_g(A_i).$$

Definition 5.1.2: For a *non-negative measurable function* f_g (f_g-$NNMF_g$) when $f_g: g^{-1}(X) \to [0, +\infty]$, the bi-pseudo-integral $\int_{g^{-1}(X)}^{(\oplus_g, \odot_g)}$ is defined on $g^{-1}(X)$ with respect to m_g in form as follows: $\left(\boldsymbol{BP}_{(g-TR)} - \int_{g^{-1}(X)}^{(\oplus_g, \odot_g)}\right)(m_g, f_g - NNMF_g) = \int_{g^{-1}(X)}^{(\oplus_g, \odot_g)} f_g \odot_g dm_g =$
$= \sup\left\{\int_{g^{-1}(X)}^{(\oplus_g, \odot_g)} s_g \odot_g dm_g : s_g \le f_g, s_g - \text{ is a } SNNMF_g\right\}$ and say that f_g is integrable if
$\left(\boldsymbol{BP}_{(g-TR)}\int_{g^{-1}(X)}^{(\oplus_g, \odot_g)}\right)(m_g, f_g - NNMF_g) = \int_{g^{-1}(X)}^{(\oplus_g, \odot_g)} f_g \odot_g dm_g < +\infty.$

Proposition 5.1.3: Let s_g be a *simple non-negative measurable* function (in short form s_g-$-SNNMF_g$) defined on X, let m be a $\oplus_g -$ *measure* on $(X, \mathcal{A})$, $(\oplus \neq \vee)$ and let $\bar{g}_{a,r}|_{[0,+\infty]} = g_{a,r}$, $\overline{\oplus}_{\bar{g}_{a,r}}|_{[0,\infty]} = \oplus_{g_{a,r}}$ be a generator of the operation $\oplus_g$.
Then $\left(\boldsymbol{BP}_{(g-TR)} - \int_{g^{-1}(X)}^{(\oplus_g, \odot_g)}\right)(m_g, s_g - SNNMF_g) = g^{-1}\left(\int_{g^{-1}(X)}^{(+,\cdot)}(g \circ s_g) \cdot d(g \circ m_g)\right)$,
where the right-hand side is the Lebesgue integral $\left(\boldsymbol{L} - \int_{g^{-1}(X)}^{(+,\ \cdot)}\right)(g \circ m_g, g \circ s_g - SNNMF_g)$.

Proof: $\left(\boldsymbol{BP}_{(g-TR)} - \int_{g^{-1}(X)}^{(\oplus_g, \odot_g)}\right)(m_g, s_g - SNNMF_g) = \oplus_{i=1}^{n} g(\propto_i) \odot_g m_g(A_i) =$

$$= g^{-1}\left(\sum_{i=1}^{n} g\left(g(\propto_i) \odot_g m_g(A_i)\right)\right) = g^{-1}\left(\sum_{i=1}^{n} g\left(g^{-1}\left(g(\propto_i) \cdot g\left(m_g(A_i)\right)\right)\right)\right) =$$

$$= g^{-1}\left(\sum_{i=1_i}^{n} g(\propto_i) \cdot g\left(m_g(A_i)\right)\right) = g^{-1}\left(\int_{g^{-1}(X)}^{(+,\cdot)}(g \circ s_g) \cdot d(g \circ m_g)\right) =$$

$$= g^{-1}\left(\left(\boldsymbol{L} - \int_{g^{-1}(X)}^{(+,\ \cdot)}\right)(g \circ m_g, g \circ s_g - SNNMF_g)\right).$$

Also, $\left(\boldsymbol{BP}_{(g-TR)} - \int_{g^{-1}(X)}^{(\oplus_g, \odot_g)}\right)(m_g, s_g - SNNMF_g) = g^{-1}\left(\sum_{i=1}^{n} g(\propto_i) \cdot m(g(A_i))\right) =$

$$= g^{-1}\left(\int_{g^{-1}(X)}^{(+,\cdot)}(s \circ g) \cdot d(m \circ g)\right) = g^{-1}\left(\left(\boldsymbol{L} - \int_{g^{-1}(X)}^{(+,\ \cdot)}\right)(s \circ g, m \circ g - SNNMF)\right).$$

The bi-$(\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}}) - integral$ in case of the $f_{\bar{g}} - RMF_{\bar{g}}$ with respect to a $\overline{\oplus}_{\bar{g}} - measure$ $m_{\bar{g}}$ [2], [6], [8], [9], [11], [16], [18], [21], will be given below.

