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Eulerian integral associated with product of two multivariable I-functions

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ABSTRACT
The present paper is evaluated a new Eulerian integral associated with the product of two multivariable I-functions with general arguments . Several particular cases are given .

Keywords: Eulerian integral, multivariable I-function, generalized Lauricella function of several variables, multivariable H-function, generalized hypergeometric function.

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1. Introduction

In this paper, we evaluate a new Eulerian integral of most general characters associated with the products of two multivariable I-functions defined by Prasad [1] but of greater order. Several particular cases are given.

The multivariable I-function of r-variables is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1,\cdots,z_r) = I^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;\cdots;p_r,q_r:p^{(1)},q^{(1)};\cdots;p^{(r)},q^{(r)}}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};\cdots; \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};\cdots; \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj})_{1,p_r}:(a^{(1)}_j,\alpha^{(1)}_j)_{1,p^{(1)}};\cdots;(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}} \\ \\ (\mathrm{b}_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj})_{1,q_r}:(b^{(1)}_j,\beta^{(1)}_j)_{1,q^{(1)}};\cdots;(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}} \end{array}\right) \tag{1.1}$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.2}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [1]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_i| < \frac{1}{2}\Omega_i\pi$, where

$$\Omega_i = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)} \right) + \cdots +$$

$$\left(\sum_{k=1}^{n_s} \alpha_{sk}^{(i)} - \sum_{k=n_s+1}^{p_s} \alpha_{sk}^{(i)} \right) - \left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_s} \beta_{sk}^{(i)} \right) \tag{1.3}$$

where $i = 1, \cdots, r$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$$I(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$$

$$I(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$$

where $k = 1, \cdots, r : \alpha'_k = min[Re(b_j^{(k)}/\beta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta'_k = max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \cdots, n_k$$

We will use these following notations in this section :

$$U_r = p_2, q_2; p_3, q_3; \cdots; p_{r-1}, q_{r-1}; V_r = 0, n_2; 0, n_3; \cdots; 0, n_{r-1} \tag{1.4}$$

$$W_r = (p^{(1)}, q^{(1)}); \cdots; (p^{(r)}, q^{(r)}); X_r = (m^{(1)}, n^{(1)}); \cdots; (m^{(r)}, n^{(r)}) \tag{1.5}$$

$$A = (a_{2k}; \alpha_{2k}^{(1)}, \alpha_{2k}^{(2)}); \cdots; (a_{(r-1)k}; \alpha_{(r-1)k}^{(1)}, \alpha_{(r-1)k}^{(2)}, \cdots, \alpha_{(r-1)k}^{(r-1)}) \tag{1.6}$$

$$B = (b_{2k}; \beta_{2k}^{(1)}, \beta_{2k}^{(2)}); \cdots; (b_{(r-1)k}; \beta_{(r-1)k}^{(1)}, \beta_{(r-1)k}^{(2)}, \cdots, \beta_{(r-1)k}^{(r-1)}) \tag{1.7}$$

$$\mathfrak{A} = (a_{rk}; \alpha_{rk}^{(1)}, \alpha_{rk}^{(2)}, \cdots, \alpha_{rk}^{(r)}) : \mathfrak{B} = (b_{rk}; \beta_{rk}^{(1)}, \beta_{rk}^{(2)}, \cdots, \beta_{rk}^{(r)}) \tag{1.8}$$

$$A_1 = (a_k^{(1)}, \alpha_k^{(1)})_{1,p^{(1)}}; \cdots; (a_k^{(r)}, \alpha_k^{(r)})_{1,p^{(r)}}; B_1 = (b_k^{(1)}, \beta_k^{(1)})_{1,p^{(1)}}; \cdots; (b_k^{(r)}, \beta_k^{(r)})_{1,p^{(r)}} \tag{1.9}$$

The multivariable I-function of r-variables write :

$$I(z_1, \cdots, z_r) = I_{U_r:p_r,q_r;W_r}^{V_r;0,n_r;X_r} \left(\begin{array}{c|c} z_1 & A ; \mathfrak{A}; A_1 \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r & B; \mathfrak{B}; B_1 \end{array} \right) \tag{1.10}$$

$$I(z'_1, z'_2, ...z'_s) = I^{0,n'_2;0,n'_3;\cdots;0,n'_r:m'^{(1)},n'^{(1)};\cdots;m'^{(s)},n'^{(s)}}_{p'_2,q'_2,p'_3,q'_3;\cdots;p'_s,q'_s:p'^{(1)},q'^{(1)};\cdots;p'^{(s)},q'^{(s)}} \left(\begin{array}{c|c} \text{z'}_1 & (\text{a'}_{2j};\alpha'^{(1)}_{2j},\alpha'^{(2)}_{2j})_{1,p'_2};\cdots; \\ \cdot & \\ \cdot & \\ \cdot & \\ \text{z'}_s & (\text{b'}_{2j};\beta'^{(1)}_{2j},\beta'^{(2)}_{2j})_{1,q'_2};\cdots; \end{array} \right.$$

$$\left. \begin{array}{c} (\text{a'}_{sj};\alpha'^{(1)}_{sj},\cdots,\alpha'_{sj}{}^{(s)})_{1,p'_s} : (a'^{(1)}_j,\alpha'^{(1)}_j)_{1,p'^{(1)}};\cdots;(a'_j{}^{(s)},\alpha'_j{}^{(s)})_{1,p'^{(s)}} \\ \\ (\text{b'}_{sj};\beta'^{(1)}_{sj},\cdots,\beta'_{sj}{}^{(s)})_{1,q'_s} : (b'^{(1)}_j,\beta'^{(1)}_j)_{1,q'^{(1)}};\cdots;(b'_j{}^{(s)},\beta'_j{}^{(s)})_{1,q'^{(s)}} \end{array} \right) \tag{1.11}$$

$$= \frac{1}{(2\pi\omega)^s} \int_{L_1} \cdots \int_{L_s} \psi(t_1,\cdots,t_s) \prod_{i=1}^{s} \xi_i(t_i) z_i^{t_i} \mathrm{d}t_1 \cdots \mathrm{d}t_s \tag{1.12}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [1]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

