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Eulerian integral associated with product of two multivariable Aleph-functions and a class of polynomials of several variables

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ABSTRACT
The present paper is evaluated a new Eulerian integral associated with the product of two multivariable Aleph-functions with general arguments and general class of polynomials. Several particular cases are given .

Keywords: Eulerian integral, multivariable Aleph-function, multivariable I-function, generalized Lauricella function of several variables, multivariable H-function, generalized hypergeometric function, class of polynomials.

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1. Introduction

In this paper, we evaluate a new Eulerian integral of most general characters associated with the products of two multivariable Aleph-functions and general class of polynomials of several variables, but of greater order. Several particular cases are given.
The Aleph-function of several variables generalize the multivariable I-function defined by Sharma and Ahmad [2] , itself is an a generalisation of G and H-functions of several variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We have : $\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|}\mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\end{array}\right.$

$$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}]\ ,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:$$
$$\cdots\cdots\cdots\cdots\ ,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}]:$$

$$\left.\begin{array}{c}[(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;\ ;[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}]\\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;;[(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

and $\theta_k(s_k) = \dfrac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]}$ (1.3)

Suppose , as usual , that the parameters

$a_j, j = 1,\cdots,p; b_j, j = 1,\cdots,q;$

$c_j^{(k)}, j = 1,\cdots,n_k; c_{ji^{(k)}}^{(k)}, j = n_k+1,\cdots,p_{i^{(k)}};$

$d_j^{(k)}, j = 1,\cdots,m_k; d_{ji^{(k)}}^{(k)}, j = m_k+1,\cdots,q_{i^{(k)}};$

with $k = 1\cdots,r, i = 1,\cdots,R$, $i^{(k)} = 1,\cdots,R^{(k)}$

are complex numbers , and the $\alpha's, \beta's, \gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)} + \tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} + \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)} - \tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)} - \sum_{j=1}^{m_k}\delta_j^{(k)}$$

$$-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)}s_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r}\alpha_j^{(k)}s_k)$ with $j = 1$ to n and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)}s_k)$ with $j = 1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)} - \tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)} - \tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} - \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)} - \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1\cdots,r, i = 1,\cdots,R \text{ , } i^{(k)} = 1,\cdots,R^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,), max(\,|z_1|,\cdots,|z_r|\,) \to 0$

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,), min(\,|z_1|,\cdots,|z_r|\,) \to \infty$

where $k = 1,\cdots,r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1,\cdots,m_k$ and

$$\beta_k = max[Re((c_j^{(k)}-1)/\gamma_j^{(k)})], j=1,\cdots,n_k$$

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R \ ; \ V = m_1, n_1; \cdots ; m_r, n_r \tag{1.6}$$

$$\mathrm{W} = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \cdots, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.7}$$

$$A = \{(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.8}$$

$$B = \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.9}$$

$$C = \{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\}, \tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}, \cdots, \{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.10}$$

$$D = \{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}, \cdots, \{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.11}$$

The multivariable Aleph-function write :

$$\aleph(z_1, \cdots, z_r) = \aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A : C} \\ \cdot & \\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot & \\ \mathrm{z}_r & \mathrm{B : D} \end{array}\right) \tag{1.12}$$

Consider the Aleph-function of s variables

$$\aleph(z_1, \cdots, z_s) = \aleph_{P_i,Q_i,\iota_i;r':P_{i^{(1)}},Q_{i^{(1)}},\iota_{i^{(1)}};r^{(1)};\cdots;P_{i^{(s)}},Q_{i^{(s)}};\iota_{i^{(s)}};r^{(s)}}^{0,N:M_1,N_1,\cdots,M_s,N_s}\left(\begin{array}{c|} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_s \end{array}\right.$$

$$\begin{array}{ll} [(\mathrm{u}_j; \mu_j^{(1)}, \cdots, \mu_j^{(r')})_{1,N}] & ,[\iota_i(u_{ji}; \mu_{ji}^{(1)}, \cdots, \mu_{ji}^{(r')})_{N+1,P_i}] : \\ \ldots\ldots\ldots\ldots\ldots\ldots & ,[\iota_i(v_{ji}; \upsilon_{ji}^{(1)}, \cdots, \upsilon_{ji}^{(r')})_{M+1,Q_i}] : \end{array}$$

$$\left.\begin{array}{l} [(\mathrm{a}_j^{(1)}); \alpha_j^{(1)})_{1,N_1}], [\iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)}; \alpha_{ji^{(1)}}^{(1)})_{N_1+1,P_i^{(1)}}]; \cdots; [(\mathrm{a}_j^{(s)}); \alpha_j^{(s)})_{1,N_s}], [\iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)}; \alpha_{ji^{(s)}}^{(s)})_{N_s+1,P_i^{(s)}}] \\ [(\mathrm{b}_j^{(1)}); \beta_j^{(1)})_{1,M_1}], [\iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)}; \beta_{ji^{(1)}}^{(1)})_{M_1+1,Q_i^{(1)}}]; \cdots; [(\mathrm{b}_j^{(s)}); \beta_j^{(s)})_{1,M_s}], [\iota_{i^{(s)}}(b_{ji^{(s)}}^{(s)}; \beta_{ji^{(s)}}^{(s)})_{M_s+1,Q_i^{(s)}}] \end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^s}\int_{L_1}\cdots\int_{L_s}\zeta(t_1, \cdots, t_s)\prod_{k=1}^{s}\phi_k(t_k) z_k^{t_k}\,\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.13}$$

with $\omega = \sqrt{-1}$

$$\zeta(t_1, \cdots, t_s) = \frac{\prod_{j=1}^{N}\Gamma(1-u_j+\sum_{k=1}^{s}\mu_j^{(k)}t_k)}{\sum_{i=1}^{r'}[\iota_i\prod_{j=N+1}^{P_i}\Gamma(u_{ji}-\sum_{k=1}^{s}\mu_{ji}^{(k)}t_k)\prod_{j=1}^{Q_i}\Gamma(1-\upsilon_{ji}+\sum_{k=1}^{s}\upsilon_{ji}^{(k)}t_k)]} \tag{1.14}$$

