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Eulerian integral associated with product of two multivariable Aleph-functions

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ABSTRACT
The present paper is evaluated a new Eulerian integral associated with the product of two multivariable Aleph-functions with general arguments. Several particular cases are given .

Keywords: Eulerian integral, multivariable Aleph-function, multivariable I-function, generalized Lauricella function of several variables, multivariable H-function, generalized hypergeometric function.

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1. Introduction

In this paper, we evaluate a new Eulerian integral of most general characters associated with the products of two multivariable Aleph-functions and general class of polynomials of several variables, but of greater order. Several particular cases are given.
The Aleph-function of several variables generalize the multivariable I-function defined by Sharma and Ahmad [2] , itself is an a generalisation of G and H-functions of several variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We have : $\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i(1)},q_{i(1)},\tau_{i(1)};R^{(1)};\cdots;p_{i(r)},q_{i(r)};\tau_{i(r)};R^{(r)}}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{array}\right|$

$$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}] \quad ,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:$$
$$\cdots\cdots\cdots\cdots \quad ,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}]:$$

$$\left.\begin{array}{l}[(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;\ ;[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;;[(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^r\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^r\alpha_j^{(k)}s_k)}{\sum_{i=1}^R[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^r\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^r\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

and $\theta_k(s_k) = \dfrac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)} - \delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]}$ (1.3)

Suppose , as usual , that the parameters

$a_j, j = 1,\cdots,p; b_j, j = 1,\cdots,q;$

$c_j^{(k)}, j = 1,\cdots,n_k; c_{ji^{(k)}}^{(k)}, j = n_k+1,\cdots,p_{i^{(k)}};$

$d_j^{(k)}, j = 1,\cdots,m_k; d_{ji^{(k)}}^{(k)}, j = m_k+1,\cdots,q_{i^{(k)}};$

with $k = 1\cdots,r, i = 1,\cdots,R$, $i^{(k)} = 1,\cdots,R^{(k)}$

are complex numbers , and the $\alpha's, \beta's, \gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)} + \tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} + \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)} - \tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)} - \sum_{j=1}^{m_k}\delta_j^{(k)}$$

$$-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)}s_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r}\alpha_j^{(k)}s_k)$ with $j = 1$ to n and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)}s_k)$ with $j = 1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)} - \tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)} - \tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} - \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)} - \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1\cdots,r, i = 1,\cdots,R \text{ , } i^{(k)} = 1,\cdots,R^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(y_1,\cdots,y_r) = 0(\,|y_1|^{\alpha_1},\cdots,|y_r|^{\alpha_r}\,), max(\,|y_1|,\cdots,|y_r|\,) \rightarrow 0$

$\aleph(y_1,\cdots,y_r) = 0(\,|y_1|^{\beta_1},\cdots,|y_r|^{\beta_r}\,), min(\,|y_1|,\cdots,|y_r|\,) \rightarrow \infty$

where $k = 1,\cdots,r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1,\cdots,m_k$ and

$$\beta_k = max[Re((c_j^{(k)}-1)/\gamma_j^{(k)})], j=1,\cdots,n_k$$

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R \ ; \ V = m_1, n_1; \cdots ; m_r, n_r \tag{1.6}$$

$$\mathrm{W} = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \cdots, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.7}$$

$$A = \{(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.8}$$

$$B = \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.9}$$

$$C = \{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\}, \tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}, \cdots, \{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.10}$$

$$D = \{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}, \cdots, \{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.11}$$

The multivariable Aleph-function write :

$$\aleph(z_1, \cdots, z_r) = \aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c} z_1 & \mathrm{A : C} \\ \cdot & \\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot & \\ z_r & \mathrm{B : D} \end{array}\right) \tag{1.12}$$

Consider the Aleph-function of s variables

$$\aleph(z_1, \cdots, z_s) = \aleph_{P_i,Q_i,\iota_i;r':P_{i^{(1)}},Q_{i^{(1)}},\iota_{i^{(1)}};r^{(1)};\cdots;P_{i^{(s)}},Q_{i^{(s)}};\iota_{i^{(s)}};r^{(s)}}^{0,N:M_1,N_1,\cdots,M_s,N_s}\left(\begin{array}{c|} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_s \end{array}\right.$$

$$\begin{array}{ll} [(\mathrm{u}_j; \mu_j^{(1)}, \cdots, \mu_j^{(r')})_{1,N}] & ,[\iota_i(u_{ji}; \mu_{ji}^{(1)}, \cdots, \mu_{ji}^{(r')})_{N+1,P_i}] : \\ \cdots\cdots\cdots\cdots\cdots\cdots & ,[\iota_i(v_{ji}; \upsilon_{ji}^{(1)}, \cdots, \upsilon_{ji}^{(r')})_{M+1,Q_i}] : \end{array}$$