where $|arg z'_i| < \frac{1}{2}\Omega'_i \pi$,

$$\Omega'_i = \sum_{k=1}^{n'^{(i)}} \alpha'_k{}^{(i)} - \sum_{k=n'^{(i)}+1}^{p'^{(i)}} \alpha'_k{}^{(i)} + \sum_{k=1}^{m'^{(i)}} \beta'_k{}^{(i)} - \sum_{k=m^{(i)}+1}^{q'^{(i)}} \beta'_k{}^{(i)} + \left(\sum_{k=1}^{n'_2} \alpha'_{2k}{}^{(i)} - \sum_{k=n_2+1}^{p'_2} \alpha'_{2k}{}^{(i)} \right)$$

$$+ \cdots + \left(\sum_{k=1}^{n'_s} \alpha'_{sk}{}^{(i)} - \sum_{k=n'_s+1}^{p'_s} \alpha'_{sk}{}^{(i)} \right) - \left(\sum_{k=1}^{q'_2} \beta'_{2k}{}^{(i)} + \sum_{k=1}^{q'_3} \beta'_{3k}{}^{(i)} + \cdots + \sum_{k=1}^{q'_s} \beta'_{sk}{}^{(i)} \right) \tag{1.13}$$

where $i = 1, \cdots, s$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$$I(z'_1,\cdots,z'_s) = 0(\,|z'_1|^{\alpha'_1},\cdots,|z'_s|^{\alpha'_s}\,)\,,max(\,|z'_1|\,,\cdots,|z'_s|\,) \rightarrow 0$$

$$I(z'_1,\cdots,z'_s) \;= 0(\,|z'_1|^{\beta'_1}\;,\cdots,|z'_s|^{\beta'_s}\,)\,,min(\,|z'_1|\,,\cdots,|z'_s|\,) \rightarrow \infty$$

where $k = 1,\cdots,z : \alpha''_k = min[Re(b'^{(k)}_j/\beta'^{(k)}_j)], j = 1,\cdots,m'_k$ and

$$\beta''_k = max[Re((a'^{(k)}_j - 1)/\alpha'^{(k)}_j)], j = 1,\cdots,n'_k$$

We will use these following notations in this section :

$$U_s = p'_2, q'_2; p'_3, q'_3; \cdots ; p'_{s-1}, q'_{s-1}; V_s = 0, n_2; 0, n_3; \cdots ; 0, n_{s-1} \tag{1.14}$$

$$W_s = (p'^{(1)}, q'^{(1)}); \cdots ; (p'^{(s)}, q'^{(s)}); X_s = (m'^{(1)}, n'^{(1)}); \cdots ; (m'^{(s)}, n'^{(s)}) \tag{1.15}$$

$$A' = (a'_{2k}; \alpha'^{(1)}_{2k}, \alpha'^{(2)}_{2k}); \cdots ; (a'_{(s-1)k}; \alpha'^{(1)}_{(s-1)k}, \alpha'^{(2)}_{(s-1)k}, \cdots , \alpha'^{(s-1)}_{(s-1)k}) \tag{1.16}$$

$$B' = (b'_{2k}; \beta'^{(1)}_{2k}, \beta'^{(2)}_{2k}); \cdots ; (b'_{(s-1)k}; \beta'^{(1)}_{(s-1)k}, \beta'^{(2)}_{(s-1)k}, \cdots , \beta'^{(s-1)}_{(s-1)k}) \tag{1.17}$$

$$\mathfrak{A}' = (a'_{sk}; \alpha'^{(1)}_{sk}, \alpha'^{(2)}_{sk}, \cdots , \alpha'^{(s)}_{sk}) : \mathfrak{B}' = (b'_{sk}; \beta'^{(1)}_{sk}, \beta'^{(2)}_{sk}, \cdots , \beta'^{(s)}_{sk}) \tag{1.18}$$

$$A'_1 = (a'^{(1)}_k, \alpha^{(1)}_k)_{1,p^{(1)}}; \cdots ; (a'^{(s)}_k, \alpha'^{(s)}_k)_{1,p'^{(s)}}; B'_1 = (b'^{(1)}_k, \beta'^{(1)}_k)_{1,p'^{(1)}}; \cdots ; (b'^{(s)}_k, \beta'^{(s)}_k)_{1,p'^{(s)}} \tag{1.19}$$

The multivariable I-function write :

$$I(z'_1, \cdots , z'_s) = I^{V_s;0,n'_s;X_s}_{U_s:p'_s,q'_s;W_s} \left(\begin{array}{c|c} z'_1 & A' ;\mathfrak{A}';A'_1 \\ \cdot & \\ \cdot & \\ \cdot & \\ z'_s & B'; \mathfrak{B}'; B'_1 \end{array} \right) \tag{1.20}$$

2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [6 ,page 39 eq .30]

$$\frac{\prod_{j=1}^{P}\Gamma(A_j)}{\prod_{j=1}^{Q}\Gamma(B_j)}\, {}_PF_Q\left[(A_P);(B_Q);-(x_1+\cdots+x_r)\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\frac{\prod_{j=1}^{P}\Gamma(A_j+s_1+\cdots+s_r)}{\prod_{j=1}^{Q}\Gamma(B_j+s_1+\cdots+s_r)}\Gamma(-s_1)\cdots\Gamma(-s_r)x_1^{s_1}\cdots x_r^{s_r}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.1}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j + s_1 + \cdots + s_r)$ are separated from those of $\Gamma(-s_j), j = 1, \cdots , r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j = 1, \cdots , r$

In order to evaluate a number of integrals of multivariable I-function, we first establish the formula

