Int Jr. of Mathematics Sciences & Applications
Vol. 2, No. 2, May 2012

ISSN No: 2230-9888
www.journalshub.com

SECOND HANKEL DETERMINANT FOR SOME SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract

Sharp bounds for the functional $\left|a_2a_4 - a_3^2\right|$ ***are derived for functions belonging to the classes M (α) and N (λ). Also certain application of the main results for a class of functions defined by convolution is given. As a special case, the coefficient bounds for a class of functions defined through fractional derivatives are obtained.***

Key Words: Analytic functions, Starlike functions, convex functions, Hankel determinant, Fractional derivative.

2000 Mathematics Subject Classification: Primary 30C45.

1 INTRODUCTION

Let ***A*** denote the class of normalized analytic Univalent functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

Where $z \in D = \{z : |z| < 1\}$. In [10], the qth Hankel determinant for $q \geq 1$ and $n \geq 0$ is stated by Noonan and Thomas as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has also been considered by several authors. For example, Noor in [11] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions f given by (1.1) with bounded boundary. Ehrenborg in [3] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Leyman in [7].

Easily, one can observe that the Fekete and Szegö functional is $H_2(1)$. Fekete and Szegö [4] then further generalized the estimate $\left|a_3 - \mu a_2^2\right|$ where μ is real and $f \in \boldsymbol{A}$. For our discussion in this paper, we consider the Hankel determinant in the case q =2 and n=2, which $H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}$.

A function $f \in \boldsymbol{A}$ is said to be starlike and convex respectively if and only if, for $z \in D$, $\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ and $\Re\left\{\frac{(zf'(z))'}{f'(z)}\right\} > 0$. By usual notations, we denote these classes of functions respectively by $\boldsymbol{S}^*$ and $\boldsymbol{C}$.

In the present paper, we obtain an upper bound for the functional $\left|a_2 a_4 - a_3^2\right|$ for functions belongs to the class $\boldsymbol{M}(\alpha)$ and $\boldsymbol{N}(\lambda)$, which are defined as follows.

Definition 1.1 Let f be given by (1.1). Then $f \in \boldsymbol{M}(\alpha)$ if and only if

$$\Re\left\{\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)}\right\} > 0, \quad 0 \le \alpha \le 1, \ z \in D. \tag{1.2}$$

We see that $\boldsymbol{M}(0) = \boldsymbol{S}^*$, the class of starlike functions and $\boldsymbol{M}(1) = \boldsymbol{C}$, the class of convex functions.

Definition 1.2 : Let f be given by (1.1). Then $f \in \boldsymbol{N}(\lambda)$ if and only if

$$\Re\left\{\frac{\lambda z^3 f'''(z) + (1+2\lambda) z^2 f''(z) + z f'(z)}{\lambda z^2 f''(z) + z f'(z)}\right\} > 0, \ 0 \le \lambda \le 1, \ z \in D. \tag{1.3}$$

We note that $\boldsymbol{N}(0) = \boldsymbol{C}$, the class of convex functions.

The definitions of the classes $\boldsymbol{M}(\alpha)$ and $\boldsymbol{N}(\lambda)$ are motivated by the classes studied by [2, 17] and [6, 18] respectively.

For fixed $g \in \boldsymbol{A}$, we define the class $\boldsymbol{M}_g(\alpha)$ be the class of functions $f \in \boldsymbol{A}$ for which $(f * g) \in \boldsymbol{M}(\alpha)$ and the class $\boldsymbol{N}_g(\lambda)$ be the class of functions $f \in \boldsymbol{A}$ for which $(f * g) \in \boldsymbol{N}(\lambda)$.

In order to introduce the above mentioned classes we need the following:

Definition 1.3 (See [10,11]; see also [13,14]) Let the function $f(z)$ be analytic in a simply connected region of the z-plane containing the origin. Then the fractional derivative of f of order γ is defined by

$$D_z^{\gamma} f(z) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \int_0^z \frac{f(\varsigma)}{(z-\varsigma)} d\varsigma, \quad (0 \le \gamma < 1). \tag{1.4}$$

Where the multiplicity of $(z-\varsigma)^{\gamma}$ is removed by requiring $\log(z-\varsigma)$ to be real when $(z-\varsigma) > 0$.