$$\left.\begin{array}{l} [(\mathrm{a}_j^{(1)}); \alpha_j^{(1)})_{1,N_1}], [\iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)}; \alpha_{ji^{(1)}}^{(1)})_{N_1+1,P_i^{(1)}}]; \cdots; [(\mathrm{a}_j^{(s)}); \alpha_j^{(s)})_{1,N_s}], [\iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)}; \alpha_{ji^{(s)}}^{(s)})_{N_s+1,P_i^{(s)}}] \\ [(\mathrm{b}_j^{(1)}); \beta_j^{(1)})_{1,M_1}], [\iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)}; \beta_{ji^{(1)}}^{(1)})_{M_1+1,Q_i^{(1)}}]; \cdots; [(\mathrm{b}_j^{(s)}); \beta_j^{(s)})_{1,M_s}], [\iota_{i^{(s)}}(b_{ji^{(s)}}^{(s)}; \beta_{ji^{(s)}}^{(s)})_{M_s+1,Q_i^{(s)}}] \end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^s}\int_{L_1}\cdots\int_{L_s}\zeta(t_1, \cdots, t_s)\prod_{k=1}^{s}\phi_k(t_k)z_k^{t_k}\,\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.13}$$

with $\omega = \sqrt{-1}$

$$\zeta(t_1, \cdots, t_s) = \frac{\prod_{j=1}^{N}\Gamma(1-u_j+\sum_{k=1}^{s}\mu_j^{(k)}t_k)}{\sum_{i=1}^{r'}[\iota_i\prod_{j=N+1}^{P_i}\Gamma(u_{ji}-\sum_{k=1}^{s}\mu_{ji}^{(k)}t_k)\prod_{j=1}^{Q_i}\Gamma(1-v_{ji}+\sum_{k=1}^{s}\upsilon_{ji}^{(k)}t_k)]} \tag{1.14}$$

and $\phi_k(t_k) = \dfrac{\prod_{j=1}^{M_k}\Gamma(b_j^{(k)}-\beta_j^{(k)}t_k)\prod_{j=1}^{N_k}\Gamma(1-a_j^{(k)}+\alpha_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{r^{(k)}}[\iota_{i^{(k)}}\prod_{j=M_k+1}^{Q_{i^{(k)}}}\Gamma(1-b_{ji^{(k)}}^{(k)}+\beta_{ji^{(k)}}^{(k)}t_k)\prod_{j=N_k+1}^{P_{i^{(k)}}}\Gamma(a_{ji^{(k)}}^{(k)}-\alpha_{ji^{(k)}}^{(k)}s_k)]}$ (1.15)

Suppose , as usual , that the parameters

$u_j, j=1,\cdots,P; v_j, j=1,\cdots,Q;$

$a_j^{(k)}, j=1,\cdots,N_k; a_{ji^{(k)}}^{(k)}, j=n_k+1,\cdots,P_{i^{(k)}};$

$b_{ji^{(k)}}^{(k)}, j=m_k+1,\cdots,Q_{i^{(k)}}; b_j^{(k)}, j=1,\cdots,M_k;$

with $k=1\cdots,s, i=1,\cdots,r'$, $i^{(k)}=1,\cdots,r^{(k)}$

are complex numbers , and the $\alpha's, \beta's, \gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{N}\mu_j^{(k)} + \iota_i\sum_{j=N+1}^{P_i}\mu_{ji}^{(k)} + \sum_{j=1}^{N_k}\alpha_j^{(k)} + \iota_{i^{(k)}}\sum_{j=N_k+1}^{P_{i^{(k)}}}\alpha_{ji^{(k)}}^{(k)} - \iota_i\sum_{j=1}^{Q_i}\upsilon_{ji}^{(k)} - \sum_{j=1}^{M_k}\beta_j^{(k)}$$

$$-\iota_{i^{(k)}}\sum_{j=M_k+1}^{Q_{i^{(k)}}}\beta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.16}$$

The reals numbers τ_i are positives for $i=1,\cdots,r$, $\iota_{i^{(k)}}$ are positives for $i^{(k)}=1\cdots r^{(k)}$

The contour L_k is in the t_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(b_j^{(k)}-\beta_j^{(k)}t_k)$ with $j=1$ to M_k are separated from those of $\Gamma(1-u_j+\sum_{i=1}^{s}\mu_j^{(k)}t_k)$with $j=1$ to N and $\Gamma(1-a_j^{(k)}+\alpha_j^{(k)}t_k)$ with $j=1$ to N_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2}B_i^{(k)}\pi$, where

$$B_i^{(k)} = \sum_{j=1}^{N}\mu_j^{(k)} - \iota_i\sum_{j=N+1}^{P_i}\mu_{ji}^{(k)} - \iota_i\sum_{j=1}^{Q_i}\upsilon_{ji}^{(k)} + \sum_{j=1}^{N_k}\alpha_j^{(k)} - \iota_{i^{(k)}}\sum_{j=N_k+1}^{P_{i^{(k)}}}\alpha_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{M_k}\beta_j^{(k)} - \iota_{i^{(k)}}\sum_{j=M_k+1}^{q_{i^{(k)}}}\beta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k=1,\cdots,s, i=1,\cdots,r , i^{(k)}=1,\cdots,r^{(k)} \tag{1.17}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_s) = 0(\,|z_1|^{\alpha_1'},\cdots,|z_s|^{\alpha_s'}\,), max(\,|z_1|,\cdots,|z_s|\,) \rightarrow 0$