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}\mathrm{d}t \ = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}$$

$$F^{1:1,\cdots,1;1\cdots,1}_{1:0,\cdots,0;0,\cdots,0}\left(\begin{array}{c}(\alpha:h_1,\cdots,h_l,1,\cdots,1):(\lambda_1:1),\cdots,(\lambda_l:1);(-\sigma_1:1),\cdots,(-\sigma_k:1)\\ \cdot\ \cdot\ \cdot\\ (\alpha+\beta:h_1,\cdots,h_l,1,\cdots,1):-,\cdots,-;-,\cdots,-\end{array}\right.$$

$$\left.;\tau_1(b-a)^{h_1},\cdots,\tau_l(b-a)^{h_l},-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right) \tag{2.2}$$

where $a,b\in\mathbb{R}(a<b),\alpha,\beta,f_i,g_i,\sigma_i,\tau_j,h_j\in\mathbb{C},\lambda_j\in\mathbb{R}^+(i=1,\cdots,k;j=1,\cdots,l)$

$$min(Re(\alpha),Re(\beta))>0,\ \max_{1\leqslant j\leqslant l}\left\{\left|\tau_j(b-a)^{h_j}\right|\right\}<1,\ \max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1,$$

and $F^{1:1,\cdots,1;1\cdots,1}_{1:0,\cdots,0;0,\cdots,0}$ is a particular case of the generalized Lauricella function introduced by Srivastava-Daoust[4,page 454] and [5] given by :

$$F^{1:1,\cdots,1;1\cdots,1}_{1:0,\cdots,0;0,\cdots,0}\left(\begin{array}{c}(\alpha:h_1,\cdots,h_l,1,\cdots,1):(\lambda_1:1),\cdots,(\lambda_l:1);(-\sigma_1:1),\cdots,(-\sigma_k:1)\\ \cdot\ \cdot\ \cdot\\ (\alpha+\beta:h_1,\cdots,h_l,1,\cdots,1):-,\cdots,-;-,\cdots,-\end{array}\right.$$

$$\left.;\tau_1(b-a)^{h_1},\cdots,\tau_l(b-a)^{h_l},-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\prod_{j=1}^{l}\Gamma(\lambda_j)\prod_{j=1}^{k}\Gamma(-\sigma_j)}$$

$$\frac{1}{(2\pi\omega)^{l+k}}\int_{L_1}\cdots\int_{L_{l+k}}\frac{\Gamma\left(\alpha+\sum_{j=1}^{l}h_jw_j+\sum_{j=1}^{k}w_{l+j}\right)}{\Gamma\left(\alpha+\beta+\sum_{j=1}^{l}h_jw_j+\sum_{j=1}^{k}w_{l+j}\right)}\prod_{j=1}^{l}\Gamma(\lambda_j+w_j)\prod_{j=1}^{k}\Gamma(-\sigma_j+w_{l+j})$$

$$\prod_{j=1}^{l+k}\Gamma(-w_j)z_1^{w_1}\cdots z_l^{w_l}z_{l+1}^{w_{l+1}}\cdots,w_{l+k}^{w_{l+k}}\,\mathrm{d}w_1\cdots\mathrm{d}w_{l+k} \tag{2.3}$$

Here the contour $L_j's$ are defined by $L_j=L_{w\zeta_j\infty}(Re(\zeta_j)=v_j'')$ starting at the point $v_j''-\omega\infty$ and terminating at the point $v_j''+\omega\infty$ with $v_j''\in\mathbb{R}(j=1,\cdots,l)$ and each of the remaining contour $L_{l+1},\cdots,L_{l+k}$ run from $-\omega\infty$ to $\omega\infty$

(2.2) can be easily established by expanding $\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}$ by means of the formula :

$$(1-z)^{-\alpha}=\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}z^r(|z|<1) \tag{2.4}$$

integrating term by term with the help of the integral given by Saigo and Saxena [2, page 93, eq.(3.2)] and applying the definition of the generalized Lauricella function [4, page 454].

3. Eulerian integral

In this section , we denote :

$$\theta_i = \prod_{j=1}^{l} \left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{(i)}} \ (i=1,\cdots,r)\,;\theta'_i = \prod_{j=1}^{l} \left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j'^{(i)}} \ (i=1,\cdots,s) \quad (3.1)$$

$$U = p_2,q_2;p_3,q_3;\cdots;p_{r-1},q_{r-1};p'_2,q'_2;p'_3,q'_3;\cdots;p'_{s-1},q'_{s-1};0,0;\cdots;0,0;0,0;\cdots;0,0 \quad (3.2)$$

$$V = 0,n_2;0,n_3;\cdots;0,n_{r-1};0,n'_2;0,n'_3;\cdots;0,n'_{s-1};0,0;\cdots;0,0;0,0;\cdots;0,0 \quad (3.3)$$

$$X = m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)};m'^{(1)},n'^{(1)};\cdots;m'^{(s)},n'^{(s)};1,0;\cdots;1,0;1,0;\cdots;1,0 \quad (3.4)$$

$$Y = p^{(1)},q^{(1)};\cdots;p^{(r)},q^{(r)};p'^{(1)},q'^{(1)};\cdots;p'^{(s)},q'^{(s)};0,1;\cdots;0,1;0,1;\cdots;0,1 \quad (3.5)$$

$$A = (a_{2k};\alpha_{2k}^{(1)},\alpha_{2k}^{(2)});\cdots;(a_{(r-1)k};\alpha_{(r-1)k}^{(1)},\alpha_{(r-1)k}^{(2)},\cdots,\alpha_{(r-1)k}^{(r-1)})\,;(a'_{2k};\alpha_{2k}'^{(1)},\alpha_{2k}'^{(2)});\cdots;$$

$$(a'_{(s-1)k};\alpha_{(s-1)k}'^{(1)},\alpha_{(s-1)k}'^{(2)},\cdots,\alpha_{(s-1)k}'^{(s-1)}) \quad (3.6)$$

$$;(b'_{2k};\beta_{2k}'^{(1)},\beta_{2k}'^{(2)});\cdots;B = (b_{2k};\beta_{2k}^{(1)},\beta_{2k}^{(2)});\cdots;(b_{(r-1)k};\beta_{(r-1)k}^{(1)},\beta_{(r-1)k}^{(2)},\cdots,\beta_{(r-1)k}^{(r-1)})$$

