SECOND HANKEL DETERMINANT FOR SOME SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract

Sharp bounds for the functional \( \left| a_2 a_4 - a_3^2 \right| \) are derived for functions belonging to the classes \( M (\alpha) \) and \( N (\lambda) \). Also certain application of the main results for a class of functions defined by convolution is given. As a special case, the coefficient bounds for a class of functions defined through fractional derivatives are obtained.

Key Words: Analytic functions, Starlike functions, convex functions, Hankel determinant, Fractional derivative.

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1 INTRODUCTION

Let \( A \) denote the class of normalized analytic Univalent functions \( f \) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

(1.1)

Where \( z \in D = \{z : |z| < 1\} \). In [10], the \( q^{th} \) Hankel determinant for \( q \geq 1 \) and \( n \geq 0 \) is stated by Noonan and Thomas as
This determinant has also been considered by several authors. For example, Noor in [11] determined the rate of growth of \( H_q(n) \) as \( n \to \infty \) for functions \( f \) given by (1.1) with bounded boundary. Ehrenborg in [3] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Leyman in [7].

Easily, one can observe that the Fekete and Szegő functional is \( H_2(1) \). Fekete and Szegő [4] then further generalized the estimate \( \left| a_3 - \mu a_2^2 \right| \) where \( \mu \) is real and \( f \in A \). For our discussion in this paper, we consider the Hankel determinant in the case \( q = 2 \) and \( n = 2 \), which is

\[
H_2(n) = \begin{vmatrix}
a_n & a_{n+1} & \ldots & a_{n+q-1} \\
a_{n+1} & \ldots & \ldots & \\
\ldots & \ldots & \ldots & \\
a_{n+q-1} & \ldots & \ldots & a_{n+2q-2} \\
\end{vmatrix}.
\]

A function \( f \in A \) is said to be starlike and convex respectively if and only if, for \( z \in D \),

\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad \text{and} \quad \Re \left\{ \frac{(zf'(z))'}{f'(z)} \right\} > 0.
\]

By usual notations, we denote these classes of functions respectively by \( S^* \) and \( C \).

In the present paper, we obtain an upper bound for the functional \( |a_3 a_4 - a_5^2| \) for functions belongs to the class \( M(\alpha) \) and \( N(\lambda) \), which are defined as follows.

**Definition 1.1** Let \( f \) be given by (1.1). Then \( f \in M(\alpha) \) if and only if

\[
\Re \left\{ \frac{zf'(z) + \alpha z^2 f''(z)}{(1 - \alpha) f(z) + \alpha zf'(z)} \right\} > 0, \quad 0 \leq \alpha \leq 1, \quad z \in D.
\]

(1.2)

We see that \( M(0) = S^* \), the class of starlike functions and \( M(1) = C \), the class of convex functions.

**Definition 1.2** : Let \( f \) be given by (1.1). Then \( f \in N(\lambda) \) if and only if

\[
\Re \left\{ \frac{\lambda z^3 f'''(z) + (1 + 2\lambda)z^2 f''(z) + zf'(z)}{\lambda z^2 f''(z) + zf'(z)} \right\} > 0, \quad 0 \leq \lambda \leq 1, \quad z \in D.
\]

(1.3)

We note that \( N(0) = C \), the class of convex functions.
The definitions of the classes \( M(\alpha) \) and \( N(\lambda) \) are motivated by the classes studied by \([2, 17]\) and \([6, 18]\) respectively.

For fixed \( g \in A \), we define the class \( M_g(\alpha) \) to be the class of functions \( f \in A \) for which \((f \ast g) \in M(\alpha)\) and the class \( N_g(\lambda) \) to be the class of functions \( f \in A \) for which \((f \ast g) \in N(\lambda)\).

In order to introduce the above mentioned classes we need the following:

**Definition 1.3** (See \([10,11]\); see also \([13,14]\)) Let the function \( g(\zeta) \) be analytic in a simply connected region of the \( \zeta \)-plane containing the origin. Then the fractional derivative of \( f \) of order \( \gamma \) is defined by

\[
D_\gamma f(z) = \frac{1}{\Gamma(1-\gamma)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\gamma}} d\zeta, \quad (0 \leq \gamma < 1). \tag{1.4}
\]

Where the multiplicity of \( (z-\zeta)^\gamma \) is removed by requiring \( \log(z-\zeta) \) to be real when \( (z-\zeta) > 0 \).