Using the above definition (1.3) and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [13] introduced the operator

$\Omega^{\gamma} : A \to A$ defined by $(\Omega^{\gamma} f)(z) = \Gamma(2-\gamma) z^{\gamma} D_z^{\gamma} f(z), \quad (\gamma \neq 2, 3, 4.....).$

The class $\boldsymbol{M}_{\gamma}(\alpha)$ consists of functions $f \in \boldsymbol{A}$ for which $\Omega^{\gamma} f \in \boldsymbol{M}(\alpha)$ and the class $\boldsymbol{N}_{\gamma}(\lambda)$ consists of functions $f \in \boldsymbol{A}$ for which $\Omega^{\gamma} f \in \boldsymbol{M}(\alpha)$. $\boldsymbol{M}_{\gamma}(\alpha)$ and $\boldsymbol{N}_{\gamma}(\lambda)$ are the special cases of the classes $\boldsymbol{M}_g(\alpha)$ and $\boldsymbol{N}_g(\lambda)$ when $g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\gamma)}{\Gamma(n+1-\gamma)} z^n$.

Let P be the family of all functions p analytic in D for which $\Re\{p(z)\} > 0$ and

$$p(z) = 1 + c_1 z + c_2 z^2 + ..., \qquad z \in D. \tag{1.5}$$

To prove our main result, we need the following:

Lemma 1.1 ([14]) If $p \in P$ then $|c_k| \le 2$ for each k.

Lemma 1.2 ([5]) The power series for p(z) given by (1.5) converges in D to a function in P if and only if the Toeplitz determinats

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \dots & c_n \\ c_{-1} & 2 & c_1 & \dots & c_{n-1} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ c_{-n} & c_{-n+1} & c_{-n+1} & \dots & 2 \end{vmatrix}, \qquad n = 1,2,3,\dots \tag{1.6}$$

and $c_{-k} = \overline{c}_k$ are all nonnegative. They are strictly positive except for

$$p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k} z), \quad \rho_k > 0, \quad t_k \quad real$$

and $t_k \neq t_j$ for $k \neq j$, in this case $D_n > 0$ for $n < m-1$ and $D_n = 0$ for $n \geq m$.

Lemma 1.3 ([8 ,9]) Let the function p ϵ P be given by power series (1.5).

Then $2c_2 = c_1^2 + x(4 - c_1^2)$ (1.7)

for some x , $|x| \leq 1$ and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z. \quad (1.8)$$

2. Main Results

Theorem 2.1 Let $f(z)$ ϵ $\boldsymbol{M}$ (α). Then

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{1}{(1+\alpha)(1+3\alpha)}. \quad (2.1)$$

The result obtained is sharp.

Proof: For $f(z)$ ϵ $\boldsymbol{M}$ (α) given by (1.1), there exists a p ϵ P such that

$$\left[zf'(z) + \alpha z^2 f''(z)\right] = \left[(1-\alpha)f(z) + \alpha z f'(z)\right] p(z).$$

Then equating the coefficients, we obtain

$$a_2 = \frac{c_1}{(1+\alpha)}, \quad a_3 = \frac{c_1^2 + c_2}{2(1+2\alpha)}, \quad a_4 = \frac{c_1^3}{6(1+3\alpha)} + \frac{c_1 c_2}{2(1+3\alpha)} + \frac{c_3}{3(1+3\alpha)}.$$

Thus we have

$$\left|a_2 a_4 - a_3^2\right|$$

$$= \frac{1}{12}\left|\frac{4c_1c_3}{(1+\alpha)(1+3\alpha)} + c_1^2 c_2\left(\frac{6}{(1+\alpha)(1+3\alpha)} - \frac{6}{(1+2\alpha)^2}\right) - \frac{3c_2^2}{(1+2\alpha)^2} - c_1^4\left(\frac{3}{(1+2\alpha)^2} - \frac{2}{(1+\alpha)(1+3\alpha)}\right)\right|. \quad (2.2)$$