$$(b'_{(s-1)k};\beta_{(s-1)k}'^{(1)},\beta_{(s-1)k}'^{(2)},\cdots,\beta_{(s-1)k}'^{(s-1)}) \quad (3.7)$$

$$\mathfrak{A} = (a_{rk};\alpha_{rk}^{(1)},\alpha_{rk}^{(2)},\cdots,\alpha_{rk}^{(r)},0,\cdots,0,0\cdots,0,0,\cdots,0) \quad (3.8)$$

$$\mathfrak{A}' = (a'_{sk};0,\cdots,0,\alpha_{sk}'^{(1)},\alpha_{sk}'^{(2)},\cdots,\alpha_{sk}'^{(s)},0,\cdots,0,0,\cdots,0) \quad (3.9)$$

$$\mathfrak{B} = (b_{rk};\beta_{rk}^{(1)},\beta_{rk}^{(2)},\cdots,\beta_{rk}^{(r)},0,\cdots,0,0,\cdots,0,0,\cdots,0) \quad (3.10)$$

$$\mathfrak{B}' = (b'_{sk};0,\cdots,0,\beta_{sk}'^{(1)},\beta_{sk}'^{(2)},\cdots,\beta_{sk}'^{(s)},0,\cdots,0,0,\cdots,0) \quad (3.11)$$

$$\mathfrak{A}_1 = (a_k^{(1)},\alpha_k^{(1)})_{1,p^{(1)}};\cdots;(a_k^{(r)},\alpha_k^{(r)})_{1,p^{(r)}};(a_k'^{(1)},\alpha_k^{(1)})_{1,p^{(1)}};\cdots;(a_k'^{(s)},\alpha_k'^{(s)})_{1,p'^{(s)}};$$
$$(1,0);\cdots;(1,0);(1.0);\cdots;(1.0) \quad (3.12)$$

$$\mathfrak{B}_1 = (b_k^{(1)},\beta_k^{(1)})_{1,p^{(1)}};\cdots;(b_k^{(r)},\beta_k^{(r)})_{1,p^{(r)}};(b_k'^{(1)},\beta_k'^{(1)})_{1,p'^{(1)}};\cdots;(b_k'^{(s)},\beta_k'^{(s)})_{1,p'^{(s)}};$$

$$(0,1);\cdots;(0,1);(0,1);\cdots;(0,1) \quad (3.13)$$

$$K_1 = (1-\alpha;\mu_1,\cdots,\mu_r,\mu'_1,\cdots,\mu'_s,h_1,\cdots,h_l,1,\cdots,1) \tag{3.14}$$

$$K_2 = (1-\beta;\rho_1,\cdots,\rho_r,\rho'_1,\cdots,\rho'_s,0,\cdots,0,0\cdots,0) \tag{3.15}$$

$$K_j = [1-\lambda_j;\zeta_j^{(1)},\cdots,\zeta_j^{(r)},\zeta_j'^{(1)}\cdots,\zeta_j'^{(s)},0,\cdots,\underset{\text{j}}{1},\cdots,0,0\cdots,0]_{1,l} \tag{3.16}$$

$$K'_j = [1+\sigma_j;\lambda_j^{(1)},\cdots,\lambda_j^{(r)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(s)},0,\cdots,0,0\cdots,\underset{\text{j}}{1},\cdots,0]_{1,k} \tag{3.17}$$

$$L_1 = (1-\alpha-\beta;\mu_1+\rho_1,\cdots,\mu_r+\rho_r,\mu'_1+\rho'_1,\cdots,\mu'_s+\rho'_s,h_1,\cdots,h_l,1,\cdots,1) \tag{3.18}$$

$$L_j = [1-\lambda_j;\zeta_j^{(1)},\cdots,\zeta_j^{(r)},\zeta_j'^{(1)}\cdots,\zeta_j'^{(s)},0,\cdots,0,0\cdots,0]_{1,l} \tag{3.19}$$

$$L'_j = [1+\sigma_j;\lambda_j^{(1)},\cdots,\lambda_j^{(r)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(s)},0,\cdots,0,0,\cdots,0]_{1,k} \tag{3.20}$$

We the following generalized Eulerian integral :

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$I_{U_r:p_r,q_r;W_r}^{V_r;0,n_r;X_r}\left(\begin{array}{c} \mathrm{z}_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$$I_{U_s:p'_s,q'_s;W_s}^{V_s;0,n'_s;X_s}\left(\begin{array}{c} \mathrm{z'}_1\theta'_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z'}_s\theta'_s(t-a)^{\mu'_s}(b-t)^{\rho'_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{array}\right)\mathrm{d}t =$$

$$(b-a)^{\alpha+\beta-1}\left\{\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}\right\}$$

$$I_{U:p_r+p'_s+l+k+2,q_r+q'_s+l+k+1;Y}^{V;0,n_r+n'_s+l+k+2;X}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}\ ;\ \mathrm{K}_1,K_2,K_j,K'_j,\mathfrak{A},\mathfrak{A}';\mathfrak{A}_1 \\ \cdots & \cdot \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \cdot \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(1)}_j}} & \cdot \\ \cdots & \cdot \\ \frac{z'_s(b-a)^{\mu'_s+\rho'_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(s)}_j}} & \cdot \\ \tau_1(b-a)^{h_1} & \cdot \\ \cdots & \cdot \\ \tau_l(b-a)^{h_l} & \mathrm{B}\ ,\ \mathrm{L}_1,L_j,L'_j,\mathfrak{B},\mathfrak{B}';\mathfrak{B}_1 \\ \frac{(b-a)f_1}{af_1+g_1} & \\ \cdots & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \quad (3.21)$$

We obtain the I-function of $r+s+k+l$ variables. The quantities $U,V,X,Y,A,B,K_1,K_2,K_j,K'_j,\mathfrak{A},\mathfrak{A}',\mathfrak{A}_1,L_1,L_j,L'_j,$ $\mathfrak{B},\mathfrak{B}'$ and $\mathfrak{B}_1$ are defined above.