and $\phi_k(t_k) = \dfrac{\prod_{j=1}^{M_k}\Gamma(b_j^{(k)}-\beta_j^{(k)}t_k)\prod_{j=1}^{N_k}\Gamma(1-a_j^{(k)}+\alpha_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{r^{(k)}}[\iota_{i^{(k)}}\prod_{j=M_k+1}^{Q_{i^{(k)}}}\Gamma(1-b_{ji^{(k)}}^{(k)}+\beta_{ji^{(k)}}^{(k)}t_k)\prod_{j=N_k+1}^{P_{i^{(k)}}}\Gamma(a_{ji^{(k)}}^{(k)}-\alpha_{ji^{(k)}}^{(k)}s_k)]}$ (1.15)

Suppose , as usual , that the parameters

$u_j, j=1,\cdots,P; v_j, j=1,\cdots,Q;$

$a_j^{(k)}, j=1,\cdots,N_k; a_{ji^{(k)}}^{(k)}, j=n_k+1,\cdots,P_{i^{(k)}};$

$b_{ji^{(k)}}^{(k)}, j=m_k+1,\cdots,Q_{i^{(k)}}; b_j^{(k)}, j=1,\cdots,M_k;$

with $k=1\cdots,s, i=1,\cdots,r' , i^{(k)}=1,\cdots,r^{(k)}$

are complex numbers , and the $\alpha's, \beta's, \gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{N}\mu_j^{(k)} + \iota_i\sum_{j=N+1}^{P_i}\mu_{ji}^{(k)} + \sum_{j=1}^{N_k}\alpha_j^{(k)} + \iota_{i^{(k)}}\sum_{j=N_k+1}^{P_{i^{(k)}}}\alpha_{ji^{(k)}}^{(k)} - \iota_i\sum_{j=1}^{Q_i}\upsilon_{ji}^{(k)} - \sum_{j=1}^{M_k}\beta_j^{(k)}$$

$$-\iota_{i^{(k)}}\sum_{j=M_k+1}^{Q_{i^{(k)}}}\beta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.16}$$

The reals numbers τ_i are positives for $i=1,\cdots,r$, $\iota_{i^{(k)}}$ are positives for $i^{(k)}=1\cdots r^{(k)}$

The contour L_k is in the t_k-p lane and run from $\sigma-i\infty$ to $\sigma+i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(b_j^{(k)}-\beta_j^{(k)}t_k)$ with $j=1$ to M_k are separated from those of $\Gamma(1-u_j+\sum_{i=1}^{s}\mu_j^{(k)}t_k)$with $j=1$ to N and $\Gamma(1-a_j^{(k)}+\alpha_j^{(k)}t_k)$ with $j=1$ to N_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2}B_i^{(k)}\pi$, where

$$B_i^{(k)} = \sum_{j=1}^{N}\mu_j^{(k)} - \iota_i\sum_{j=N+1}^{P_i}\mu_{ji}^{(k)} - \iota_i\sum_{j=1}^{Q_i}\upsilon_{ji}^{(k)} + \sum_{j=1}^{N_k}\alpha_j^{(k)} - \iota_{i^{(k)}}\sum_{j=N_k+1}^{P_{i^{(k)}}}\alpha_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{M_k}\beta_j^{(k)} - \iota_{i^{(k)}}\sum_{j=M_k+1}^{q_{i^{(k)}}}\beta_{ji^{(k)}}^{(k)} > 0, \text{ with } k=1,\cdots,s, i=1,\cdots,r , i^{(k)}=1,\cdots,r^{(k)} \tag{1.17}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_s) = 0(|z_1|^{\alpha_1'},\cdots,|z_s|^{\alpha_s'}), max(|z_1|,\cdots,|z_s|) \rightarrow 0$

$\aleph(z_1,\cdots,z_s) = 0(|z_1|^{\beta_1'},\cdots,|z_s|^{\beta_s'}), min(|z_1|,\cdots,|z_s|) \rightarrow \infty$

where $k = 1, \cdots , z : \alpha'_k = min[Re(b_j^{(k)}/\beta_j^{(k)})], j = 1, \cdots , M_k$ and

$$\beta'_k = max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \cdots , N_k$$

We will use these following notations in this paper

$$U' = P_i, Q_i, \iota_i; r' ;\ V' = M_1, N_1; \cdots ; M_s, N_s \tag{1.18}$$

$$W' = P_{i^{(1)}}, Q_{i^{(1)}}, \iota_{i^{(1)}}; r^{(1)}, \cdots , P_{i^{(r)}}, Q_{i^{(r)}}, \iota_{i^{(s)}}; r^{(s)} \tag{1.19}$$

$$A' = \{(u_j; \mu_j^{(1)}, \cdots , \mu_j^{(s)})_{1,N}\}, \{\iota_i(u_{ji}; \mu_{ji}^{(1)}, \cdots , \mu_{ji}^{(s)})_{N+1,P_i}\} \tag{1.20}$$

$$B' = \{\iota_i(v_{ji}; v_{ji}^{(1)}, \cdots , v_{ji}^{(s)})_{M+1,Q_i}\} \tag{1.21}$$

$$C' = (a_j^{(1)}; \alpha_j^{(1)})_{1,N_1} , \iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)}; \alpha_{ji^{(1)}}^{(1)})_{N_1+1,P_{i^{(1)}}}, \cdots , (a_j^{(s)}; \alpha_j^{(s)})_{1,N_s} , \iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)}; \alpha_{ji^{(s)}}^{(s)})_{N_s+1,P_{i^{(s)}}} \tag{1.22}$$

$$D' = (b_j^{(1)}; \beta_j^{(1)})_{1,M_1}, \iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)}; \beta_{ji^{(1)}}^{(1)})_{M_1+1,Q_{i^{(1)}}}, \cdots , (b_j^{(s)}; \beta_j^{(s)})_{1,M_s}, \iota_{i^{(s)}}(\beta_{ji^{(s)}}^{(s)}; \beta_{ji^{(s)}}^{(s)})_{M_s+1,Q_{i^{(s)}}} \tag{1.23}$$