Substituting for c_2 and c_3 using Lemma 1.3, we obtain

$$\left|a_2a_4-a_3^2\right|$$
$$=\frac{1}{12}\left|xc_1^2(4-c_1^2)\left(\frac{5}{(1+\alpha)(1+3\alpha)}-\frac{9}{2(1+2\alpha)^2}\right)-c_1^4\left(\frac{27}{4(1+2\alpha)^2}-\frac{6}{(1+\alpha)(1+3\alpha)}\right)+\right. \tag{2.3}$$
$$\left.\frac{2(4-c_1^2)c_1(1-|x|^2)z}{(1+\alpha)(1+3\alpha)}-x^2(4-c_1^2)\left(\frac{c_1^2}{(1+\alpha)(1+3\alpha)}+\frac{3(4-c_1^2)}{4(1+2\alpha)^2}\right)\right|.$$

With $|c_1|\le 2$ from Lemma 1.1, we get c_1=c and assume without restriction that c ϵ [0, 2]. Thus applying the triangle inequality on (2.3) with ρ = $|x|\le 1$, we obtain

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{12}\left\{\rho c^2(4-c^2)\left(\frac{5}{(1+\alpha)(1+3\alpha)}-\frac{9}{2(1+2\alpha)^2}\right)+c^4\left(\frac{27}{4(1+2\alpha)^2}-\frac{6}{(1+\alpha)(1+3\alpha)}\right)\right.$$
$$\left.+\frac{2(4-c^2)c}{(1+\alpha)(1+3\alpha)}+\rho^2(4-c^2)\left(\frac{c^2-2c}{(1+\alpha)(1+3\alpha)}+\frac{3(4-c^2)}{4(1+2\alpha)^2}\right)\right\}$$
$$=F(\rho)\ . \tag{2.4}$$

Furthermore,

$$F'(\rho)=\frac{1}{12}\left\{c^2(4-c^2)\left(\frac{5}{(1+\alpha)(1+3\alpha)}-\frac{9}{2(1+2\alpha)^2}\right)+2\rho\ (4-c)\left(\frac{c^2-2c}{(1+\alpha)(1+3\alpha)}+\frac{3(4-c)}{4(1+2\alpha)^2}\right)\right\}. \tag{2.5}$$

It can be easily shown that $F'(\rho)>0$ and thus is an increasing function implying $\underset{\rho\le 1}{Max}\,F(\rho)=F(1)$. Now, let

$$G(c)=F(1)$$
$$=\frac{1}{12}\left\{c^2(4-c^2)\left(\frac{5}{(1+\alpha)(1+3\alpha)}-\frac{9}{2(1+2\alpha)^2}\right)+c^4\left(\frac{27}{4(1+2\alpha)^2}-\frac{6}{(1+\alpha)(1+3\alpha)}\right)\right. \tag{2.6}$$
$$\left.+\frac{2(4-c^2)c}{(1+\alpha)(1+3\alpha)}+x^2(4-c_1^2)\left(\frac{c^2-2c}{(1+\alpha)(1+3\alpha)}+\frac{3(4-c_1^2)}{4(1+2\alpha)^2}\right)\right\}\ .$$

Trivially one can show that G(c) has maximum attained at c =1. The upper bound for $\left|a_2a_4-a_3^2\right|$ is attained for ρ = 1 and c = 1. That is

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{(1+\alpha)(1+3\alpha)}.$$

Letting c_1= 1, c_2= -1 and c_3= -2 in (2.2), shows that the result is sharp.

Theorem 2.2 Let $f(z)$ ϵ $\boldsymbol{N}$ (λ). Then

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{8(1+\lambda)(1+3\lambda)}. \tag{2.7}$$

The result obtained is sharp.