Using the above definition (1.3) and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava \([13]\) introduced the operator

\[ \Omega^\gamma : A \rightarrow A \]

defined by \((\Omega^\gamma f)(z) = \Gamma(2-\gamma)z^\gamma D_\gamma f(z), \quad (\gamma \neq 2, 3, 4, \ldots)\).

The class \( M_\gamma(\alpha) \) consists of functions \( f \in A \) for which \( \Omega^\gamma f \in M(\alpha) \) and the class \( N_\gamma(\lambda) \) consists of functions \( f \in A \) for which \( \Omega^\gamma f \in M(\alpha) \). \( M_\gamma(\alpha) \) and \( N_\gamma(\lambda) \) are the special cases of the classes \( M_g(\alpha) \) and \( N_g(\lambda) \) when \( g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\gamma)}{\Gamma(n+1-\gamma)} \frac{z^n}{n} \).

Let \( P \) be the family of all functions \( p \) analytic in \( D \) for which \( \Re\{p(z)\} > 0 \) and

\[ p(z) = 1 + c_1 z + c_2 z^2 + \ldots, \quad z \in D. \tag{1.5} \]

To prove our main result, we need the following:

**Lemma 1.1** (\([14]\)) If \( p \in P \) then \( |c_k| \leq 2 \) for each \( k \).

**Lemma 1.2** (\([5]\)) The power series for \( p(z) \) given by (1.5) converges in \( D \) to a function in \( P \) if and only if the Toeplitz determinants

\[
D_n = \begin{vmatrix}
2 & c_1 & c_2 & \ldots & c_n \\
c_{-1} & 2 & c_1 & \ldots & c_{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_{-n} & c_{-n+1} & c_{-n+1} & \ldots & 2
\end{vmatrix}, \quad n = 1, 2, 3, \ldots \tag{1.6}
\]
and \( c_{-k} = \overline{c_k} \) are all nonnegative. They are strictly positive except for

\[
p(z) = \sum_{k=1}^{m} \rho_k \rho_0 e^{i \theta k z}, \quad \rho_k > 0, \quad t_k \text{ real}
\]

and \( t_k \neq t_j \) for \( k \neq j \), in this case \( D_n > 0 \) for \( n < m - 1 \) and \( D_n = 0 \) for \( n \geq m \).

**Lemma 1.3** ([8,9]) Let the function \( p \in P \) be given by power series (1.5).

Then

\[
2c_2 = c_1^2 + x(4 - c_1^2) \quad (1.7)
\]

for some \( x \), \( |x| \leq 1 \) and

\[
4c_2 = c_1^2 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z. \quad (1.8)
\]

**2. Main Results**

**Theorem 2.1** Let \( f(z) \in M(\alpha) \). Then

\[
|a_2a_4 - a_3^2| \leq \frac{1}{(1 + \alpha)(1 + 3\alpha)}. \quad (2.1)
\]

The result obtained is sharp.

**Proof:** For \( f(z) \in M(\alpha) \) given by (1.1), there exists a \( p \in P \) such that

\[
\left[ zf'(z) + \alpha z^2 f''(z) \right] = \left[ (1 - \alpha)f(z) + \alpha z f'(z) \right] p(z).
\]

Then equating the coefficients, we obtain

\[
a_2 = \frac{c_1}{1 + \alpha}, \quad a_3 = \frac{c_1^2 + c_2}{2(1 + 2\alpha)}, \quad a_4 = \frac{c_1^3}{6(1 + 3\alpha)} + \frac{c_1c_2}{2(1 + 3\alpha)} + \frac{c_3}{3(1 + 3\alpha)}.
\]

Thus we have

\[
|a_2a_4 - a_3^2| \leq \frac{1}{12} \left[ \frac{4c_1c_3}{(1 + \alpha)(1 + 3\alpha)} + c_1^2c_2 \left( \frac{6}{(1 + \alpha)(1 + 3\alpha)} - \frac{6}{(1 + 2\alpha)^2} \right) - \frac{3c_2^2}{(1 + 2\alpha)^2} - c_1^4 \left( \frac{3}{(1 + 2\alpha)^2} - \frac{2}{(1 + \alpha)(1 + 3\alpha)} \right) \right]. \quad (2.2)
\]