The multivariable Aleph-function write :

$$\aleph(z_1, \cdots , z_s) = \aleph_{U':W'}^{0,N:V'} \left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A'} : \mathrm{C'} \\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot & \\ \mathrm{z}_s & \mathrm{B'} : \mathrm{D'} \end{array} \right) \tag{1.24}$$

Srivastava and Garg introduced and defined a general class of multivariable polynomials [3] as follows

$$S_L^{h_1,\cdots,h_u}[z_1, \cdots , z_u] = \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u \leqslant L} (-L)_{h_1R_1+\cdots+h_uR_u} B(E; R_1, \cdots , R_u)\frac{z_1^{R_1} \cdots z_u^{R_u}}{R_1! \cdots R_u!} \tag{1.25}$$

2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [4 ,page 39 eq .30]

$$\frac{\prod_{j=1}^{P} \Gamma(A_j)}{\prod_{j=1}^{Q} \Gamma(B_j)}\, {}_PF_Q\left[(A_P); (B_Q); -(x_1 + \cdots + x_r)\right]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \frac{\prod_{j=1}^{P} \Gamma(A_j + s_1 + \cdots + s_r)}{\prod_{j=1}^{Q} \Gamma(B_j + s_1 + \cdots + s_r)} \Gamma(-s_1) \cdots \Gamma(-s_r) x_1^{s_1} \cdots x_r^{s_r} \mathrm{d}s_1 \cdots \mathrm{d}s_r \tag{2.1}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j + s_1 + \cdots + s_r)$ are separated from those of $\Gamma(-s_j), j = 1, \cdots , r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j = 1, \cdots , r$

The Lauricella function $F_D^{(k)}$ is defined as

$$F_D^{(k)}\left[a,b_1,\cdots,b_k;c;x_1,\cdots,x_k\right]=\frac{\Gamma(c)}{\Gamma(a)\prod_{j=1}^{k}\Gamma(b_j)}\frac{1}{(2\pi\omega)^k}\int_{L_1}\cdots\int_{L_k}\frac{\Gamma\left(a+\sum_{j=1}^{k}\zeta_j\right)\Gamma(b_1+\zeta_1),\cdots,\Gamma(b_k+\zeta_k)}{\Gamma\left(c+\sum_{j=1}^{k}\zeta_j\right)}$$

$$\prod_{j=1}^{k}\Gamma(-\zeta_j)(-x_j)^{\zeta_i}\,\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_k \tag{2.1}$$

where $max\left[|arg(-x_1)|,\cdots,|arg(-x_k)|\right]<\pi, c\neq 0,-1,-2,\cdots$.

In order to evaluate a number of integrals of multivariable I-function, we first establish the formula

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}\mathrm{d}t=(b-a)^{\alpha+\beta-1}B(\alpha,\beta)\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}$$

$$\times F_D^{(k)}\left[\alpha,-\sigma_1,\cdots,-\sigma_k;\alpha+\beta;-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right] \tag{2.2}$$

where $a,b\in\mathbb{R}(a<b),\alpha,\beta,f_i,g_i,\sigma_i\in\mathbb{C},(i=1,\cdots,k)$; $min(Re(\alpha),Re(\beta))>0$ and

$$\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1$$

$F_D^{(k)}$ is a Lauricella's function of k-variables, see Srivastava et al ([5], page60)

The formula (2.2) can be establish by expanding $\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$ by means of the formula :

$$(1-z)^{-\alpha}=\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}z^r(|z|<1) \tag{2.3}$$

integrating term by term with the help of the integral given by Saigo and Saxena [1, page 93, eq.(3.2)] and applying the definition of the Lauricella function $F_D^{(k)}$ [4, page 454].

3. Eulerian integral

In this section , we note :

$$K_1=(1-\alpha-\sum_{i=1}^{u}R_i(\mu_i+\mu_i');\mu_1,\cdots,\mu_r,\mu_1',\cdots,\mu_s',1,\cdots,1,\upsilon_1,\cdots,\upsilon_l) \tag{3.1}$$

$$K_2=(1-\beta-\sum_{i=1}^{u}R_i(\rho_i+\rho_i');\rho_1,\cdots,\rho_r,\rho_1',\cdots,\rho_s',0,\cdots,0,\tau_1,\cdots,\tau_l) \tag{3.2}$$

$$K_P=[1-A_j;0,\cdots,0,0,\cdots,0,0,\cdots,0,1,\cdots,1]_{1,P} \tag{3.3}$$

$$K_j=[1+\sigma_j-\sum_{i=1}^{u}R_i(\lambda_i^{(j)}+\lambda_i'^{(j)});\lambda_j^{(1)},\cdots,\lambda_j^{(r)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(s)},0\cdots,1,\cdots,0,\zeta_j',\cdots,\zeta_j^{(l)}]_{1,k} \tag{3.4}$$

j

$$L_1=(1-\alpha-\beta-\sum_{i=1}^{u}R_i(\mu_i+\mu_i'+\rho_i+\rho_i');\mu_1+\rho_1,\cdots,\mu_r+\rho_r,\mu_1'+\rho_1',\cdots,\mu_s'+\rho_s',$$

$$1,\cdots,1,\upsilon_1+\tau_1,\cdots,\upsilon_l+\tau_l) \tag{3.5}$$

$$L_Q=[1-B_j;0,\cdots,0,0,\cdots,0,0,\cdots,0,1\cdots,1]_{1,Q} \tag{3.6}$$

$$L_j=[1+\sigma_j-\sum_{i=1}^{u}R_i(\lambda_i^{(j)}+\lambda_i'^{(j)});\lambda_j^{(1)},\cdots,\lambda_j^{(r)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(s)},0\cdots,0,\zeta_j',\cdots,\zeta_j^{(l)}]_{1,k} \tag{3.7}$$

$$V_1=V;V';1,0;\cdots;1,0;1,0;\cdots;1,0\ ;W_1=W;W';0,1;\cdots;0,1;0,1;\cdots;0,1 \tag{3.8}$$

$$C_1=C;C';(1,0),\cdots,(1,0);(1,0),\cdots,(1,0)\ ;D_1=D;D';(0,1),\cdots,(0,1);(0,1),\cdots,(0,1) \tag{3.9}$$