$\aleph(z_1,\cdots,z_s) = 0(\,|z_1|^{\beta_1'},\cdots,|z_s|^{\beta_s'}\,), min(\,|z_1|,\cdots,|z_s|\,) \rightarrow \infty$

where $k=1,\cdots,z:\alpha'_k=min[Re(b_j^{(k)}/\beta_j^{(k)})], j=1,\cdots,M_k$ and

$$\beta'_k=max[Re((a_j^{(k)}-1)/\alpha_j^{(k)})], j=1,\cdots,N_k$$

We will use these following notations in this paper

$$U'=P_i,Q_i,\iota_i;r'\,;\;V'=M_1,N_1;\cdots;M_s,N_s \tag{1.18}$$

$$W'=P_{i^{(1)}},Q_{i^{(1)}},\iota_{i^{(1)}};r^{(1)},\cdots,P_{i^{(r)}},Q_{i^{(r)}},\iota_{i^{(s)}};r^{(s)} \tag{1.19}$$

$$A'=\{(u_j;\mu_j^{(1)},\cdots,\mu_j^{(s)})_{1,N}\},\{\iota_i(u_{ji};\mu_{ji}^{(1)},\cdots,\mu_{ji}^{(s)})_{N+1,P_i}\} \tag{1.20}$$

$$B'=\{\iota_i(v_{ji};\upsilon_{ji}^{(1)},\cdots,\upsilon_{ji}^{(s)})_{M+1,Q_i}\} \tag{1.21}$$

$$C'=(a_j^{(1)};\alpha_j^{(1)})_{1,N_1}\;,\iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)};\alpha_{ji^{(1)}}^{(1)})_{N_1+1,P_{i^{(1)}}},\cdots,(a_j^{(s)};\alpha_j^{(s)})_{1,N_s}\;,\iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)};\alpha_{ji^{(s)}}^{(s)})_{N_s+1,P_{i^{(s)}}} \tag{1.22}$$

$$D'=(b_j^{(1)};\beta_j^{(1)})_{1,M_1},\iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)};\beta_{ji^{(1)}}^{(1)})_{M_1+1,Q_{i^{(1)}}},\cdots,(b_j^{(s)};\beta_j^{(s)})_{1,M_s},\iota_{i^{(s)}}(\beta_{ji^{(s)}}^{(s)};\beta_{ji^{(s)}}^{(s)})_{M_s+1,Q_{i^{(s)}}} \tag{1.23}$$

The multivariable Aleph-function write :

$$\aleph(z_1,\cdots,z_s)=\;\aleph_{U':W'}^{0,N:V'}\left(\begin{array}{c|c} \mathrm{z}_1 & \text{A' : C'} \\ \cdot & \cdot\;\cdot\;\cdot \\ \cdot & \\ \mathrm{z}_s & \text{B' : D'}\end{array}\right) \tag{1.24}$$

2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [3 ,page 39 eq .30]

$$\frac{\prod_{j=1}^{P}\Gamma(A_j)}{\prod_{j=1}^{Q}\Gamma(B_j)}\,{}_PF_Q\left[(A_P);(B_Q);-(x_1+\cdots+x_r)\right]$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\frac{\prod_{j=1}^{P}\Gamma(A_j+s_1+\cdots+s_r)}{\prod_{j=1}^{Q}\Gamma(B_j+s_1+\cdots+s_r)}\Gamma(-s_1)\cdots\Gamma(-s_r)x_1^{s_1}\cdots x_r^{s_r}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.1}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j+s_1+\cdots+s_r)$ are separated from those of $\Gamma(-s_j), j=1,\cdots,r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j=1,\cdots,r$

In order to evaluate a number of integrals of multivariable I-function, we first establish the formula

$$\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}\mathrm{d}t\;=(b-a)^{\alpha+\beta-1}B(\alpha,\beta)\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}$$

$$F^{1:1,\cdots,1;1\cdots,1}_{1:0,\cdots,0;0,\cdots,0}\left(\begin{array}{c}(\alpha:h_1,\cdots,h_l,1,\cdots,1):(\lambda_1:1),\cdots,(\lambda_l:1);(-\sigma_1:1),\cdots,(-\sigma_k:1)\\ \cdots\\ (\alpha+\beta:h_1,\cdots,h_l,1,\cdots,1):-,\cdots,-;-,\cdots,-\end{array}\right.$$

$$\left.;\tau_1(b-a)^{h_1},\cdots,\tau_l(b-a)^{h_l},-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right) \tag{2.2}$$

where $a,b\in\mathbb{R}(a<b),\alpha,\beta,f_i,g_i,\sigma_i,\tau_j,h_j\in\mathbb{C},\lambda_j\in\mathbb{R}^+(i=1,\cdots,k;j=1,\cdots,l)$

$$min(Re(\alpha),Re(\beta))>0,\ \max_{1\leqslant j\leqslant l}\left\{\left|\tau_j(b-a)^{h_j}\right|\right\}<1,\ \max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1,$$

and $F^{1:1,\cdots,1;1\cdots,1}_{1:0,\cdots,0;0,\cdots,0}$ is a particular case of the generalized Lauricella function introduced by Srivastava-Daoust[3,page 454] and [4] given by :