Definition 5.1.4: For *a real measurable* function $f_{\bar{g}}$ (short form $f_{\bar{g}} - RMF_{\bar{g}}$) in case of $f_{\bar{g}}: \bar{g}^{-1}(X) \to (-\infty, +\infty)$, $(\overline{\oplus} \neq \overline{\vee})$, if at least one of the functions $f_{\bar{g}}^{+}$, $f_{\bar{g}}^{-}$ is integrable, then

$$\left(\boldsymbol{BP}_{(\bar{g}-TR)} - \int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}})}\right)(m_{\bar{g}}, f_{\bar{g}} - RMF_{\bar{g}}) =$$

$$= \left(\int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}})} f_{\bar{g}}^{+} \overline{\odot}_{\bar{g}} dm_{\bar{g}}\right) \overline{\ominus}_{\bar{g}} \left(\int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}})} f_{\bar{g}}^{-} \overline{\odot}_{\bar{g}} dm_{\bar{g}}\right) =$$

$$= \left(\boldsymbol{BP}_{(\bar{g}-TR)} - \int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}})}\right)(m_{\bar{g}}, f_{\bar{g}}^{+}) \overline{\ominus}_{\bar{g}} \left(\boldsymbol{BP}_{(\bar{g}-TR)} - \int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}})}\right)(m_{\bar{g}}, f_{\bar{g}}^{-}).$$

A function $f_{\bar{g}}$ is called integrable iff

$$-\infty < \left(\boldsymbol{BP}_{(\bar{g}-TR)} - \int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})}\right)\left(m_{\bar{g}},\ f_{\bar{g}} - RMF_{\bar{g}}\right) = \int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})} f_{\bar{g}}\ \overline{\odot}_{\bar{g}}\ dm_{\bar{g}} < +\infty.$$

Proposition 5.1.5: Let $(X, \mathcal{A}, m)$ be a $\overline{\oplus}_{\bar{g}} - measure$ space. The integral of a real measurable $f_{\bar{g}}$- $RMF_{\bar{g}}$ with respect to a $\overline{\oplus}_{\bar{g}}$- measure m, for $\overline{\oplus}_{\bar{g}} \neq \overline{\nabla}$ (if $\int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})}$ is defined) is given by:

$$\left(\boldsymbol{BP}_{(\bar{g}-TR)} - \int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})}\right)\left(m_{\bar{g}},\ f_{\bar{g}} - RMF_{\bar{g}}\right) = \bar{g}^{-1}\left(\left(\boldsymbol{L} - \int_{\bar{g}^{-1}(X)}^{(+,\ \cdot\ )}\right)\left(\bar{g} \circ m_{\bar{g}}, \bar{g} \circ f_{\bar{g}} - RMF_{\bar{g}}\right)\right)$$

and the integral in the right-hand side $\left(\boldsymbol{L} - \int_{\bar{g}^{-1}(X)}^{(+,\ \cdot\ )}\right)\left(\bar{g} \circ m_{\bar{g}}, \bar{g} \circ f_{\bar{g}} - RMF_{\bar{g}}\right)$ is *Lebesgue integral*, also $\bar{g} \circ m_{\bar{g}}$ is the *Lebesgue measure*.