Proof: For $f(z)$ ϵ $\boldsymbol{N}$ (λ) given by (1.1), there exists a p ϵ P such that

$$\left[\lambda z^3 f'''(z)+(1+2\lambda)z^2 f''(z)+zf'(z)\right]=\left[\lambda z^2 f''(z)+zf'(z)\right]p(z).$$

Then equating the coefficients, we obtain

$$a_2=\frac{c_1}{2(1+\lambda)}\ ,\ a_3=\frac{c_1^2+c_2}{6(1+2\lambda)},\quad a_4=\frac{c_1^3}{24(1+3\lambda)}+\frac{c_1c_2}{8(1+3\lambda)}+\frac{c_3}{12(1+3\lambda)}.$$

Thus we have

$$\left|a_2a_4-a_3^2\right|$$

$$=\frac{1}{144}\left|\frac{6c_1c_3}{(1+\lambda)(1+3\lambda)}+c_1^2c_2\left(\frac{9}{(1+\lambda)(1+3\lambda)}-\frac{8}{(1+2\lambda)^2}\right)-\frac{4c_2^2}{(1+2\lambda)^2}-c_1^4\left(\frac{4}{(1+2\lambda)^2}-\frac{3}{(1+\lambda)(1+3\lambda)}\right)\right|. \tag{2.8}$$

Substituting for c_2 and c_3 using Lemma 2.3, we obtain

$$\left|a_2a_4-a_3^2\right|=\frac{1}{144}\left|xc_1^2(4-c_1^2)\left(\frac{15}{2(1+\lambda)(1+3\lambda)}-\frac{6}{(1+2\lambda)^2}\right)-c_1^4\left(\frac{9}{(1+2\lambda)^2}-\frac{9}{(1+\lambda)(1+3\lambda)}\right)\right.$$
$$\left.+\frac{6(4-c_1^2)c_1(1-|x|^2)z}{2(1+\lambda)(1+3\lambda)}-x^2(4-c_1^2)\left(\frac{3c_1^2}{2(1+\lambda)(1+3\lambda)}+\frac{(4-c_1^2)}{(1+2\lambda)^2}\right)\right|. \tag{2.9}$$

With $|c_1|\le 2$ from Lemma 1.1, we get c_1=c and assume without restriction that c ϵ [0, 2].

Thus applying the triangle inequality on (2.9) with ρ = $|x|\le 1$, we obtain

$$\left|a_2a_4-a_3^2\right|$$
$$\le\frac{1}{144}\left\{\rho c^2(4-c^2)\left(\frac{15}{2(1+\lambda)(1+3\lambda)}-\frac{6}{(1+2\lambda)^2}\right)+c^4\left(\frac{9}{(1+2\lambda)^2}-\frac{9}{(1+\lambda)(1+3\lambda)}\right)\right.$$
$$\left.+\frac{3(4-c^2)c}{(1+\lambda)(1+3\lambda)}+\rho^2\ (4-c^2)\left(\frac{3c^2-6c}{2(1+\lambda)(1+3\lambda)}+\frac{(4-c^2)}{(1+2\lambda)^2}\right)\right\}$$

$$= F(\rho). \tag{2.10}$$

Furthermore,

$$F'(\rho) = \frac{1}{144}\left\{c^2(4-c^2)\left\{\frac{15}{2(1+\lambda)(1+3\lambda)} - \frac{6}{(1+2\alpha)^2}\right\} + 2\rho\ (4-c^2)\left\{\frac{3c^2-6c}{2(1+\lambda)(1+3\lambda)} + \frac{(4-c^2)}{(1+2\lambda)^2}\right\}\right\} \tag{2.11}$$

It can be easily shown that $F'(\rho) > 0$ and thus is an increasing function implying $\underset{\rho\le 1}{Max}\, F(\rho) = F(1)$. Now, let

$$G(c) = F(1) =$$

$$\frac{1}{144}\left\{c^2(4-c^2)\left(\frac{15}{2(1+\lambda)(1+3\lambda)} - \frac{6}{(1+2\lambda)^2}\right) + c^4\left(\frac{9}{(1+2\lambda)^2} - \frac{9}{(1+\lambda)(1+3\lambda)}\right)\right.$$
$$\left. + \frac{3(4-c^2)c}{(1+\lambda)(1+3\lambda)} + (4-c^2)\left(\frac{3c^2-6c}{2(1+\lambda)(1+3\lambda)} + \frac{(4-c^2)}{(1+2\lambda)^2}\right)\right\}.$$

Trivially one can show that G(c) has maximum attained at c =1. The upper bound for $\left|a_2a_4 - a_3^2\right|$ is attained for ρ = 1 and c = 1. That is,

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{8(1+\lambda)(1+3\lambda)}.$$

Letting c_1= 1, c_2= -1 and c_3= -2 in (2.8), shows that the result is sharp.