$$F^{1:1,\cdots,1;1\cdots,1}_{1:0,\cdots,0;0,\cdots,0}\left(\begin{array}{c}(\alpha:h_1,\cdots,h_l,1,\cdots,1):(\lambda_1:1),\cdots,(\lambda_l:1);(-\sigma_1:1),\cdots,(-\sigma_k:1)\\ \cdots\\ (\alpha+\beta:h_1,\cdots,h_l,1,\cdots,1):-,\cdots,-;-,\cdots,-\end{array}\right.$$

$$\left.;\tau_1(b-a)^{h_1},\cdots,\tau_l(b-a)^{h_l},-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\prod_{j=1}^{l}\Gamma(\lambda_j)\prod_{j=1}^{k}\Gamma(-\sigma_j)}$$

$$\frac{1}{(2\pi\omega)^{l+k}}\int_{L_1}\cdots\int_{L_{l+k}}\frac{\Gamma\left(\alpha+\sum_{j=1}^{l}h_js_j+\sum_{j=1}^{k}s_{l+j}\right)}{\Gamma\left(\alpha+\beta+\sum_{j=1}^{l}h_js_j+\sum_{j=1}^{k}s_{l+j}\right)}\prod_{j=1}^{l}\Gamma(\lambda_j+s_j)\prod_{j=1}^{k}\Gamma(-\sigma_j+s_{l+j})$$

$$\prod_{j=1}^{l+k}\Gamma(-s_j)z_1^{s_1}\cdots z_l^{s_l}z_{l+1}^{s_{l+1}}\cdots,z_{l+k}^{s_{l+k}}\,\mathrm{d}s_1\cdots\mathrm{d}s_{l+k} \tag{2.3}$$

Here the contour $L_j's$ are defined by $L_j=L_{w\zeta_j\infty}(Re(\zeta_j)=v_j'')$ starting at the point $v_j''-\omega\infty$ and terminating at the point $v_j''+\omega\infty$ with $v_j''\in\mathbb{R}(j=1,\cdots,l)$ and each of the remaining contour $L_{l+1},\cdots,L_{l+k}$ run from $-\omega\infty$ to $\omega\infty$

(2.2) can be easily established by expanding $\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}$ by means of the formula :

$$(1-z)^{-\alpha}=\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}z^r(|z|<1) \tag{2.4}$$

integrating term by term with the help of the integral given by Saigo and Saxena [1, page 93, eq.(3.2)] and applying the definition of the generalized Lauricella function [4, page 454].

3. Eulerian integral

In this section , we note :

$$\theta_i = \prod_{j=1}^{l} \left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{(i)}} \; (i=1,\cdots,r)\,; \theta'_i = \prod_{j=1}^{l} \left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j'^{(i)}} \; (i=1,\cdots,s) \quad (3.1)$$

$$K_1 = (1-\alpha;\mu_1,\cdots,\mu_r,\mu'_1,\cdots,\mu'_s,h_1,\cdots,h_l,1,\cdots,1) \quad (3.2)$$

$$K_2 = (1-\beta;\rho_1,\cdots,\rho_r,\rho'_1,\cdots,\rho'_s,0,\cdots,0,0\cdots,0) \quad (3.3)$$

$$K_j = [1-\lambda_j;\zeta_j^{(1)},\cdots,\zeta_j^{(r)},\zeta_j'^{(1)}\cdots,\zeta_j'^{(s)},0,\cdots,\underset{\text{j}}{1},\cdots,0,0\cdots,0]_{1,l} \quad (3.4)$$

$$K'_j = [1+\sigma_j;\lambda_j^{(1)},\cdots,\lambda_j^{(r)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(s)},0,\cdots,0,0\cdots,\underset{\text{j}}{1},\cdots,0]_{1,k} \quad (3.5)$$

$$L_1 = (1-\alpha-\beta;\mu_1+\rho_1,\cdots,\mu_r+\rho_r,\mu'_1+\rho'_r,\cdots,\mu'_s+\rho'_s,h_1,\cdots,h_l,1,\cdots,1) \quad (3.6)$$

$$L_j = [1-\lambda_j;\zeta_j^{(1)},\cdots,\zeta_j^{(r)},\zeta_j'^{(1)}\cdots,\zeta_j'^{(s)},0,\cdots,0,0\cdots,0]_{1,l} \quad (3.7)$$

$$L'_j = [1+\sigma_j;\lambda_j^{(1)},\cdots,\lambda_j^{(r)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(s)},0,\cdots,0,0,\cdots,0]_{1,k} \quad (3.8)$$

$$V_1 = V;V';1,0;\cdots;1,0;1,0;\cdots;1,0\;\;;W_1 = W;W';0,1;\cdots;0,1;0,1;\cdots;0,1 \quad (3.9)$$

$$C_1 = C;C';(1,0);\cdots;(1,0);(1,0);\cdots;(1,0)\,;D_1 = D;D';(0,1);\cdots;(0,1);(0,1);\cdots;(0,1) \quad (3.10)$$

We the following generalized Eulerian integral :

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} \text{z}_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \text{z}_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$$\aleph_{U':W'}^{0,N:V'}\left(\begin{array}{c} \text{z'}_1\theta'_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \text{z'}_s\theta'_s(t-a)^{\mu'_s}(b-t)^{\rho'_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{array}\right)\mathrm{d}t =$$