$$\left(\boldsymbol{BP}_{(\bar{g}-TR)} - \int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})}\right)\left(m_{\bar{g}},\ f_{\bar{g}}^{\ +}\right) \overline{\ominus}_{\bar{g}} \left(\boldsymbol{BP}_{(\bar{g}-TR)} - \int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})}\right)\left(m_{\bar{g}},\ f_{\bar{g}}^{\ -}\right) =$$

$$= \left(\int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})} f_{\bar{g}}^{\ +}\ \overline{\odot}_{\bar{g}}\ dm_{\bar{g}}\right) \overline{\ominus}_{\bar{g}} \left(\int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})} f_{\bar{g}}^{\ -}\ \overline{\odot}_{\bar{g}}\ dm_{\bar{g}}\right) =$$

$$= \bar{g}^{-1}\left[\bar{g}\left(\left(\int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})} f_{\bar{g}}^{\ +}\ \overline{\odot}_{\bar{g}}\ dm_{\bar{g}}\right)\right) - \bar{g}\left(\left(\int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})} f_{\bar{g}}^{\ -}\ \overline{\odot}_{\bar{g}}\ dm_{\bar{g}}\right)\right)\right] =$$

$$= \bar{g}^{-1}\left\{\bar{g}\left(\bar{g}^{-1}\left(\int_{g^{-1}(X)}^{(+,\cdot\ )} \left(\bar{g} \circ f_{\bar{g}}^{\ +}\right) \cdot d\left(\bar{g} \circ m_{g}\right)\right)\right)\right.$$

$$\left. - \bar{g}\left(\bar{g}^{-1}\left(\int_{g^{-1}(X)}^{(+,\cdot\ )} \left(\bar{g} \circ f_{\bar{g}}^{\ -}\right) \cdot d\left(\bar{g} \circ m_{g}\right)\right)\right)\right\} =$$

$$= \int_{g^{-1}(X)}^{(+,\cdot\ )} \left(\bar{g} \circ f_{\bar{g}}\right) \cdot d\left(\bar{g} \circ m_{g}\right) = \bar{g}^{-1}\left(\left(\boldsymbol{L} - \int_{\bar{g}^{-1}(X)}^{(+,\ \cdot\ )}\right)\left(\bar{g} \circ m_{\bar{g}}, \bar{g} \circ f_{\bar{g}} - RMF_{\bar{g}}\right)\right) =$$

$$= \left(\boldsymbol{BP}_{(\bar{g}-TR)} - \int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})}\right)\left(m_{\bar{g}},\ f_{\bar{g}} - RMF_{\bar{g}}\right).$$

$$\left(\boldsymbol{BP}_{(\bar{g}-TR)} - \int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})}\right)\left(m_{\bar{g}},\ f_{\bar{g}} - RMF_{\bar{g}}\right) = \bar{g}^{-1}\left(\left(\boldsymbol{L} - \int_{g^{-1}(X)}^{(+,\ \cdot\ )}\right)\left(f \circ \bar{g}, m \circ \bar{g} - RMF\right)\right).$$

Conclusion:

Based on properties of $f_{\bar{g}}$ [2], [9], $\bar{g} - calculus$ introduced by E.Pap [4], [6], [12], [13], [17] and $g - integral$ can write [3], [6], [12], [13], [14], [16], [17], [18], [20]:

$$\left(\boldsymbol{BP}_{(\bar{g}-TR)} - \int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})}\right)\left(m_{\bar{g}},\ f_{\bar{g}} - RMF_{\bar{g}}\right) = \int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})} f_{\bar{g}}\ \overline{\odot}_{\bar{g}}\ dm_{\bar{g}} =$$

$$= \bar{g}^{-1}\left(\int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})} \bar{g}(f_{\bar{g}}) \cdot d\left(\bar{g}(m_{\bar{g}})\right)\right) = \bar{g}^{-1}\left(\int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})} f(\bar{g}) \cdot d\big(m(\bar{g})\big)\right).$$