Provided that

(A) $a,b\in\mathbb{R}(a<b);\mu_i,\mu'_u,\rho_i,\rho'_u,\lambda_j^{(i)},\lambda_j'^{(u)},h_v\in\mathbb{R}^+$, $f_i,g_j,\tau_v,\sigma_j,\lambda_v\in\mathbb{C}$ $(i=1,\cdots,r;j=1,\cdots;k;$ $u=1,\cdots,s;v=1,\cdots,l)$

(B) $a_{ij},b_{ik},\in\mathbb{C}\ (i=1,\cdots,r;j=1,\cdots,p_i;k=1,\cdots,q_i);a_j^{(i)},b_j^{(k)}\in\mathbb{C}$

$(i=1,\cdots,r;j=1,\cdots,p^{(i)};k=1,\cdots,q^{(i)})$

$a'_{ij},b'_{ik},\in\mathbb{C}\ (i=1,\cdots,s;j=1,\cdots,p'_i;k=1,\cdots,q'_i);a_j'^{(i)},b_j'^{(k)},\in\mathbb{C}$

$(i=1,\cdots,r;j=1,\cdots,p'^{i)};k=1,\cdots,q'^{(i)})$

$\alpha_{ij}^{(k)},\beta_{ij}^{(k)}\in\mathbb{R}^+$ ($(i=1,\cdots,r,j=1,\cdots,p_i,k=1,\cdots,r)$; $\alpha_j^{(i)},\beta_i^{(i)}\in\mathbb{R}^+\ (i=1,\cdots,r;j=1,\cdots,p_i)$

$\alpha_{ij}'^{(k)},\beta_{ij}'^{(k)}\in\mathbb{R}^+$ ($(i=1,\cdots,s,j=1,\cdots,p'_i,k=1,\cdots,s)$; $\alpha_j'^{(i)},\beta_i'^{(i)}\in\mathbb{R}^+\ (i=1,\cdots,s;j=1,\cdots,p'_i)$

(C) $\max\limits_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1,\ \max\limits_{1\leqslant j\leqslant l}\left\{\left|\tau_j(b-a)^{h_j}\right|\right\}<1$

(D) $Re\big[\alpha+\sum_{i=1}^{r}\mu_i\min\limits_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}+\sum_{i=1}^{s}\mu^{(i)}\min\limits_{1\leqslant j\leqslant m'^{(i)}}\frac{b_j'^{(i)}}{\beta_j'^{(i)}}\big]>0$

$$Re\big[\beta+\sum_{i=1}^{r}\rho_i\min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}+\sum_{i=1}^{s}\rho'_i\min_{1\leqslant j\leqslant m'^{(i)}}\frac{b_j'^{(i)}}{\beta_j'^{(i)}}\big]>0$$

(E) $Re\left(\alpha+\sum_{i=1}^{r}v_i\mu_i+\sum_{i=1}^{s}v'_i\mu'_i+\sum_{i=1}^{l}h_iv''_i\right)>0\,;Re\left(\beta+\sum_{i=1}^{r}v_i\rho_i+\sum_{i=1}^{s}v'_i\rho'_i\right)>0$

$$Re\left(\lambda_j+\sum_{i=1}^{r}v_i\zeta_j^{(i)}+\sum_{i=1}^{s}v_i'\zeta_j'^{(i)}\right)>0(j=1,\cdots,l);\ Re\left(-\sigma_j+\sum_{i=1}^{r}v_i\lambda_j^{(i)}+\sum_{i=1}^{s}v_i'\lambda_j'^{(i)}\right)>0(j=1,\cdots,k);$$

(F) $$\Omega_i=\sum_{k=1}^{n^{(i)}}\alpha_k^{(i)}-\sum_{k=n^{(i)}+1}^{p^{(i)}}\alpha_k^{(i)}+\sum_{k=1}^{m^{(i)}}\beta_k^{(i)}-\sum_{k=m^{(i)}+1}^{q^{(i)}}\beta_k^{(i)}+\left(\sum_{k=1}^{n_2}\alpha_{2k}^{(i)}-\sum_{k=n_2+1}^{p_2}\alpha_{2k}^{(i)}\right)+\cdots+$$

$$\left(\sum_{k=1}^{n_s}\alpha_{sk}^{(i)}-\sum_{k=n_s+1}^{p_s}\alpha_{sk}^{(i)}\right)-\left(\sum_{k=1}^{q_2}\beta_{2k}^{(i)}+\sum_{k=1}^{q_3}\beta_{3k}^{(i)}+\cdots+\sum_{k=1}^{q_s}\beta_{sk}^{(i)}\right)-\mu_i-\rho_i$$

$$-\sum_{l=1}^{k}\lambda_j^{(i)}-\sum_{l=1}^{l}\zeta_j^{(i)}>0\quad(i=1,\cdots,r)$$

$$\Omega_i'=\sum_{k=1}^{n'^{(i)}}\alpha_k'^{(i)}-\sum_{k=n'^{(i)}+1}^{p'^{(i)}}\alpha_k'^{(i)}+\sum_{k=1}^{m'^{(i)}}\beta_k'^{(i)}-\sum_{k=m^{(i)}+1}^{q'^{(i)}}\beta_k'^{(i)}+\left(\sum_{k=1}^{n_2'}\alpha_{2k}'^{(i)}-\sum_{k=n_2+1}^{p_2'}\alpha_{2k}'^{(i)}\right)+$$

$$\cdots+\left(\sum_{k=1}^{n_s'}\alpha_{sk}'^{(i)}-\sum_{k=n_s'+1}^{p_s'}\alpha_{sk}'^{(i)}\right)-\left(\sum_{k=1}^{q_2'}\beta_{2k}'^{(i)}+\sum_{k=1}^{q_3'}\beta_{3k}'^{(i)}+\cdots+\sum_{k=1}^{q_s'}\beta_{sk}^{(i)}\right)-\mu_i'-\rho_i'$$

$$-\sum_{l=1}^{k}\lambda_j'^{(i)}-\sum_{l=1}^{l}\zeta_j'^{(i)}>0\quad(i=1,\cdots,s)$$