We have the following result

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_u}\left(\begin{array}{c} \mathrm{x}_1(t-a)^{\mu_1+\mu_1'}(b-t)^{\rho_1+\rho_1'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}} \\ . \\ . \\ . \\ \mathrm{x}_u(t-a)^{\mu_u+\mu_u'}(b-t)^{\rho_u+\rho_u'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(u)}-\lambda_j'^{(u)}} \end{array}\right)$$

$$\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} \mathrm{z}_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ . \\ . \\ . \\ \mathrm{z}_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$$\aleph_{U':W'}^{0,N:V'}\left(\begin{array}{c} \mathrm{z'}_1(t-a)^{\mu_1'}(b-t)^{\rho_1'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ . \\ . \\ . \\ \mathrm{z'}_s(t-a)^{\mu_s'}(b-t)^{\rho_s'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{array}\right)$$

$$
{}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{l} z_i''(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t=
$$

$$
(b-a)^{\alpha+\beta+\sum_1^u R_i(\mu_i+\mu_i'+\rho_i+\rho_i')}\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j-\sum_{i=1}^{u}R_i(\lambda_j^{(i)}+\lambda_j'^{(i)})}
$$

$$
\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}(-L)_{h_1R_1+\cdots+h_uR_u}B(E;R_1,\cdots,R_u)\frac{x_1^{R_1}\cdots x_u^{R_u}}{R_1!\cdots R_u!}
$$

$$
\aleph_{U;U;l+k+2,l+k+1:W_1}^{0,n;0,N;l+k+2:V_1}\left(\begin{array}{c|c}
\frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \\
\cdot\ \cdot\ \cdot & \\
\frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \text{A ; A'; K}_1,K_2,K_P,K_j:C_1 \\
\frac{z_1'(b-a)^{\mu_1'+\rho_1'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\
\cdot\ \cdot\ \cdot & \cdot \\
\frac{z_s'(b-a)^{\mu_s'+\rho_s'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(s)}}} & \cdot \\
\frac{(b-a)f_1}{af_1+g_1} & \cdot \\
\cdot\ \cdot\ \cdot & \cdot \\
\frac{(b-a)f_k}{af_k+g_k} & \cdot \\
\frac{z_1''(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(1)}}} & \text{B ;B'; L}_1,L_j,L_Q:D_1 \\
\cdot\ \cdot\ \cdot & \\
\frac{z_l''(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(l)}}} &
\end{array}\right) \qquad (3.10)
$$

We obtain the Aleph-function of $r+s+k+l$ variables. The quantities $A,B,A',B',C_1,D_1,K_1,K_2,K_P,K_j,L_1,L_j$, L_Q and $V_1,W_1$are defined above.

Provided that

(A) $a,b\in\mathbb{R}(a<b);\mu_i,\rho_i,\mu_j',\rho_j'\lambda_v^{(i)};\lambda_v'^{(i)}\in\mathbb{R}^+$, $f_i,g_j,\tau_v,\sigma_j\in\mathbb{C}$ $(i=1,\cdots,r;j=1,\cdots;s;v=1,\cdots,k)$

(B) See I

(C) $\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1$

(D) $Re\left[\alpha + \sum_{i=1}^{r} \mu_i \min_{1\leqslant j\leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}} + \sum_{i=1}^{s} \mu'_i \min_{1\leqslant j\leqslant M_i} \frac{b_j^{(i)}}{\beta_j^{(i)}}\right] > 0$

$$Re\left[\beta + \sum_{i=1}^{r} \rho_i \min_{1\leqslant j\leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}} + \sum_{i=1}^{s} \rho'_i \min_{1\leqslant j\leqslant M_i} \frac{b_j'^{(i)}}{\beta_j'^{(i)}}\right] > 0$$

(E) $U_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} + \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} + \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{m_k} \delta_j^{(k)}$

$$-\tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0$$

$$U_i'^{(k)} = \sum_{j=1}^{N} \mu_j^{(k)} + \iota_i \sum_{j=N+1}^{P_i} \mu_{ji}^{(k)} + \sum_{j=1}^{N_k} \alpha_j^{(k)} + \iota_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \alpha_{ji^{(k)}}^{(k)} - \iota_i \sum_{j=1}^{Q_i} \upsilon_{ji}^{(k)} - \sum_{j=1}^{M_k} \beta_j^{(k)}$$

$$-\iota_{i^{(k)}} \sum_{j=M_k+1}^{Q_{i^{(k)}}} \beta_{ji^{(k)}}^{(k)} \leqslant 0$$

(F) $A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$

$+\sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} - \mu_k - \rho_k > 0$, with $k = 1 \cdots, r, i = 1, \cdots, R$, $i^{(k)} = 1, \cdots, R^{(k)}$

$B_i^{(k)} = \sum_{j=1}^{N} \mu_j^{(k)} - \iota_i \sum_{j=N+1}^{P_i} \mu_{ji}^{(k)} - \iota_i \sum_{j=1}^{Q_i} \upsilon_{ji}^{(k)} + \sum_{j=1}^{N_k} \alpha_j^{(k)} - \iota_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \alpha_{ji^{(k)}}^{(k)}$