By the restriction of the pseudo-operations $\overline{\oplus}_{\bar{g}}\,|_{[0,+\infty]} = \oplus_g$, $\overline{\odot}_{\bar{g}}\,|_{[0,+\infty]} = \odot_g$ in conditions introduced by E.Pap, and of the generator $\bar{g}|_{[0,+\infty]} = g$, our $(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}}) - integral$ is of Pap`s integral on the interval $[-\infty,+\infty]$ (in case $\overline{\oplus}_{\bar{g}} \neq \overline{\vee}$) i.e., belongs to the second class of bi-pseudo-integral where pseudo-operations are generated [3], [8], [16], [18], [13], [17], [19].

$$\left(\boldsymbol{BP}_{(\bar{g}-TR)} - \int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})}\right)(m_{\bar{g}},\ f_{\bar{g}} - RMF_{\bar{g}}) = \left(\boldsymbol{Pap} - \int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})}\right)(m_{\bar{g}},\ f_{\bar{g}} - RMF_{\bar{g}})$$

$$\left(\boldsymbol{Pap} - \int_{g^{-1}(X)}^{(\oplus_g,\odot_g)}\right)(m_g,\ f_g - RMF_g)|_{[-\infty,+\infty]} = \left(\boldsymbol{BP}_{(\bar{g}-TR)} - \int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})}\right)(m_{\bar{g}},\ f_{\bar{g}} - RMF_{\bar{g}}).$$

When the situation is in the conditions of mentined at the beginning, i.e., $\bar{g}(x) = \bar{g}_{a,r}(x) =$
$= ax^r\ for\ r \geq 1,\ a > 0$ and the system of axioms ($\oplus$,$\odot$-A.1$\div A.7$) is fulfilled, we conclude:

$$\left(\boldsymbol{BP}_{(\bar{g}-TR)} - \int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}\right)(m_{\bar{g}},\ f_{\bar{g}} - RMF_{\bar{g}}) = \left(\boldsymbol{M},\boldsymbol{K} - \int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}\right)(m_{\bar{g}},\ f_{\bar{g}} - RMF_{\bar{g}}),$$

$$\left(\boldsymbol{BP}_{(\bar{g}-TR)} - \int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}\right)(m_{\bar{g}},\ f_{\bar{g}} - RMF_{\bar{g}}) =$$

$$= \ \bar{g}_{a,r}^{\ -1}\left(a^{-2/r} \odot_{g_{a,r}} \left(\boldsymbol{L} - \int_{\bar{g}^{-1}(X)}^{(+,\ \cdot\ )}\right)(\bar{g}_{a,r} \circ m_{\bar{g}}, \bar{g}_{a,r} \circ f_{\bar{g}} - RMF_{\bar{g}})\right),$$

$$\left(\boldsymbol{BP}_{(g-TR)}\int_{g^{-1}(X)}^{(\vee,\odot_{g_{1,r}})}\right)(m_g,\ f_g - NNMF_g) = \left(\boldsymbol{M} - \int_{g^{-1}(X)}^{(\vee,\ \odot_{g_{1,r}})}\right)(m_g,\ f_g - NNMF_g) =$$

$$= \left(\boldsymbol{S}h - \int_{g^{-1}(X)}^{(\vee,\ \odot_{g_{1,r}})}\right)(m_g,\ f_g - NNMF_g).$$

Moreover, for $\bar{g} - transform$ of the pseudo-probability measure $(\oplus - \mathrm{P}) - m$, $(m_{\bar{g}})$ and the induced probability measure P $(P_{\bar{g}})$, can be presented the $(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})$– $integral$ with respect of $m_{\bar{g}}$ and relation with the $(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})$– $integral$ with repect of $P_{\bar{g}}$ [9]:

$$\left(\boldsymbol{BP}_{(\bar{g}-TR)} - \int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})}\right)(m_{\bar{g}},\ f_{\bar{g}} - RMF_{\bar{g}}) = \int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})} f_{\bar{g}}\ \overline{\odot}_{\bar{g}}\ dm_{\bar{g}} =$$

$$= \bar{g}^{-1}\left(\int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})} f(\bar{g}) \cdot d(P_{\bar{g}})\right).$$

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