(G) $$\left|arg\left(z_i\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{(i)}}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(i)}}\right)\right|<\frac{1}{2}\Omega_i\pi\quad(a\leqslant t\leqslant b;i=1,\cdots,r)$$

$$\left|arg\left(z_i'\prod_{j=1}^{l}\left[1-\tau_j'(t-a)^{h_i'}\right]^{-\zeta_j'^{(i)}}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(i)}}\right)\right|<\frac{1}{2}\Omega_i'\pi\quad(a\leqslant t\leqslant b;i=1,\cdots,s)$$

Proof

To prove (3.21) first espressing the I-function of r variables by the Mellin-Barnes contour integral with the help of the equation (1.2), the I-function of s variables by the Mellin-Barnes contour integral with the help of the equation (1.12). Now collect the power of $\left[1-\tau_j(t-a)^{h_i}\right]$ with $(i=1,\cdots,r;j=1,\cdots,l)$ and collect the power of (f_jt+g_j) with $j=1,\cdots,k$. Use the equations (2.2) and (2.3) and express the result in Mellin-Barnes contour integral. Interpreting the $(r+s+k+l)$ dimensional Mellin-Barnes integral in multivariable I-function, we obtain the equation (3.21)

Remark : If $U_r=V_r=U_s=V_s=U=V=A=B=0$, the multivariable I-function defined by Prasad degenere in multivariable H-function defined by Srivastava et al [6], see Saigo et al [3 page 45-68]

4. Particular cases

a) If $\rho_1=\cdots,\rho_r=\rho'_1=\cdots,\rho'_s=0$, we obtain

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$I_{U_r:p_r,q_r;W_r}^{V_r;0,n_r;X_r}\left(\begin{array}{c} \mathrm{z}_1\theta_1(t-a)^{\mu_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r\theta_r(t-a)^{\mu_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$$I_{U_s:p'_s,q'_s;W_s}^{V_s;0,n'_s;X_s}\left(\begin{array}{c} \mathrm{z'}_1\theta'_1(t-a)^{\mu'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z'}_s\theta'_s(t-a)^{\mu'_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{array}\right)\mathrm{d}t=(b-a)^{\alpha+\beta-1}\left\{\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}\right\}$$

$$I_{U:p_r+p'_s+l+k+1,q_r+q'_s+l+k+1;Y}^{V;0,n_r+n'_s+l+k+1;X}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \\ \cdots & \\ \frac{z_r(b-a)^{\mu_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \mathrm{A}\ ;\ \mathrm{K}_1,K_j,K'_j,\mathfrak{A},\mathfrak{A}';\mathfrak{A}_1 \\ \frac{z'_1(b-a)^{\mu'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdots & \cdot \\ \frac{z'_s(b-a)^{\mu'_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(s)}}} & \cdot \\ \tau_1(b-a)^{h_1} & \cdot \\ \cdots & \cdot \\ \tau_l(b-a)^{h_l} & \mathrm{B}\ ,\ \mathrm{L'}_1,L_j,L'_j,\mathfrak{B},\mathfrak{B}';\mathfrak{B}_1 \\ \frac{(b-a)f_1}{af_1+g_1} & \\ \cdots & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \tag{4.1}$$

where $L'_1=(1-\alpha-\beta;\mu_1,\cdots,\mu_r,\mu'_1,\cdots,\mu'_s,h_1,\cdots,h_l,1,\cdots,1)$ (4.2)

the others quantities are the same that (3.21)

The validities conditions are the same that (3.21) with $\rho_1=\cdots,\rho_r=\rho'_1=\cdots,\rho'_s=0$

b) if $\mu_1 = \cdots, \mu_r = \mu'_1 = \cdots, \mu'_s = 0$, we have the following result

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$I_{U_r:p_r,q_r;W_r}^{V_r;0,n_r;X_r}\left(\begin{array}{c} \mathrm{z}_1\theta_1(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r\theta_r(b-t)^{\rho_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$$I_{U_s:p'_s,q'_s;W_s}^{V_s;0,n'_s;X_s}\left(\begin{array}{c} \mathrm{z'}_1\theta'_1(b-t)^{\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z'}_s\theta'_s s(b-t)^{\rho'_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{array}\right)\mathrm{d}t = (b-a)^{\alpha+\beta-1}\left\{\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}\right\}$$

$$I_{U:p_r+p'_s+l+k+1,q_r+q'_s+l+k+1;Y}^{V;0,n_r+n'_s+l+k+1;X}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \\ \cdots & \mathrm{A}\ ;\ \mathrm{K}_2, K_j, K'_j, \mathfrak{A}, \mathfrak{A}'; \mathfrak{A}_1 \\ \frac{z_r(b-a)^{\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \cdot \\ \frac{z'_1(b-a)^{\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdots & \cdot \\ \frac{z'_s(b-a)^{\rho'_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(s)}}} & \cdot \\ \tau_1(b-a)^{h_1} & \cdot \\ \cdots & \cdot \\ \tau_l(b-a)^{h_l} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \mathrm{B}\ ,\ \mathrm{L"}_1, L_j, L'_j, \mathfrak{B}, \mathfrak{B}'; \mathfrak{B}_1 \\ \cdots & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \tag{4.3}$$

where $L''_1 = (1-\alpha-\beta;\rho_1,\cdots,\rho_r,\rho'_r,\cdots,\rho'_s,h_1,\cdots,h_l,1,\cdots,1)$ (4.4)