$(b-a)^{\alpha+\beta-1}\left\{\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}\right\}$

$$\aleph_{U;U';l+k+2,l+k+1:W_1}^{0,n;0,N;l+k+2:V_1}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \\ \cdots & \mathrm{A}\ ;\ \mathrm{A}\ ;\ \mathrm{K}_1, K_2, K_j, K'_j : C_1 \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \cdot \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(1)}_j}} & \cdot \\ \cdots & \cdot \\ \frac{z'_s(b-a)^{\mu'_s+\rho'_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(s)}_j}} & \cdot \\ \tau_1(b-a)^{h_1} & \cdot \\ \cdots & \cdot \\ \tau_l(b-a)^{h_l} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \mathrm{B}\ ;\mathrm{B'};\ \mathrm{L}_1, L_j, L'_j : D_1 \\ \cdots & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \tag{3.11}$$

We obtain the Aleph-function of $r+s+k+l$ variables. The quantities $A, B, A', B', C', D', K_1, K_2, K_j, K'_j, L_1, L_j$ and L'_j are defined above.

Provided that

(A) $a, b \in \mathbb{R}(a<b); \mu_i, \mu'_u, \rho_i, \rho'_u, \lambda_j^{(i)}, \lambda_j'^{(u)}, h_v \in \mathbb{R}^+$, $f_i, g_j, \tau_v, \sigma_j, \lambda_v \in \mathbb{C}$ $(i=1,\cdots,r; j=1,\cdots;k;$

$u=1,\cdots,s; v=1,\cdots,l)$

(B) See I

(C) $\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1,\ \max_{1\leqslant j\leqslant l}\left\{\left|\tau_j(b-a)^{h_j}\right|\right\}<1$

(D) $Re\big[\alpha+\sum_{i=1}^{r}\mu_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}+\sum_{i=1}^{s}\mu'_i\min_{1\leqslant j\leqslant M_i}\frac{b_j^{(i)}}{\beta_j^{(i)}}\big]>0$

$$Re\big[\beta+\sum_{i=1}^{r}\rho_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}+\sum_{i=1}^{s}\rho'_i\min_{1\leqslant j\leqslant M_i}\frac{b_j^{(i)}}{\beta_j^{(i)}}\big]>0$$

(E) $U_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}+\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}+\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}-\sum_{j=1}^{m_k}\delta_j^{(k)}$

$$-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}\leqslant 0$$

$$U_i'^{(k)}=\sum_{j=1}^{N}\mu_j^{(k)}+\iota_i\sum_{j=N+1}^{P_i}\mu_{ji}^{(k)}+\sum_{j=1}^{N_k}\alpha_j^{(k)}+\iota_{i^{(k)}}\sum_{j=N_k+1}^{P_{i^{(k)}}}\alpha_{ji^{(k)}}^{(k)}-\iota_i\sum_{j=1}^{Q_i}\upsilon_{ji}^{(k)}-\sum_{j=1}^{M_k}\beta_j^{(k)}$$

$$-\iota_{i^{(k)}}\sum_{j=M_k+1}^{Q_{i^{(k)}}}\beta_{ji^{(k)}}^{(k)}\leqslant 0$$

(F) $$A_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}-\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}-\mu_k-\rho_k>0,\quad \text{with } k=1\cdots,r, i=1,\cdots,R\ , i^{(k)}=1,\cdots,R^{(k)}$$

$$B_i^{(k)}=\sum_{j=1}^{N}\mu_j^{(k)}-\iota_i\sum_{j=N+1}^{P_i}\mu_{ji}^{(k)}-\iota_i\sum_{j=1}^{Q_i}\upsilon_{ji}^{(k)}+\sum_{j=1}^{N_k}\alpha_j^{(k)}-\iota_{i^{(k)}}\sum_{j=N_k+1}^{P_{i^{(k)}}}\alpha_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{M_k}\beta_j^{(k)}-\iota_{i^{(k)}}\sum_{j=M_k+1}^{q_{i^{(k)}}}\beta_{ji^{(k)}}^{(k)}-\mu_k'-\rho_k'>0,\quad \text{with } k=1,\cdots,s, i=1,\cdots,r\ , i^{(k)}=1,\cdots,r^{(k)}$$

(G) $$Re\left(\alpha+\sum_{i=1}^{r}v_i\mu_i+\sum_{i=1}^{s}v_i'\mu_i'+\sum_{i=1}^{l}h_iv_i''\right)>0\,;Re\left(\beta+\sum_{i=1}^{r}v_i\rho_i+\sum_{i=1}^{s}v_i'\rho_i'\right)>0$$

$$Re\left(\lambda_j+\sum_{i=1}^{r}v_i\zeta_j^{(i)}+\sum_{i=1}^{s}v_i'\zeta_j'^{(i)}\right)>0(j=1,\cdots,l);\,Re\left(-\sigma_j+\sum_{i=1}^{r}v_i\lambda_j^{(i)}+\sum_{i=1}^{s}v_i'\lambda_j'^{(i)}\right)>0(j=1,\cdots,k);$$