Substituting for \( c_2 \) and \( c_3 \) using Lemma 1.3, we obtain
\[ \left| a_2 a_4 - a_3^2 \right| \]
\[ = \frac{1}{12} \left| x c_1^2 (4 - c_1^2) \left( \frac{5}{(1 + \alpha)(1 + 3 \alpha)} - \frac{9}{2(1 + 2 \alpha)^2} \right) - c_1^4 \left( \frac{27}{4(1 + 2 \alpha)^2} - \frac{6}{(1 + \alpha)(1 + 3 \alpha)} \right) + \frac{2(4 - c_1^2) c_3 (1 - |x|^2) z}{(1 + \alpha)(1 + 3 \alpha)} - x^2 (4 - c_1^2) \left( \frac{c_1^2}{(1 + \alpha)(1 + 3 \alpha)} + \frac{3(4 - c_1^2)}{4(1 + 2 \alpha)^2} \right) \right| . \] (2.3)

With \( |c_1| \leq 2 \) from Lemma 1.1, we get \( c_1 = c \) and assume without restriction that \( c \in [0, 2] \). Thus applying the triangle inequality on (2.3) with \( \rho = |x| \leq 1 \), we obtain

\[ \left| a_2 a_4 - a_3^2 \right| \leq \frac{1}{12} \left\{ \rho c^2 (4 - c^2) \left( \frac{5}{(1 + \alpha)(1 + 3 \alpha)} - \frac{9}{2(1 + 2 \alpha)^2} \right) + c^4 \left( \frac{27}{4(1 + 2 \alpha)^2} - \frac{6}{(1 + \alpha)(1 + 3 \alpha)} \right) + \frac{2(4 - c^2) \rho c^3 (1 - |x|^2) z}{(1 + \alpha)(1 + 3 \alpha)} + x^2 (4 - c^2) \left( \frac{c^2 - 2c}{(1 + \alpha)(1 + 3 \alpha)} + \frac{3(4 - c^2)}{4(1 + 2 \alpha)^2} \right) \right\} = F(\rho) . \] (2.4)

Furthermore,

\[ F'(\rho) = \frac{1}{12} \left\{ c^2 (4 - c^2) \left( \frac{5}{(1 + \alpha)(1 + 3 \alpha)} - \frac{9}{2(1 + 2 \alpha)^2} \right) + 2 \rho (4 - c^2) \left( \frac{c^2 - 2c}{(1 + \alpha)(1 + 3 \alpha)} + \frac{3(4 - c^2)}{4(1 + 2 \alpha)^2} \right) \right\} . \] (2.5)

It can be easily shown that \( F'(\rho) > 0 \) and thus is an increasing function implying \( \max_{\rho \leq 1} F(\rho) = F(1) \). Now, let

\[ G(c) = F(1) \]
\[ = \frac{1}{12} \left\{ c^2 (4 - c^2) \left( \frac{5}{(1 + \alpha)(1 + 3 \alpha)} - \frac{9}{2(1 + 2 \alpha)^2} \right) + c^4 \left( \frac{27}{4(1 + 2 \alpha)^2} - \frac{6}{(1 + \alpha)(1 + 3 \alpha)} \right) + \frac{2(4 - c^2) \rho c^3 (1 - |x|^2) z}{(1 + \alpha)(1 + 3 \alpha)} + x^2 (4 - c^2) \left( \frac{c^2 - 2c}{(1 + \alpha)(1 + 3 \alpha)} + \frac{3(4 - c^2)}{4(1 + 2 \alpha)^2} \right) \right\} . \] (2.6)

Trivially one can show that \( G(c) \) has maximum attained at \( c = 1 \). The upper bound for \( \left| a_2 a_4 - a_3^2 \right| \) is attained for \( \rho = 1 \) and \( c = 1 \). That is

\[ \left| a_2 a_4 - a_3^2 \right| \leq \frac{1}{(1 + \alpha)(1 + 3 \alpha)}. \]

Letting \( c_1 = 1, c_2 = -1 \) and \( c_3 = -2 \) in (2.2), shows that the result is sharp.

**Theorem 2.2** Let \( f(z) \in N(\lambda) \). Then
\[ \left| a_2a_4 - a_1^2 \right| \leq \frac{1}{8(1 + \lambda)(1 + 3\lambda)}. \] (2.7)

The result obtained is sharp.