The validities conditions are the same that (3.21) with $\mu_1 = \cdots, \mu_r = \mu'_1 = \cdots, \mu'_s = 0$

c)Let $U = p_2, q_2; p_3, q_3; \cdots; p_{r-1}, q_{r-1}; 0, 0; \cdots; 0, 0; 0, 0; \cdots; 0, 0; 0, 0; \cdots; 0, 0$ (4.5)

$$V = 0, n_2; 0, n_3; \cdots ; 0, n_{r-1}; 0, 0; \cdots ; 0, 0; 0, 0; \cdots ; 0, 0; 0, 0; \cdots ; 0, 0 \tag{4.6}$$

$$X = m^{(1)}, n^{(1)}; \cdots ; m^{(r)}, n^{(r)}; 1, 0; \cdots ; 1, 0; 1, 0; \cdots ; 1, 0; 1, 0; \cdots ; 1, 0 \tag{4.7}$$

$$Y = p^{(1)}, q^{(1)}; \cdots ; p^{(r)}, q^{(r)}; 0, 1; \cdots ; 0, 1; 0, 1; \cdots ; 0, 1; 0, 1; \cdots ; 0, 1 \tag{4.8}$$

$$A = (a_{2k}; \alpha_{2k}^{(1)}, \alpha_{2k}^{(2)}); \cdots ; (a_{(r-1)k}; \alpha_{(r-1)k}^{(1)}, \alpha_{(r-1)k}^{(2)}, \cdots , \alpha_{(r-1)k}^{(r-1)}) \tag{4.9}$$

$$B = (b_{2k}; \beta_{2k}^{(1)}, \beta_{2k}^{(2)}); \cdots ; (b_{(r-1)k}; \beta_{(r-1)k}^{(1)}, \beta_{(r-1)k}^{(2)}, \cdots , \beta_{(r-1)k}^{(r-1)}) \tag{4.10}$$

$$\mathfrak{A} = (a_{rk}; \alpha_{rk}^{(1)}, \alpha_{rk}^{(2)}, \cdots , \alpha_{rk}^{(r)}, 0, \cdots , 0, 0 \cdots , 0, 0, \cdots , 0) \tag{4.11}$$

$$\mathfrak{B} = (b_{rk}; \beta_{rk}^{(1)}, \beta_{rk}^{(2)}, \cdots , \beta_{rk}^{(r)}, 0, \cdots , 0, 0 \cdots , 0, 0, \cdots , 0) \tag{4.12}$$

$$\mathfrak{A}_1 = (a_k^{(1)}, \alpha_k^{(1)})_{1,p^{(1)}}; \cdots ; (a_k^{(r)}, \alpha_k^{(r)})_{1,p^{(r)}}; (1, 0); \cdots ; (1, 0); (1, 0); \cdots ; (1, 0);$$
$$(1, 0); \cdots ; (1, 0); \tag{4.13}$$

$$\mathfrak{B}_1 = (b_k^{(1)}, \beta_k^{(1)})_{1,p^{(1)}}; \cdots ; (b_k^{(r)}, \beta_k^{(r)})_{1,p^{(r)}}; (0, 1); \cdots ; (0, 1); (0, 1); \cdots ; (0, 1);$$

$$(0, 1); \cdots ; (0, 1) \tag{4.14}$$

$$K_1 = (1 - \alpha; \mu_1, \cdots , \mu_r, \mu'_1, \cdots , \mu'_s, h_1, \cdots , h_l, 1, \cdots , 1) \tag{4.15}$$

$$K_2 = (1 - \beta; \rho_1, \cdots , \rho_r, \rho'_1, \cdots , \rho'_s, 0, \cdots , 0, 0 \cdots , 0) \tag{4.16}$$

$$K_3 = [1 - A_j; 0, \cdots , 0, 1, \cdots , 1, 0, \cdots , 0, 0, \cdots , 0]_{1,P} \tag{3.17}$$

$$K_j = [1 - \lambda_j; \zeta_j^{(1)}, \cdots , \zeta_j^{(r)}, \zeta_j'^{(1)} \cdots , \zeta_j'^{(s)}, 0, \cdots , \underset{\text{j}}{1}, \cdots , 0, 0 \cdots , 0]_{1,l} \tag{4.18}$$

$$K'_j = [1 + \sigma_j; \lambda_j^{(1)}, \cdots , \lambda_j^{(r)}, \lambda_j'^{(1)} \cdots , \lambda_j'^{(s)}, 0, \cdots , 0, 0 \cdots , \underset{\text{j}}{1}, \cdots , 0]_{1,k} \tag{4.19}$$

$$L_1 = (1 - \alpha - \beta; \mu_1 + \rho_1, \cdots , \mu_r + \rho_r, \mu'_1 + \rho'_r, \cdots , \mu'_s + \rho'_s, h_1, \cdots , h_l, 1, \cdots , 1) \tag{4.20}$$

$$L_2 = [1 - B_j; 0, \cdots , 0, 1, \cdots , 1, 0 \cdots , 0, 0, \cdots , 0]_{1,Q} \tag{4.21}$$

$$L_j = [1 - \lambda_j; \zeta_j^{(1)}, \cdots , \zeta_j^{(r)}, \zeta_j'^{(1)} \cdots , \zeta_j'^{(s)}, 0, \cdots , 0, 0 \cdots , 0]_{1,l} \tag{4.22}$$

$$L'_j = [1 + \sigma_j; \lambda_j^{(1)}, \cdots , \lambda_j^{(r)}, \lambda_j'^{(1)} \cdots , \lambda_j'^{(s)}, 0, \cdots , 0, 0, \cdots , 0]_{1,k} \tag{4.23}$$