Remark 2.1: For α = 0 and α = 1 in Theorem 2.1 and λ = 0 in Theorem 2.2, we get the results of Aini Janteng et al. [1].

3. Application to Functions Defined by Fractional Derivatives

Let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$, $(g_n > 0)$. Since

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \boldsymbol{M}_g(\alpha) \text{ if and only if } (f*g) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in \boldsymbol{M}(\alpha)$$

$$\textit{and } f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \boldsymbol{N}_g(\lambda) \text{ if and only if } (f*g) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in \boldsymbol{N}(\lambda),$$

we obtain coefficient bounds for functions in the classes $\boldsymbol{M}_g(\alpha)$ and $\boldsymbol{N}_g(\lambda)$ from the corresponding coefficient bounds for the functions $\boldsymbol{M}$ (α) and $\boldsymbol{N}$ (λ). Applying Theorems 2.1 and Theorem 2.2 for the functions $(f*g)(z) = z + \sum_{n=2}^{\infty} g_n a_n z^n$, we get the following Theorem 3.1 and Theorem 3.2.

Theorem 3.1: If $f(z)$ given by (1.1) belongs to $\boldsymbol{M}_g(\alpha)$, then

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{g_2g_4(1+\alpha)(1+3\alpha)}. \tag{3.1}$$

The result is sharp.

Proof : The proof of this theorem is much similar to the proof of Theorem 2.1 and hence we omit the details.

Theorem 3.2: If $f(z)$ given by (1.1) belongs to $\boldsymbol{N}_g(\lambda)$, then

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{8g_2g_4(1+\lambda)(1+3\lambda)}. \tag{3.2}$$

The result is sharp.

Proof: The proof of this theorem is much similar to the proof of Theorem 2.2 and hence we omit the details.

$$Since\ \ (\Omega^{\gamma} f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\gamma)}{\Gamma(n+1-\gamma)} a_n z^n, \text{ we have}$$

$$g_2 = \frac{\Gamma(3)\Gamma(2-\gamma)}{\Gamma(3-\gamma)} = \frac{2}{2-\gamma}, \tag{3.3}$$

$$g_3 = \frac{\Gamma(4)\Gamma(2-\gamma)}{\Gamma(4-\gamma)} = \frac{6}{(2-\gamma)(3-\gamma)}, \tag{3.4}$$

and

$$g_4 = \frac{\Gamma(5)\Gamma(2-\gamma)}{\Gamma(6-\gamma)} = \frac{24}{(6-\gamma)(5-\gamma)(4-\gamma)(3-\gamma)}. \tag{3.5}$$

For g_2 and g_4 given by (3.3) and (3.5), Theorems 3.1 and 3.2 reduces to the following :

Theorem 3.3 :

If $f(z)$ given by (1.1) belongs to $\boldsymbol{M}_g(\alpha)$, then

$$\left|a_2a_4 - a_3^2\right| \le \frac{(2-\gamma)(3-\gamma)(4-\gamma)(5-\gamma)(6-\gamma)}{48(1+\alpha)(1+3\alpha)}. \tag{3.6}$$

The result is sharp.

Theorem 3.4 :

If $f(z)$ given by (1.1) belongs to $N_g(\lambda)$, then

$$\left|a_2 a_4 - a_3^2\right| \le \frac{(2-\gamma)(3-\gamma)(4-\gamma)(5-\gamma)(6-\gamma)}{384(1+\alpha)(1+3\alpha)}. \tag{3.7}$$

The result is sharp.

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