**Proof:** For \( f(z) \in N(\lambda) \) given by (1.1), there exists a \( p \in P \) such that
\[
\left[ \hat{\lambda}z^2f^{\prime\prime}(z) + (1 + 2\hat{\lambda})z^2f^{\prime\prime\prime}(z) + zf^{\prime\prime\prime\prime}(z) \right] = \left[ \hat{\lambda}z^2f^{\prime\prime}(z) + zf^{\prime\prime\prime}(z) \right]p(z).
\]

Then equating the coefficients, we obtain
\[
a_2 = \frac{c_1}{2(1 + \lambda)}, \quad a_3 = \frac{c_1^2 + c_2}{6(1 + 2\lambda)}, \quad a_4 = \frac{c_1^3 + c_1c_2}{24(1 + 3\lambda)} + \frac{c_1c_2}{8(1 + 3\lambda)} + \frac{c_3}{12(1 + 3\lambda)}.
\]

Thus we have
\[
\left| a_2a_4 - a_1^2 \right| = \frac{1}{144} \left| \frac{6c_1c_3}{(1 + \lambda)(1 + 3\lambda)} + c_1^2c_2 \left( \frac{9}{(1 + \lambda)(1 + 3\lambda)} - \frac{8}{(1 + 2\lambda)^2} \right) - \frac{4c_2^2}{(1 + 2\lambda)^2} - c_1^4 \left( \frac{4}{(1 + 2\lambda)^2} - \frac{3}{(1 + \lambda)(1 + 3\lambda)} \right) \right|.
\] (2.8)

Substituting for \( c_2 \) and \( c_3 \) using Lemma 2.3, we obtain
\[
\left| a_2a_4 - a_1^2 \right| = \frac{1}{144} \left| x^2(4 - c_1^2) \right| \left[ \frac{15}{2(1 + \lambda)(1 + 3\lambda)} - \frac{6}{(1 + 2\lambda)^2} \right] - c_1^4 \left( \frac{9}{(1 + 2\lambda)^2} - \frac{9}{(1 + \lambda)(1 + 3\lambda)} \right) + \frac{6(4 - c_1^2)c_1(1 - |x|^2)z}{2(1 + \lambda)(1 + 3\lambda)} - x^2(4 - c_1^2) \left( \frac{3c_1^2}{2(1 + \lambda)(1 + 3\lambda)} + \frac{(4 - c_1^2)}{(1 + 2\lambda)^2} \right) \right|.
\] (2.9)

With \( |c_1| \leq 2 \) from Lemma 1.1, we get \( c_1 = c \) and assume without restriction that \( c \in [0, 2] \).

Thus applying the triangle inequality on (2.9) with \( \rho = |x| \leq 1 \), we obtain
\[
\left| a_2a_4 - a_1^2 \right| \leq \frac{1}{144} \left\{ \rho c^2 (4 - c^2) \left( \frac{15}{2(1 + \lambda)(1 + 3\lambda)} - \frac{6}{(1 + 2\lambda)^2} \right) + c^4 \left( \frac{9}{(1 + 2\lambda)^2} - \frac{9}{(1 + \lambda)(1 + 3\lambda)} \right) + \frac{3(4 - c^2)c}{(1 + \lambda)(1 + 3\lambda)} + \rho^2 (4 - c^2) \left( \frac{3c^2 - 6c}{2(1 + \lambda)(1 + 3\lambda)} + \frac{(4 - c^2)}{(1 + 2\lambda)^2} \right) \right\}.
\]
Furthermore,

\[ F'(\rho) = \frac{1}{144} \left\{ c^2 (4-c^2)^2 \left( \frac{15}{2(1+\lambda)(1+3\lambda)} - \frac{6}{(1+2\lambda)^2} \right) + 2\rho \left( \frac{3c^2 - 6c}{2(1+\lambda)(1+3\lambda)} + \frac{(4-c^2)}{(1+2\lambda)^2} \right) \right\} \quad (2.11) \]

It can be easily shown that \( F'(\rho) > 0 \) and thus is an increasing function implying \( \max_{\rho \leq 1} F(\rho) = F(1) \). Now, let

\[ G(c) = F(1) = \frac{1}{144} \left\{ c^2 (4-c^2)^2 \left( \frac{15}{2(1+\lambda)(1+3\lambda)} - \frac{6}{(1+2\lambda)^2} \right) + c^4 \left( \frac{9}{(1+2\lambda)^2} - \frac{9}{(1+\lambda)(1+3\lambda)} \right) \right\} \]