(H) $$\left|arg\left(z_i\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{(i)}}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(i)}}\right)\right|<\frac{1}{2}A_i^{(k)}\pi\quad(a\leqslant t\leqslant b;i=1,\cdots,r)$$

$$\left|arg\left(z_i'\prod_{j=1}^{l}\left[1-\tau_j'(t-a)^{h_i'}\right]^{-\zeta_j'^{(i)}}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(i)}}\right)\right|<\frac{1}{2}B_i^{(k)}\pi\quad(a\leqslant t\leqslant b;i=1,\cdots,s)$$

To prove (3.11) first espressing the Aleph-function of r variables by the Mellin-Barnes contour integral with the help of the equation (1.1), the Aleph-function of s variables by the Mellin-Barnes contour integral with the help of the equation (1.13). Now collect the power of $\left[1-\tau_j(t-a)^{h_i}\right]$ with $(i=1,\cdots,r;j=1,\cdots,l)$ and collect the power of (f_jt+g_j) with $j=1,\cdots,k$. Use the equations (2.2) and (2.3) and express the result in Mellin-Barnes contour integral. Interpreting the $(r+s+k+l)$ dimensional Mellin-Barnes integral in multivariable Aleph-function, we obtain the equation (3.9).

4. Particular cases

a) If $\rho_1 = \cdots , \rho_r = \rho'_1 = \cdots , \rho'_s = 0$, we obtain

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} \mathrm{z}_1\theta_1(t-a)^{\mu_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ . \\ . \\ . \\ \mathrm{z}_r\theta_r(t-a)^{\mu_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$$\aleph_{U':W'}^{0,N:V'}\left(\begin{array}{c} \mathrm{z'}_1\theta'_1(t-a)^{\mu'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ . \\ . \\ . \\ \mathrm{z'}_s\theta'_s(t-a)^{\mu'_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{array}\right)\mathrm{d}t = (b-a)^{\alpha+\beta-1}\left\{\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}\right\}$$

$$\aleph_{U;U';l+k+2,l+k+1:W_1}^{0,n;0,N;l+k+2:V_1}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \\ \cdots & \\ \frac{z_r(b-a)^{\mu_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \mathrm{A}\ ;\ \mathrm{A}\ ;\ \mathrm{K}_1, K_2, K_j, K'_j : C_1 \\ \frac{z'_1(b-a)^{\mu'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & . \\ \cdots & . \\ \frac{z'_s(b-a)^{\mu'_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(s)}}} & . \\ \tau_1(b-a)^{h_1} & . \\ \cdots & . \\ \tau_l(b-a)^{h_l} & . \\ \frac{(b-a)f_1}{af_1+g_1} & . \\ \cdots & \mathrm{B}\ ;\mathrm{B'};\ \mathrm{L'}_1, L_j, L'_j : D_1 \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \tag{4.1}$$

where $L'_1 = (1-\alpha-\beta;\mu_1,\cdots,\mu_r,\mu'_1,\cdots,\mu'_s,h_1,\cdots,h_l,1,\cdots,1)$ (4.2)

the others quantities are the same that (3.9).

The validities conditions are the same that (3.21) with $\rho_1 = \cdots , \rho_r = \rho'_1 = \cdots , \rho'_s = 0$

b) if $\mu_1 = \cdots , \mu_r = \mu'_1 = \cdots , \mu'_s = 0$, we have the following result

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} \mathrm{z}_1\theta_1(b-t)^{\rho_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r\theta_r(b-t)^{\rho_r}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$$\aleph_{U':W'}^{0,N:V'}\left(\begin{array}{c} \mathrm{z'}_1\theta'_1(b-t)^{\rho'_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z'}_s\theta'_s(b-t)^{\rho'_s}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{array}\right)\mathrm{d}t=(b-a)^{\alpha+\beta-1}\left\{\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}\right\}$$

$$\aleph_{U;U';l+k+2,l+k+2,l+k+1:W_1}^{0,n;0,N;l+k+2:V_1}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\rho_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(1)}}} & \\ \cdots & \\ \frac{z_r(b-a)^{\rho_r}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(r)}}} & \mathrm{A}\ ;\ \mathrm{A}\ ;\ \mathrm{K}_1,K_2,K_j,K'_j:C_1 \\ \frac{z'_1(b-a)^{\rho'_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdots & \cdot \\ \frac{z'_s(b-a)^{\rho'_s}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(s)}}} & \cdot \\ \tau_1(b-a)^{h_1} & \cdot \\ \cdots & \cdot \\ \tau_l(b-a)^{h_l} & \mathrm{B}\ ;\mathrm{B'};\ \mathrm{L''}_1,L_j,L'_j:D_1 \\ \frac{(b-a)f_1}{af_1+g_1} & \\ \cdots & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \quad (4.3)$$

where $L''_1=(1-\alpha-\beta;\rho_1,\cdots,\rho_r,\rho'_r,\cdots,\rho'_s,h_1,\cdots,h_l,1,\cdots,1)$ (4.4)

The validities conditions are the same that (3.21) with $\mu_1=\cdots,\mu_r=\mu'_1=\cdots,\mu'_s=0$

c) If $\tau_i,\tau_{i^{(1)}},\cdots,\tau_{i^{(r)}}\rightarrow 1$ and $\iota_i,\iota_{i^{(1)}},\cdots,\iota_{i^{(s)}}\rightarrow 1$, the multivariable Aleph-functions degenere in multivariable I-function defined by Sharma and al and we have the General Eulerian integral of product of two multivariable I-functions.

