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ON REGULAR LOCAL NEAR-RINGS

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ABSTRACT

The main aim of this paper is to derive a result on dimension theory in Regular Local near rings.

KEYWORDS: Ring, Near-ring, Local near ring, Regular near ring, Regular -local –ring, Regular local near ring, Krull Dimension, noetherian local ring

1. INTRODUCTION

Let A be a noetherian local near ring, with maximal ideal m and residue field k. Then for each i, A/m^{i+1} as an A-module of finite length, $l_A(i)$. In fact for each i, m^i/m^{i+1} is a is a finite dimensional k-vector space, and $l_A(i) = \sum \dim(m^j / m^{j+1}) : j \le i$ .It turns out that there is a polynomial p_A with rational coefficients such that $p_A(i) = l_A(i)$ for i sufficiently large. Let d_A be the degree of p_A.

2. PRELIMINARIES

In this section we give the definitions and examples which are used in next sections.

2.1 Definition: A non empty set R with two binary operations '+'and '.' Is said to be a ring if i)(R,+) is a commutative ring

ii)(R, .) is a semi group

iii) distributive laws hold good.

2.2 Example: The set of all integers modulo m under addition and multiplication modulo m is a ring

2.3 Definition: A nonempty set N is said to be a Right near-ring with two binary operations '+'and '.' If

i) (N, +) is a group (not necessarily abelian)

ii) (N, .) is a semi group and

iii) (x + y)z= xz + yz for all x ,y ,z $\in$N

2.4 Example: Let Z be the set of positive, negative integers with '0', then (z, + , .) is a near ring with usual addition and multiplication.

2.5 Definition: A near ring N is said to be Regular ring if for each element x $\in$ N then there exists an element y $\in$ N such that x= xyx.

2.6 Example: (i) M(Γ) and $M_0(\Gamma)$ are regular rings (Beidleman(10)NR Text)

(ii)Constant rings

(iii) Direct sum and product of fields.

2.7 Definition: A near ring N is said to be Regular near ring if for each element x $\in$ N then there exists an element y $\in$ N such that x= xyx.

2.8 Example:

(i) M(Γ) and $M_0(\Gamma)$ are regular near rings (Beidleman(10)NR Text)

2.9: Definition: Let (*R,M, k*) be a Noetherian local ring. (The notation means that the maximal ideal is *M* and the residue field is *k* = *R/M*.) If *d* is the dimension of *R*, then by the dimension theorem every generating set of *M* has at least *d* elements.

2.10 Note: If *M* does in fact have a generating set *S* of *d* elements, we say that *R* is *regular* and that *S* is a *regular system of parameters*

2.11 Example If *R* has dimension 0, then *R* is regular iff {0}is a maximal ideal, in other words, iff *R*is a field.
(i) M(Γ) and $M_0(\Gamma)$ are regular near rings (Beidleman(2))

2.12 Definition: A nearing N is local iff N has a unique maximal N-subgroup.

2.13 Example: $M_{aff}(v)$ is local near ring.

2.14 Definition: In a commutative algebra ,a regular local ring is a noetherian local ring having the property that the minimal number of generators of its maximal ideal is equal to its krull dimension.

2.15. Definition: A near ring N is said to be regular local near ring if it satisfies both the conditions of regular and local near rings.(or)

Let A be a noetherian local near ring with maximal ideal m and Krull dimension d. Then d is less than or equal to the dimension of m/m^2, and the ring is said to be regular local if equality holds, and in this case (*) is an isomorphism and A is an integral domain.

3. Main Results

In this section we proved main results on Local regular rings.

Theorem3.1: Let A be a noetherian local near ring, then d_A is the krull dimension of A,and this number is also the minimal length of a sequence ($a_1, \ldots a_d$) of elements of m such that m is nilpotent modulo the ideal generated by ($a_1, \ldots, a_d$).