$+\sum_{j=1}^{M_k} \beta_j^{(k)} - \iota_{i^{(k)}} \sum_{j=M_k+1}^{q_{i^{(k)}}} \beta_{ji^{(k)}}^{(k)} - \mu'_k - \rho'_k > 0$, with $k = 1, \cdots, s, i = 1, \cdots, r$, $i^{(k)} = 1, \cdots, r^{(k)}$

(G) $\left| arg\left(z_i \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(i)}} \right) \right| < \frac{1}{2} A_i^{(k)} \pi \ \ (a \leqslant t \leqslant b; i = 1, \cdots, r)$

$$\left| arg\left(z_i \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j'^{(i)}} \right) \right| < \frac{1}{2} B_i^{(k)} \pi \ \ (a \leqslant t \leqslant b; i = 1, \cdots, s)$$

(H) $P \leqslant Q + 1$. The equality holds, when , in addition, either $P > Q$ and $\left| z_i \left(\prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(i)}} \right) \right|^{\frac{1}{Q-P}} < 1 \qquad (a \leqslant t \leqslant b)$

or $P \leqslant Q$ and $\max_{1 \leqslant i \leqslant k} \left[\left| \left(z'_i \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(i)}} \right) \right| \right] < 1 \qquad (a \leqslant t \leqslant b)$

Proof

First expressing the class of polynomial $S_L^{h_1,\cdots,h_u}[.]$ in serie with the help of (1.25), expressing the Aleph-function of r variables by the Mellin-Barnes contour integral with the help of the equation (1.1), the Aleph-function of s variables by the Mellin-Barnes contour integral with the help of the equation (1.13), the generalized hypergeometric function ${}_PF_Q(.)$ in Mellin-Barnes contour integral with the help of (2.1). Now collect the power of $(f_j t + g_j)$ with $j = 1, \cdots, k$ and use the equations (2.1) and (2.2) and we obtain $k-$Mellin-Barnes contour integral. Changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) and interpreting $(r+s+k+l)-$Mellin-barnes contour integral in multivariable Aleph-function , we obtain the desired result (3.10).

4. Particular cases

a) If $\tau_i, \tau_{i^{(1)}}, \cdots, \tau_{i^{(r)}} \to 1$ and $\iota_i, \iota_{i^{(1)}}, \cdots, \iota_{i^{(s)}} \to 1$, the multivariable Aleph-functions degenere in multivariable I-function defined by Sharma and al and we have the General Eulerian integral of product of two multivariable I-functions.

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1} \prod_{j=1}^{k} (f_j t + g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_u} \left(\begin{array}{c} \mathrm{x}_1 (t-a)^{\mu_1+\mu'_1}(b-t)^{\rho_1+\rho'_1} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{x}_u (t-a)^{\mu_u+\mu'_u}(b-t)^{\rho_u+\rho'_u} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(u)}-\lambda_j'^{(u)}} \end{array} \right)$$

$$I_{U:W}^{0,\mathfrak{n}:V} \left(\begin{array}{c} \mathrm{z}_1 (t-a)^{\mu_1}(b-t)^{\rho_1} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r (t-a)^{\mu_r}(b-t)^{\rho_r} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(r)}} \end{array} \right)$$

$$I_{U':W'}^{0,N:V'} \left(\begin{array}{c} \mathrm{z'}_1 (t-a)^{\mu'_1}(b-t)^{\rho'_1} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z'}_s (t-a)^{\mu'_s}(b-t)^{\rho'_s} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j'^{(s)}} \end{array} \right)$$

$$ {}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{l} z_i''(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t= $$

$$ (b-a)^{\alpha+\beta+\sum_1^u R_i(\mu_i+\mu_i'+\rho_i+\rho_i')}\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j+\sum_{i=1}^u R_i(\lambda_j^{(i)}+\lambda_j'^{(i)})} $$

$$ \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}(-L)_{h_1R_1+\cdots+h_uR_u}B(E;R_1,\cdots,R_u)\frac{x_1^{R_1}\cdots x_u^{R_u}}{R_1!\cdots R_u!} $$

$$ I_{U;U;l+k+2,l+k+1:W_1}^{0,n;0,N;l+k+2:V_1}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(1)}}} & \\ \cdots & \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(r)}}} & \mathrm{A}\ ;\ \mathrm{A'};\ \mathrm{K}_1,K_2,K_P,K_j:C_1 \\ \frac{z_1'(b-a)^{\mu_1'+\rho_1'}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdots & \cdot \\ \frac{z_s'(b-a)^{\mu_s'+\rho_s'}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(s)}}} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \cdot \\ \frac{(b-a)f_k}{af_k+g_k} & \cdot \\ \frac{z_1''(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^k(af_j+g_j)^{\zeta_j^{(1)}}} & \mathrm{B}\ ;\mathrm{B'};\ \mathrm{L}_1,L_j,L_Q:D_1 \\ \cdots & \\ \frac{z_l''(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^k(af_j+g_j)^{\zeta_j^{(l)}}} & \end{array}\right) \tag{4.1} $$

under the same conditions that (3.8) with $\tau_i,\tau_{i^{(1)}},\cdots,\tau_{i^{(r)}}\rightarrow 1$ and $\iota_i,\iota_{i^{(1)}},\cdots,\iota_{i^{(s)}}\rightarrow 1$

a) If $\bar{B}(L;R_1,\cdots,R_u)=\dfrac{\prod_{j=1}^{\bar{A}}(\bar{a_j})_{R_1\theta_j'+\cdots+R_u\theta_j^{(u)}}\prod_{j=1}^{\bar{B}'}(\bar{b_j'})_{R_1\phi_j'}\cdots\prod_{j=1}^{\bar{B}^{(u)}}(\bar{b}_j^{(u)})_{R_u\phi_j^{(u)}}}{\prod_{j=1}^{\bar{C}}(\bar{c_j})_{m_1\psi_j'+\cdots+m_u\psi_j^{(u)}}\prod_{j=1}^{\bar{D}'}(\bar{d_j'})_{R_1'\bar{\delta'}_j}\cdots\prod_{j=1}^{\bar{D}^{(u)}}(\bar{d}_j^{(u)})_{R_u\bar{\delta}_j^{(u)}}}$ (4.2)

then the general class of multivariable polynomial $S_L^{R_1,\cdots,R_u}[x_1,\cdots,x_u]$ reduces to generalized Lauricella function defined by Srivastava et al [6].