+ \( \frac{3(4-c^2)c}{(1+\lambda)(1+3\lambda)} + (4-c^2) \left\{ \frac{3c^2 - 6c}{2(1+\lambda)(1+3\lambda)} + \frac{(4-c^2)}{(1+2\lambda)^2} \right\} \cdot \)

Trivially one can show that \( G(c) \) has maximum attained at \( c = 1 \). The upper bound for \( |a_2a_4 - a_3^2| \) is attained for \( \rho = 1 \) and \( c = 1 \). That is,

\[ |a_2a_4 - a_3^2| \leq \frac{1}{8(1+\lambda)(1+3\lambda)}. \]

Letting \( c_1 = 1, c_2 = -1 \) and \( c_3 = -2 \) in (2.8), shows that the result is sharp.

**Remark 2.1:** For \( \alpha = 0 \) and \( \alpha = 1 \) in Theorem 2.1 and \( \lambda = 0 \) in Theorem 2.2, we get the results of Aini Janteng et al. [1].

### 3. Application to Functions Defined by Fractional Derivatives

Let \( g(z) = z + \sum_{n=2}^{\infty} g_n z^n \), \( (g_n > 0) \). Since

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M_g(\alpha) \quad \text{if and only if} \quad (f^* g) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in M(\alpha) \]

and \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in N_g(\lambda) \quad \text{if and only if} \quad (f^* g) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in N(\lambda), \)

we obtain coefficient bounds for functions in the classes \( M_g(\alpha) \) and \( N_g(\lambda) \) from the corresponding coefficient bounds for the functions \( M(\alpha) \) and \( N(\lambda) \). Applying Theorems 2.1 and Theorem 2.2 for the functions \( (f^* g)(z) = z + \sum_{n=2}^{\infty} g_n a_n z^n \), we get the following Theorem 3.1 and Theorem 3.2.
Theorem 3.1: If \( f(z) \) given by (1.1) belongs to \( M_\alpha \), then

\[
|a_2a_4 - a_3^2| \leq \frac{1}{g_2g_4(1 + \alpha)(1 + 3\alpha)}.
\] (3.1)

The result is sharp.

Proof: The proof of this theorem is much similar to the proof of Theorem 2.1 and hence we omit the details.

Theorem 3.2: If \( f(z) \) given by (1.1) belongs to \( N_\lambda \), then

\[
|a_2a_4 - a_3^2| \leq \frac{1}{8g_2g_4(1 + \lambda)(1 + 3\lambda)}.
\] (3.2)

The result is sharp.

Proof: The proof of this theorem is much similar to the proof of Theorem 2.2 and hence we omit the details.

Since \( (\Omega f)(z) = z + \sum_{n=2}^\infty \frac{\Gamma(n+1)\Gamma(2-\gamma)}{\Gamma(n+1-\gamma)}a_n z^n \), we have

\[
g_2 = \frac{\Gamma(3)\Gamma(2-\gamma)}{\Gamma(3-\gamma)} = \frac{2}{2 - \gamma},
\] (3.3)

\[
g_3 = \frac{\Gamma(4)\Gamma(2-\gamma)}{\Gamma(4-\gamma)} = \frac{6}{(2 - \gamma)(3 - \gamma)},
\] (3.4)

and

\[
g_4 = \frac{\Gamma(5)\Gamma(2-\gamma)}{\Gamma(6-\gamma)} = \frac{24}{(6 - \gamma)(5 - \gamma)(4 - \gamma)(3 - \gamma)}.\] (3.5)

For \( g_2 \) and \( g_4 \) given by (3.3) and (3.5), Theorems 3.1 and 3.2 reduces to the following:

Theorem 3.3:

If \( f(z) \) given by (1.1) belongs to \( M_\alpha \), then

\[
|a_2a_4 - a_3^2| \leq \frac{(2 - \gamma)(3 - \gamma)(4 - \gamma)(5 - \gamma)(6 - \gamma)}{48(1 + \alpha)(1 + 3\alpha)}.
\] (3.6)

The result is sharp.

Theorem 3.4:
If \( f(z) \) given by (1.1) belongs to \( N_{g}(\lambda) \), then

\[
\left| a_{n}a_{q} - a_{s}^{2} \right| \leq \frac{(2-\gamma)(3-\gamma)(4-\gamma)(5-\gamma)(6-\gamma)}{384(1+\alpha)(1+3\alpha)}.
\]

(3.7)

The result is sharp.

REFERENCES