We have the following result

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$I^{V_r;0,n_r;X_r}_{U_r:p_r,q_r;W_r}\left(\begin{array}{c} \mathrm{z}_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ . \\ . \\ . \\ \mathrm{z}_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{s}z_i'\theta_i'(t-a)^{\mu_i'}(b-t)^{\rho_i'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(i)}}\right]\mathrm{d}t = B\times$$

$$I^{V;0,n_r+P+l+k+2;X}_{U:p_r+P+l+k+2,q_r+Q+l+k+1;Y}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \\ \cdots & \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \mathrm{A}\ ;\ \mathrm{K}_1,K_2,K_3,K_j,K_j',\mathfrak{A};\mathfrak{A}_1 \\ \frac{z_1'(b-a)^{\mu_1'+\rho_1'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & . \\ \cdots & . \\ \frac{z_s'(b-a)^{\mu_s'+\rho_s'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(s)}}} & . \\ \tau_1(b-a)^{h_1} & . \\ \cdots & . \\ \tau_l(b-a)^{h_l} & \mathrm{B}\ ,\ \mathrm{L}_1,L_2,L_j,L_j',\mathfrak{B};\mathfrak{B}_1 \\ \frac{(b-a)f_1}{af_1+g_1} & \\ \cdots & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \tag{4.23}$$

where $B = (b-a)^{\alpha+\beta-1}\dfrac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}(af_j+g_j)^{\sigma_j}$

Provided that

(A) $a,b\in\mathbb{R}(a<b);\mu_i,\rho_i,\lambda_j^{(i)},h_v\in\mathbb{R}^+$, $f_i,g_j,\tau_v,\sigma_j,\lambda_v\in\mathbb{C}$ $(i=1,\cdots,r;j=1,\cdots;k;v=1,\cdots,l)$

(B) $a_{ij},b_{ik},\in\mathbb{C}$ $(i=1,\cdots,r;j=1,\cdots,p_i;k=1,\cdots,q_i)$; $a_j^{(i)},b_j^{(k)}\in\mathbb{C}$

$(i=1,\cdots,r;j=1,\cdots,p^{(i)};k=1,\cdots,q^{(i)})$

$\alpha_{ij}^{(k)},\beta_{ij}^{(k)}\in\mathbb{R}^+$ ($(i=1,\cdots,r,j=1,\cdots,p_i,k=1,\cdots,r)$; $\alpha_j^{(i)},\beta_i^{(i)}\in\mathbb{R}^+$ $(i=1,\cdots,r;j=1,\cdots,p_i)$

(C) $\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1,\ \max_{1\leqslant j\leqslant l}\left\{\left|\tau_j(b-a)^{h_j}\right|\right\}<1,$

(D) $Re\big[\alpha+\sum_{i=1}^{r}\mu_i\min_{1\leqslant j\leqslant m_i}\frac{b_j^{(i)}}{\beta_j^{(i)}}\big]>0$ and $Re\big[\beta+\sum_{i=1}^{r}\rho_i\min_{1\leqslant j\leqslant m_i}\frac{b_j^{(i)}}{\beta_j^{(i)}}\big]>0$

(E) $Re\left(\alpha+\sum_{i=1}^{r}v_i\mu_i+\sum_{i=1}^{l}h_iv_i''\right)>0$; $Re\left(\beta+\sum_{i=1}^{r}v_i\rho_i\right)>0$

$Re\left(-\sigma_j+\sum_{i=1}^{r}v_i\lambda_j^{(i)}\right)>0(j=1,\cdots,k)$; $Re\left(\lambda_j+\sum_{i=1}^{r}v_i\zeta_j^{(i)}\right)>0(j=1,\cdots,l)$;

(F) $$\Omega_i=\sum_{k=1}^{n^{(i)}}\alpha_k^{(i)}-\sum_{k=n^{(i)}+1}^{p^{(i)}}\alpha_k^{(i)}+\sum_{k=1}^{m^{(i)}}\beta_k^{(i)}-\sum_{k=m^{(i)}+1}^{q^{(i)}}\beta_k^{(i)}+\left(\sum_{k=1}^{n_2}\alpha_{2k}^{(i)}-\sum_{k=n_2+1}^{p_2}\alpha_{2k}^{(i)}\right)+\cdots+$$

$$\left(\sum_{k=1}^{n_s}\alpha_{sk}^{(i)}-\sum_{k=n_s+1}^{p_s}\alpha_{sk}^{(i)}\right)-\left(\sum_{k=1}^{q_2}\beta_{2k}^{(i)}+\sum_{k=1}^{q_3}\beta_{3k}^{(i)}+\cdots+\sum_{k=1}^{q_s}\beta_{sk}^{(i)}\right)-\mu_i-\rho_i$$

$$-\sum_{l=1}^{k}\lambda_l^{(i)}>0\quad(i=1,\cdots,r)$$

(G) $$\left|arg\left(z_i\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{(i)}}\prod_{j=1}^{k}(f_jt+g_j)^{\lambda_j^{(i)}}\right)\right|<\frac{1}{2}\Omega_i\pi\quad(a\leqslant t\leqslant b;i=1,\cdots,r)$$

(H) $P\leqslant Q+1$. The equality holds, when , in addition,

either $P>Q$ and $\left|\left(z_i'\sum_{i=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{(i)}}\prod_{j=1}^{k}(f_jt+g_j)^{\lambda_j^{(i)}}\right)\right|^{\frac{1}{Q-P}}$ <1 $(a\leqslant t\leqslant b)$

or $P\leqslant Q$ and $\max_{1\leqslant i\leqslant k}\left[\left|\left(z_i'\sum_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{(i)}}\prod_{j=1}^{k}(f_jt+g_j)^{\lambda_j^{(i)}}\right)\right|\right]<1\ \ (a\leqslant t\leqslant b)$

Proof

To prove (4.23) first espressing the I-function of r variables by the Mellin-Barnes contour integral with the help of the equation (1.2), the generalized hypergeometric function ${}_PF_Q(.)$ in Mellin-Barnes contour integral with the help of (2.1).Now collect the power of $\left[1-\tau_j(t-a)^{h_i}\right]$ with $(i=1,\cdots,r;j=1,\cdots,l)$ and collect the power of (f_jt+g_j) with $j=1,\cdots,k$. Use the equations (2.2) and (2.3) and express the result in Mellin-Barnes contour integral. Interpreting the $(r+s+k+l)$ dimensional Mellin-Barnes integral in multivariable I-function, we obtain the equation (4.23).

Remark:
By the following similar procedure, the results of this document can be extented to product of any finite number of multivariable I-functions defined by Prasad [1].

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of two multivariable I-functions defined by Prasad [6] with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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