$$F^{1+\bar{A}:\bar{B}';\cdots;\bar{B}^{(u)}}_{\bar{C}:\bar{D}';\cdots;\bar{D}^{(u)}}\left(\begin{array}{c}\mathrm{x}_1\\ \cdots\\ \cdots\\ \mathrm{x}_u\end{array}\left|\begin{array}{c}[(\text{-L});\mathrm{R}_1,\cdots,R_u][(\bar{a});\bar{\theta}',\cdots,\bar{\theta}^{(u)}]:[(\bar{b}');\bar{\phi}'];\cdots;[(\bar{b}^{(u)}),\bar{\phi}^{(u)}]\\ [(\mathrm{c})\;;\psi',\cdots,\bar{\psi}^{(u)}]:[(\bar{d}');\bar{\delta}'];\cdots;[(\bar{d}^{(u)}),\bar{\delta}^{(u)}]\end{array}\right.\right) \quad (4.3)$$

and we have the formula

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^k (f_jt+g_j)^{\sigma_j}$$

$$F^{1+\bar{A}:\bar{B}';\cdots;\bar{B}^{(u)}}_{\bar{C}:\bar{D}';\cdots;\bar{D}^{(u)}}\left(\begin{array}{c}\mathrm{x}_1(t-a)^{\mu_1+\mu_1'}(b-t)^{\rho_1+\rho_1'}\prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}}\\ .\\ .\\ .\\ \mathrm{x}_u(t-a)^{\mu_u+\mu_u'}(b-t)^{\rho_u+\rho_u'}\prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(u)}-\lambda_j'^{(u)}}\end{array}\right.$$

$$\left.\begin{array}{c}[(\text{-L});\mathrm{R}_1,\cdots,R_u][(\bar{a});\bar{\theta}',\cdots,\bar{\theta}^{(u)}]:[(\bar{b}');\bar{\phi}'];\cdots;[(\bar{b}^{(u)}),\bar{\phi}^{(u)}]\\ [(\mathrm{c})\;;\psi',\cdots,\bar{\psi}^{(u)}]:[(\bar{d}');\bar{\delta}'];\cdots;[(\bar{d}^{(u)}),\bar{\delta}^{(u)}]\end{array}\right)$$

$$\aleph^{0,\mathfrak{n}:V}_{U:W}\left(\begin{array}{c}\mathrm{z}_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(1)}}\\ .\\ .\\ .\\ \mathrm{z}_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(r)}}\end{array}\right)$$

$$\aleph^{0,N:V'}_{U':W'}\left(\begin{array}{c}\mathrm{z'}_1(t-a)^{\mu_1'}(b-t)^{\rho_1'}\prod_{j=1}^k (f_jt+g_j)^{-\lambda_j'^{(1)}}\\ .\\ .\\ .\\ \mathrm{z'}_s(t-a)^{\mu_s'}(b-t)^{\rho_s'}\prod_{j=1}^k (f_jt+g_j)^{-\lambda_j'^{(s)}}\end{array}\right)$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^l z_i''(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^k (f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t=$$

$$(b-a)^{\alpha+\beta+\sum_1^u R_i(\mu_i+\mu_i'+\rho_i+\rho_i')}\frac{\prod_{j=1}^Q\Gamma(B_j)}{\prod_{j=1}^P\Gamma(A_j)}\prod_{j=1}^k(af_j+g_j)^{\sigma_j+\sum_{i=1}^u R_i(\lambda_j^{(i)}+\lambda_j'^{(i)})}$$

$$\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}(-L)_{h_1R_1+\cdots+h_uR_u}\bar{B}(E;R_1,\cdots,R_u)\frac{x_1^{R_1}\cdots x_u^{R_u}}{R_1!\cdots R_u!}$$

$$\aleph_{U;U;l+k+2,l+k+1:W_1}^{0,n;0,N;l+k+2:V_1}\left(\begin{array}{c|c}\frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(1)}}} & \\ \cdots & \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(r)}}} & \mathrm{A}\ ;\ \mathrm{A'};\ \mathrm{K}_1,K_2,K_P,K_j:C_1 \\ \frac{z_1'(b-a)^{\mu_1'+\rho_1'}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdots & \cdot \\ \frac{z_s'(b-a)^{\mu_s'+\rho_s'}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(s)}}} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \cdot \\ \frac{(b-a)f_k}{af_k+g_k} & \cdot \\ \frac{z_1''(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^k(af_j+g_j)^{\zeta_j^{(1)}}} & \mathrm{B}\ ;\mathrm{B'};\ \mathrm{L}_1,L_j,L_Q:D_1 \\ \cdots & \\ \frac{z_l''(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^k(af_j+g_j)^{\zeta_j^{(l)}}} & \end{array}\right) \tag{4.4}$$

where

$$\bar{B}(L;R_1,\cdots,R_u)=\frac{\prod_{j=1}^{\bar{A}}(\bar{a_j})_{R_1\theta_j'+\cdots+R_u\theta_j^{(u)}}\prod_{j=1}^{\bar{B}'}(\bar{b_j'})_{R_1\phi_j'}\cdots\prod_{j=1}^{\bar{B}^{(u)}}(\bar{b}_j^{(u)})_{R_u\phi_j^{(u)}}}{\prod_{j=1}^{\bar{C}}(\bar{c_j})_{m_1\psi_j'+\cdots+m_u\psi_j^{(u)}}\prod_{j=1}^{\bar{D}'}(\bar{d_j'})_{R_1'\bar{\delta'}_j}\cdots\prod_{j=1}^{\bar{D}^{(u)}}(\bar{d}_j^{(u)})_{R_u\bar{\delta}_j^{(u)}}} \tag{4.5}$$

under the dame conditions that (3.8).