Corollary3.2: If A is a Noetherian local ring and a is an element of the maximal ideal of A, then $\dim(A/(a)) \leq \dim(A)-1$ with equality if a does not belong to any minimal prime of A (if and only if a doesnot belong to any of the minimal primes which are at the bottom of a chain of length thedimension of A).

Proof: Let $(a_1, \ldots, a_d)$ be sequence of elements of m lifting a minimal sequence in $m/(a)$ such that m is nilpotent modulo (a). Then d is the dimension of $A/(a)$.
But now $(a_0, a_1, \ldots, a_d)$ is a sequence in m such that m is nilpotent. Hence the dimension of A is at most d+1.
Let A be a noetherian local ring and let $Gr_m A := \oplus m_i / m^{i+1}$ which forms a graded k-algebra, generated in degree one by $V := m/m^2$. Then there is a natural surjective map (*) $S^{\cdot} V \to Gr_m A$.
If d is the dimension of V then the dimension of $S^i V$ is just the number of monomials of degree i in d variables, which is easily seen be $\binom{d+i-1}{i}$ if $i \geq 0$. This is a polynomial of degree d -1. It follows that the dimension of $Gr_m A$ is at most $\binom{d+i-1}{i}$ and hence that $l_A(i)$ is bounded by a polynomial of degree d. It follows that the dimension of A is less than or equal to the dimension of the k-vector space V.

Proposition3.3: Let B be a regular local near ring with maximal ideal m, and $A := B/I$, where I is a proper ideal of B. Then A is regular if and only if $I \cap m^2 = mI$, that is, if and only if the map $I/mI \to m/m^2$ is injective.

Proof: let $\overline{m} = m / m \cap I$ be the maximal ideal of A, let $\overline{V} = \frac{\overline{m}}{\overline{m}^2}$ then we get a diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & S.V \arrow[r] \arrow[d] & S.V \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & Gr_m I \arrow[r] & Gr_m B \arrow[r] & Gr_m A \arrow[r] & 0 &
\end{tikzcd}$$

Since B is regular, the middle vertical arrow is bijective. Then A is regular if and only if the map on the right is injective, which is true if and only if the map on the left is surjective. Since K is generated in degree one and the map is an isomorphism in degree one, A is regular if and only if $Gr_m I$ is generated in degree one. The degree i term of $Gr_m I$ is $I \cap m^i / I \cap m^{i+1}$, so regularity of A is equivalent to saying that $I \cap m^i \subseteq I m^{i-1} + I \cap m^{i+1}$ for all i. If this is true, then we see by induction that $I \cap m^2 \subseteq Im + I \cap m^{i+1}$ for all i. The Artin–Rees lemma says that $I \cap m^{i+1} \subseteq mI$ for i sufficiently large, so we can conclude $I \cap m^2 \subseteq Im$, hence $I \cap m^2 = Im$. Conversely, say $I \cap m^2 = Im$, and choose elements $(x_1, \ldots x_r)$ of I lifting a basis of $I \cap m^2$. Then the dimension of $\overline{m}/\overline{m}^2$ is $N - r$, where N is the dimension of B. However, since $(x_1, \ldots x_r)$ generates I/mI it follows from Nakayama that it also generates I, and then that the dimension of A is at least N-r. But then we have equality, hence A is regular.

Theorem3.4: Let k be an algebraically closed field and let X/k be a scheme of finite type, and let x be a closed point of X. Then the following conditions are equivalent:

1. The local near ring $O_{X,x}$ is regular.

2. There is an open neighborhood U of x which is smooth over k.

Proof: We may assume without loss of generality that X is affine, say Spec A, where A is the quotient of a polynomial ring B over k by an ideal I. Let Z:=SpecB and let m be the maximal ideal of B corresponding to the point $x \in X \subseteq Z$. Since the local near ring B_m is regular, Proposition 4 says that A_m is regular if and only if the map

$I/mI \to m/m^2$ is injective. Since x is a k-rational point of Z, the differential induces an isomorphism $m/m^2 \to \Omega_{z/k}(x)$, and corollary 3.4 now says that this injectivity is equivalent to (2).

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