We the following generalized Eulerian integral :

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$I_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c}\mathrm{z}_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(1)}}\\ .\\ .\\ .\\ \mathrm{z}_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(r)}}\end{array}\right)$$

$$I_{U':W'}^{0,N:V'}\left(\begin{array}{c}\mathrm{z'}_1\theta'_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'^{(1)}}\\ .\\ .\\ .\\ \mathrm{z'}_s\theta'_s(t-a)^{\mu'_s}(b-t)^{\rho'_s}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'^{(s)}}\end{array}\right)\mathrm{d}t=$$

$$(b-a)^{\alpha+\beta-1}\left\{\prod_{j=1}^k(af_j+g_j)^{\sigma_j}\right\}$$

$$I_{U;U';l+k+2,l+k+1:W_1}^{0,n;0,N;l+k+2:V_1}\left(\begin{array}{c|c}\frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(1)}}} & \\ \cdots & \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(r)}}} & \mathrm{A}\ ;\ \mathrm{A}\ ;\ \mathrm{K}_1,K_2,K_j,K'_j:C_1\\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(1)}}} & .\\ \cdots & .\\ \frac{z'_s(b-a)^{\mu'_s+\rho'_s}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(s)}}} & .\\ \tau_1(b-a)^{h_1} & .\\ \cdots & .\\ \tau_l(b-a)^{h_l} & \mathrm{B}\ ;\mathrm{B'};\ \mathrm{L}_1,L_j,L'_j:D_1\\ \frac{(b-a)f_1}{af_1+g_1} & \\ \cdots & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \tag{4.5}$$

under the same conditions that (3.8) with $\tau_i,\tau_{i^{(1)}},\cdots,\tau_{i^{(r)}}\ \rightarrow 1$ and $\iota_i,\iota_{i^{(1)}},\cdots,\iota_{i^{(s)}}\ \rightarrow 1$

d)Let $K_1=(1-\alpha;\mu_1,\cdots,\mu_r,\mu'_1,\cdots,\mu'_s,h_1,\cdots,h_l,1,\cdots,1)$ (4.6)

$$K_2=(1-\beta;\rho_1,\cdots,\rho_r,\rho'_1,\cdots,\rho'_s,0,\cdots,0,0\cdots,0) \tag{4.7}$$

$$K_3=[1-A_j;0,\cdots,0,1,\cdots,1,0\cdots,0,0,\cdots,0]_{1,P} \tag{4.8}$$

$$K_j=[1-\lambda_j;\zeta_j^{(1)},\cdots,\zeta_j^{(r)},\zeta_j'^{(1)}\cdots,\zeta_j'^{(s)},0,\cdots,\underset{\mathrm{j}}{1},\cdots,0,0\cdots,0]_{1,l} \tag{4.9}$$

$$K'_j=[1+\sigma_j;\lambda_j^{(1)},\cdots,\lambda_j^{(r)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(s)},0,\cdots,0,0\cdots,\underset{\mathrm{j}}{1},\cdots,0]_{1,k} \tag{4.10}$$

$$L_1 = (1-\alpha-\beta;\mu_1+\rho_1,\cdots,\mu_r+\rho_r,\mu'_1+\rho'_r,\cdots,\mu'_s+\rho'_s,h_1,\cdots,h_l,1,\cdots,1) \tag{4;11}$$

$$L_2 = [1-B_j;0,\cdots,0,1,\cdots,1,0\cdots,0,0,\cdots,0]_{1,Q} \tag{4.12}$$

$$L_j = [1-\lambda_j;\zeta_j^{(1)},\cdots,\zeta_j^{(r)},\zeta_j'^{(1)}\cdots,\zeta_j'^{(s)},0,\cdots,0,0\cdots,0]_{1,l} \tag{4.13}$$

$$L'_j = [1+\sigma_j;\lambda_j^{(1)},\cdots,\lambda_j^{(r)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(s)},0,\cdots,0,0,\cdots,0]_{1,k} \tag{4.14}$$

$$V^* = V;1,0;\cdots;1,0;1,0;\cdots;1,0;1,0;\cdots;1,0 \tag{4.15}$$

$$W^* = W;0,1;\cdots;0,1;0,1;\cdots;0,1;0,1;\cdots;0,1 \tag{4.16}$$

$$C^{(*)} = C;(1,0);\cdots;(1,0);(1,0);\cdots;(1,0);(1,0);\cdots;(1,0) \tag{4.17}$$

$$D^{(*)} = D;(0,1),\cdots,(0,1);(0,1);\cdots;(0,1);(0,1);\cdots;(0,1) \tag{4.18}$$