c) If $\tau_i,\tau_{i^{(1)}},\cdots,\tau_{i^{(r)}}\rightarrow 1,\ \iota_i,\iota_{i^{(1)}},\cdots,\iota_{i^{(s)}}\rightarrow 1$ and $R=R^{(1)}=\cdots=R^{(r)}=r'=r^{(1)}=\cdots,r^{(s)}=1$ the multivariable Aleph-function degenere in multivariable H-function defined by Srivastava et al and we obtain :

$$\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^k(f_jt+g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_u}\left(\begin{array}{c}\mathrm{x}_1(t-a)^{\mu_1+\mu_1'}(b-t)^{\rho_1+\rho_1'}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{x}_u(t-a)^{\mu_u+\mu_u'}(b-t)^{\rho_u+\rho_u'}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(u)}-\lambda_j'^{(u)}}\end{array}\right)$$

$$H_{U;W}^{0,n;V}\left(\begin{array}{c} \mathrm{z}_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(1)}} \\ . \\ . \\ . \\ \mathrm{z}_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$$H_{U';W'}^{0,N;V'}\left(\begin{array}{c} \mathrm{z'}_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ . \\ . \\ . \\ \mathrm{z'}_s(t-a)^{\mu'_s}(b-t)^{\rho'_s}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{array}\right)$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{l}z_i''(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t=$$

$$(b-a)^{\alpha+\beta+\sum_1^u R_i(\mu_i+\mu'_i+\rho_i+\rho'_i)}\frac{\prod_{j=1}^Q\Gamma(B_j)}{\prod_{j=1}^P\Gamma(A_j)}\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j+\sum_{i=1}^u R_i(\lambda_j^{(i)}+\lambda_j'^{(i)})}$$

$$\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}(-L)_{h_1R_1+\cdots+h_uR_u}B(E;R_1,\cdots,R_u)\frac{x_1^{R_1}\cdots x_u^{R_u}}{R_1!\cdots R_u!}$$

$$H_{p+P+l+k+2,q+Q+l+k+1:W_1}^{0,n+N+l+k+2:V_1}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(1)}}} & \\ \cdots & \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(r)}}} & \mathrm{A}\ ,\mathrm{A'},\ \mathrm{K}_1,K_2,K_P,K_j;C_1 \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(1)}}} & . \\ \cdots & . \\ \frac{z'_s(b-a)^{\mu'_s+\rho'_s}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(s)}}} & . \\ \frac{(b-a)f_1}{af_1+g_1} & . \\ \cdots & . \\ \frac{(b-a)f_k}{af_k+g_k} & . \\ \frac{z''_1(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^k(af_j+g_j)^{\zeta_j^{(1)}}} & \mathrm{B,B'},\ \mathrm{L}_1,L_j,L_Q;D_1 \\ \cdots & \\ \frac{z''_l(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^k(af_j+g_j)^{\zeta_j^{(l)}}} & \end{array}\right) \tag{4.6}$$

under the same conditions that (3.8) with $\tau_i, \tau_{i^{(1)}}, \cdots, \tau_{i^{(r)}} \rightarrow 1,$ $\iota_i, \iota_{i^{(1)}}, \cdots, \iota_{i^{(s)}} \rightarrow 1$ and $R = R^{(1)} = \cdots = R^{(r)} = r' = r^{(1)} = \cdots, r^{(s)} = 1$

where :

$$A =(a_j; \alpha_j^{(1)}, \alpha_j^{(2)}, \cdots, \alpha_j^{(r)}, 0, \cdots, 0, 0 \cdots, 0, 0, \cdots, 0)_{1,p} \tag{4.7}$$

$$A' =(u_j; 0, \cdots, 0, \mu_j^{(1)}, \mu_j^{(2)}, \cdots, \mu_j^{(s)}, 0, \cdots, 0, 0, \cdots, 0)_{1,P} \tag{4.8}$$

$$B =(b_j; \beta_j^{(1)}, \beta_j^{(2)}, \cdots, \beta_j^{(r)}, 0, \cdots, 0, 0 \cdots, 0, 0, \cdots, 0)_{1,q} \tag{4.9}$$

$$B' = (v_j; 0, \cdots, 0, v_s^{(j)}, v_j^{(j)}, \cdots, v_s^{(j)}, 0, \cdots, 0, 0, \cdots, 0)_{1,Q} \tag{4.10}$$

d) Let $z_1', \cdots, z_z' \rightarrow 0$ and $\lambda_j'^{(i)} \rightarrow 0, (i = 1, \cdots, s; j = 1, \cdots, k)$ we obtain the Aleph-function of $(r + k + l)-$ variables. We note

$$K_1 = (1 - \alpha - \sum_{i=1}^{u} R_i \mu_i; \mu_1, \cdots, \mu_r; 1, \cdots, 1, v_1, \cdots, v_l) \tag{4.11}$$

$$K_2 = (1 - \beta - \sum_{i=1}^{u} R_i \rho_i; \rho_1, \cdots, \rho_r; 0, \cdots, 0, \tau_1, \cdots, \tau_l) \tag{4.12}$$

$$K_3 = [1 - A_j; 0, \cdots, 0; 0, \cdots, 0, 1 \cdots, 1]_{1,P} \tag{4.13}$$

$$K_j = [1 + \sigma_j - \sum_{i=1}^{u} R_i \lambda_i^{(j)}; \lambda_j^{(1)}, \cdots, \lambda_j^{(r)}; 0, \cdots, \underset{\text{j}}{1}, \cdots, 0, \zeta_j', \cdots, \zeta_j^{(l)}]_{1,k} \tag{4.14}$$

$$L_1 = (1 - \alpha - \beta - \sum_{i=1}^{u} R_i(\mu_i + \rho_i); \mu_1 + \rho_1, \cdots, \mu_r + \rho_r;$$

$$1, \cdots, 1, v_1 + \tau_1, \cdots, v_l + \tau_l) \tag{4.15}$$

$$L_2 = [1 - B_j; 0, \cdots, 0; 0, \cdots, 0, 1, \cdots, 1]_{1,Q} \tag{4.16}$$

$$L_j = [1 + \sigma_j - \sum_{i=1}^{u} R_i \lambda_i^{(j)}; \lambda_j^{(1)}, \cdots, \lambda_j^{(r)}; 0, \cdots, 0, \zeta_j', \cdots, \zeta_j^{(l)}]_{1,k} \tag{4.17}$$

$$V^{(*)} = V; 1, 0; \cdots; 1, 0; 1, 0; \cdots; 1, 0 \quad ; W^* = W; 0, 1; \cdots; 0, 1; 0, 1; \cdots; 0, 1 \tag{4.18}$$

$$C^* = C; (1, 0), \cdots, (1, 0); (1, 0), \cdots, (1, 0); D^* = D; (0, 1), \cdots, (0, 1); (0, 1), \cdots, (0, 1) \tag{4.19}$$

We have the following integral :

$$\int_a^b (t - a)^{\alpha - 1} (b - t)^{\beta - 1} \prod_{j=1}^{k} (f_j t + g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_u}\left(\begin{array}{c} \mathrm{x}_1(t-a)^{\mu_1+\mu'_1}(b-t)^{\rho_1+\rho'_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(1)}} \\ . \\ . \\ . \\ \mathrm{x}_u(t-a)^{\mu_u+\mu'_u}(b-t)^{\rho_u+\rho'_u}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(u)}} \end{array}\right)$$

$$\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} \mathrm{z}_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(1)}} \\ . \\ . \\ . \\ \mathrm{z}_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{l} z'_i(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t=$$

$$(b-a)^{\alpha+\beta+\sum_1^u R_i(\mu_i+\rho_i)}\frac{\prod_{j=1}^Q\Gamma(B_j)}{\prod_{j=1}^P\Gamma(A_j)}\prod_{j=1}(af_j+g_j)^{\sigma_j+\sum_{i=1}^u R_i\lambda_j^{(i)}}$$

$$\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}(-L)_{h_1R_1+\cdots+h_uR_u}B(E;R_1,\cdots,R_u)\frac{x_1^{R_1}\cdots x_u^{R_u}}{R_1!\cdots R_u!}$$

$$\aleph_{U;l+k+2,l+k+1:W^*}^{0,n;0,N;l+k+2:V^*}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}\ ;\ \mathrm{K}_1,K_2,K_P,K_j;C^* \\ \cdot\ \cdot\ \cdot & . \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(r)}}} & . \\ \frac{(b-a)f_1}{af_1+g_1} & . \\ \cdot\ \cdot\ \cdot & . \\ \frac{(b-a)f_k}{af_k+g_k} & . \\ \frac{z'_1(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^k(af_j+g_j)^{\zeta_j^{(1)}}} & . \\ \cdot\ \cdot\ \cdot & . \\ \frac{z'_l(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^k(af_j+g_j)^{\zeta_j^{(l)}}} & \mathrm{B}\ ;\ \mathrm{L}_1,L_j,L_Q;D^* \end{array}\right) \tag{4.20}$$

Provided that

(A) $a,b \in \mathbb{R}(a<b);\mu_i,\rho_i,\mu'_j,\rho'_j\lambda_v^{(i)};\lambda_v'^{(i)} \in \mathbb{R}^+$, $f_i,g_j,\tau_v,\sigma_j \in \mathbb{C}$ $(i=1,\cdots,r;j=1,\cdots;s;v=1,\cdots,k)$

(B) See I

(C) $\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1$

(D) $Re\big[\alpha+\sum_{i=1}^{r}\mu_i \min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}\big]>0$

$$Re\big[\beta+\sum_{i=1}^{r}\rho_i \min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}\big]>0$$

(E) $U_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}+\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}+\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}-\sum_{j=1}^{m_k}\delta_j^{(k)}$

$$-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}\leqslant 0$$

(F) $A_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}-\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$

$+\sum_{j=1}^{m_k}\delta_j^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}-\mu_k-\rho_k>0,$ with $k=1\cdots,r,i=1,\cdots,R$, $i^{(k)}=1,\cdots,R^{(k)}$

(G) $\left|arg\left(z_i\prod_{j=1}^{k}(f_jt+g_j)^{\lambda_j^{(i)}}\right)\right|<\frac{1}{2}A_i^{(k)}\pi\ \ (a\leqslant t\leqslant b;i=1,\cdots,r)$

(H) $P\leqslant Q+1$. The equality holds, when , in addition,

either $P>Q$ and $\left|z_i\left(\prod_{j=1}^{k}(f_jt+g_j)^{\lambda_j^{(i)}}\right)\right|^{\frac{1}{Q-P}}<1\qquad(a\leqslant t\leqslant b)$

or $P\leqslant Q$ and $\max_{1\leqslant i\leqslant k}\left[\left|\left(z'_i\prod_{j=1}^{k}(f_jt+g_j)^{\lambda_j^{(i)}}\right)\right|\right]<1\qquad(a\leqslant t\leqslant b)$

Remark

By the following similar procedure, the results of this document can be extented to product of any finite number of multivariable Aleph-functions.

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of two multivariable Aleph-functions with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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