We have the following result

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} \mathrm{z}_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{s}z'_i\theta'_i(t-a)^{\mu'_i}(b-t)^{\rho'_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(i)}}\right]\mathrm{d}t = B\times$$

$$\aleph_{U;P+l+k+2,Q+l+k+1:W^*}^{0,n;P+l+k+2:V^*}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \\ \cdot\ \cdot\ \cdot & \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \mathrm{A}\ ;\ \mathrm{K}_1,K_2,K_3,K_j,K'_j:C^* \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdot\ \cdot\ \cdot & \cdot \\ \frac{z'_s(b-a)^{\mu'_s+\rho'_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(s)}}} & \cdot \\ \tau_1(b-a)^{h_1} & \cdot \\ \cdot\ \cdot\ \cdot & \cdot \\ \tau_l(b-a)^{h_l} & \mathrm{B}\ ,\ \mathrm{L}_1,L_2,L_j,L'_j:D^* \\ \frac{(b-a)f_1}{af_1+g_1} & \\ \cdot\ \cdot\ \cdot & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \tag{4.19}$$

where $B = (b-a)^{\alpha+\beta-1} \frac{\prod_{j=1}^{Q} \Gamma(B_j)}{\prod_{j=1}^{P} \Gamma(A_j)} \prod_{j=1} (af_j + g_j)^{\sigma_j}$

Provided that

(A) $a, b \in \mathbb{R}(a < b); \mu_i, \mu'_u, \rho_i, \rho'_u, \lambda_j^{(i)}, \lambda_j'^{(u)}, h_v \in \mathbb{R}^+$, $f_i, g_j, \tau_v, \sigma_j, \lambda_v \in \mathbb{C}$ $(i = 1, \cdots, r; j = 1, \cdots; k;$ $u = 1, \cdots, s; v = 1, \cdots, l)$

(B) See I

(C) $\max_{1 \leqslant j \leqslant k} \left\{ \left| \frac{(b-a)f_i}{af_i + g_i} \right| \right\} < 1, \max_{1 \leqslant j \leqslant l} \left\{ \left| \tau_j (b-a)^{h_j} \right| \right\} < 1$

(D) $Re\left[\alpha + \sum_{i=1}^{r} \mu_i \min_{1 \leqslant j \leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}\right] > 0$

$$Re\left[\beta + \sum_{i=1}^{r} \rho_i \min_{1 \leqslant j \leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}\right] > 0$$

(E) $U_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} + \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} + \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{m_k} \delta_j^{(k)}$

$$-\tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0$$

(F) $A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$

$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} - \mu_k - \rho_k > 0$, with $k = 1 \cdots, r, i = 1, \cdots, R$, $i^{(k)} = 1, \cdots, R^{(k)}$

(G) $Re\left(\alpha + \sum_{i=1}^{r} v_i \mu_i + \sum_{i=1}^{s} v'_i \mu'_i + \sum_{i=1}^{l} h_i v''_i\right) > 0$; $Re\left(\beta + \sum_{i=1}^{r} v_i \rho_i + \sum_{i=1}^{s} v'_i \rho'_i\right) > 0$

$$Re\left(\lambda_j + \sum_{i=1}^{r} v_i \zeta_j^{(i)} + \sum_{i=1}^{s} v'_i \zeta_j'^{(i)}\right) > 0 (j = 1, \cdots, l); Re\left(-\sigma_j + \sum_{i=1}^{r} v_i \lambda_j^{(i)} + \sum_{i=1}^{s} v'_i \lambda_j'^{(i)}\right) > 0 (j = 1, \cdots, k);$$

(H) $\left| arg\left(z_i \prod_{j=1}^{l} \left[1 - \tau_j (t-a)^{h_i}\right]^{-\zeta_j^{(i)}} \prod_{j=1}^{k} (f_j t + g_j)^{\lambda_j^{(i)}} \right) \right| < \frac{1}{2} A_i^{(k)} \pi \quad (a \leqslant t \leqslant b; i = 1, \cdots, r)$

(I) $P \leqslant Q+1$. The equality holds, when , in addition,

either $P > Q$ and $\left|\left(z_i' \sum_{i=1}^{l} \left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{(i)}} \prod_{j=1}^{k} (f_j t + g_j)^{\lambda_j^{(i)}}\right)\right|^{\frac{1}{Q-P}}$ <1 $(a \leqslant t \leqslant b)$

or $P \leqslant Q$ and $\max_{1 \leqslant i \leqslant k} \left[\left|\left(z_i' \sum_{j=1}^{l} \left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{(i)}} \prod_{j=1}^{k} (f_j t + g_j)^{\lambda_j^{(i)}}\right)\right|\right] < 1 \ \ (a \leqslant t \leqslant b)$

Proof

To prove (4.17) first espressing the Aleph-function of r variables by the Mellin-Barnes contour integral with the help of the equation (1.1), the generalized hypergeometric function ${}_PF_Q(.)$ in Mellin-Barnes contour integral with the help of (2.1).Now collect the power of $\left[1-\tau_j(t-a)^{h_i}\right]$ with $(i=1,\cdots,r;j=1,\cdots,l)$ and collect the power of $(f_j t + g_j)$ with $j=1,\cdots,k$. Use the equations (2.2) and (2.3) and express the result in Mellin-Barnes contour integral. Interpreting the $(r+s+k+l)$ dimensional Mellin-Barnes integral in multivariable I-function, we obtain the equation (4.15).

Remark

By the following similar procedure, the results of this document can be extented to product of any finite number of multivariable Aleph-functions.

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of two multivariable Aleph-functions with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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