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On general multiple Eulerian integrals and multivariable I-function

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ABSTRACT
The object of this paper is first,to evaluate a general multiple Eulerian integrals with general integrands involving the product of a multivariable I-function defined by Prasad [5] and generalized hypergeometric function with general arguments. The second multiple Eulerian integral contain a general class of polynomials the general polynomial set and multivariable I-function defined by Prasad with general arguments. Our integral formulas are interesting and unified nature.

Keywords :Multivariable I-function, class of polynomial, general polynomials set, multiple Eulerian integral

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1. Introduction

The object of this paper is first,to evaluate a general multiple Eulerian integrals with general integrands involving the product of a multivariable I-function defined by Prasad [5] and generalized hypergeometric function with general arguments. The second multiple Eulerian integral contain a general class of polynomials the general polynomial set and multivariable I-function defined by Prasad [5] with general arguments.

The multivariable I-function of r-variables is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1, z_2, ...z_r) = I^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;\cdots;p_r,q_r:p^{(1)},q^{(1)};\cdots;p^{(r)},q^{(r)}} \left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{array} \middle| \begin{array}{c} (\mathrm{a}_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};\cdots; \\ \\ \\ (\mathrm{b}_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};\cdots; \end{array} \right.$$

$$\left. \begin{array}{c} (\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj})_{1,p_r} : (a^{(1)}_j,\alpha^{(1)}_j)_{1,p^{(1)}};\cdots;(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}} \\ \\ (\mathrm{b}_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj})_{1,q_r} : (b^{(1)}_j,\beta^{(1)}_j)_{1,q^{(1)}};\cdots;(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}} \end{array} \right) \tag{1.1}$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(t_i)z_i^{t_i}\mathrm{d}t_1\cdots\mathrm{d}t_r \tag{1.2}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [5]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_i| < \frac{1}{2}\Omega_i\pi$, where

$$\Omega_i = \sum_{k=1}^{n^{(i)}}\alpha^{(i)}_k - \sum_{k=n^{(i)}+1}^{p^{(i)}}\alpha^{(i)}_k + \sum_{k=1}^{m^{(i)}}\beta^{(i)}_k - \sum_{k=m^{(i)}+1}^{q^{(i)}}\beta^{(i)}_k + \left(\sum_{k=1}^{n_2}\alpha^{(i)}_{2k} - \sum_{k=n_2+1}^{p_2}\alpha^{(i)}_{2k}\right) + \cdots +$$

$$\left(\sum_{k=1}^{n_s}\alpha_{sk}^{(i)}-\sum_{k=n_s+1}^{p_s}\alpha_{sk}^{(i)}\right)-\left(\sum_{k=1}^{q_2}\beta_{2k}^{(i)}+\sum_{k=1}^{q_3}\beta_{3k}^{(i)}+\cdots+\sum_{k=1}^{q_s}\beta_{sk}^{(i)}\right) \tag{1.3}$$

where $i=1,\cdots,r$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.We may establish the the asymptotic expansion in the following convenient form :

$$I(z_1,\cdots,z_r)=0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,)\,,max(\,|z_1|,\cdots,|z_r|\,)\rightarrow 0$$

$$I(z_1,\cdots,z_r)\;=0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,)\,,min(\,|z_1|,\cdots,|z_r|\,)\rightarrow\infty$$

where $k=1,\cdots,r:\alpha_k'=min[Re(b_j^{(k)}/\beta_j^{(k)})],j=1,\cdots,m_k$ and

$$\beta_k'=max[Re((a_j^{(k)}-1)/\alpha_j^{(k)})],j=1,\cdots,n_k$$

We will use these following notations in this section :

$$U_r=p_2,q_2;p_3,q_3;\cdots;p_{r-1},q_{r-1};V_r=0,n_2;0,n_3;\cdots;0,n_{r-1} \tag{1.4}$$

$$W_r=(p^{(1)},q^{(1)});\cdots;(p^{(r)},q^{(r)});X_r=(m^{(1)},n^{(1)});\cdots;(m^{(r)},n^{(r)}) \tag{1.5}$$

$$A=(a_{2k};\alpha_{2k}^{(1)},\alpha_{2k}^{(2)});\cdots;(a_{(r-1)k};\alpha_{(r-1)k}^{(1)},\alpha_{(r-1)k}^{(2)},\cdots,\alpha_{(r-1)k}^{(r-1)}) \tag{1.6}$$

$$B=(b_{2k};\beta_{2k}^{(1)},\beta_{2k}^{(2)});\cdots;(b_{(r-1)k};\beta_{(r-1)k}^{(1)},\beta_{(r-1)k}^{(2)},\cdots,\beta_{(r-1)k}^{(r-1)}) \tag{1.7}$$

$$\mathfrak{A}=(a_{rk};\alpha_{rk}^{(1)},\alpha_{rk}^{(2)},\cdots,\alpha_{rk}^{(r)}):\mathfrak{B}=(b_{rk};\beta_{rk}^{(1)},\beta_{rk}^{(2)},\cdots,\beta_{rk}^{(r)}) \tag{1.8}$$

$$A_1=(a_k^{(1)},\alpha_k^{(1)})_{1,p^{(1)}};\cdots;(a_k^{(r)},\alpha_k^{(r)})_{1,p^{(r)}};B_1=(b_k^{(1)},\beta_k^{(1)})_{1,p^{(1)}};\cdots;(b_k^{(r)},\beta_k^{(r)})_{1,p^{(r)}} \tag{1.9}$$

The multivariable I-function of r-variables write :

$$I(z_1,\cdots,z_r)=\;I_{U_r:p_r,q_r;W_r}^{V_r;0,n_r;X_r}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}\;;\mathfrak{A};\mathrm{A}_1\\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & \mathrm{B};\;\mathfrak{B};\;\mathrm{B}_1\end{array}\right) \tag{1.10}$$

Srivastava [8] introduced the general class of polynomials :

$$S_N^M(x)=\sum_{K=0}^{[N/M]}\frac{(-N)_{MK}}{K!}A_{N,k}x^K,N=0,1,2,... \tag{1.11}$$

where M is an arbtrary positive integer and the coefficient $A_{N,k}$ are arbitrary constants, real or complex.

By suitably specialized the coefficient $A_{N,k}$ the polynomials $S_N^M(x)$ can be reduced to the classical orthogonal polynomials such as Jacobi, Hermite, Legendre and Laguerre polynomials etc.

2. Sequence of function

Agarwal and Chaubey [1], Salim [7] and several others have studied a general sequence of functions. In the present document we shall study the following useful series formula for a general sequence of functions.

$$R_n^{\alpha,\beta}[x;E,F,g,h;p,q;\gamma;\delta;e^{-sx^r}]=\sum_{w,v,u,t,e,k_1,k_2,}\psi(w,v,u,t,e,k_1,k_2)x^R \tag{2.1}$$

where $$\sum_{w,v,u,t,e,k_1,k_2}=\sum_{w=0}^{\infty}\sum_{v=0}^{n}\sum_{u=0}^{v}\sum_{t=0}^{n}\sum_{c=0}^{t}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \tag{2.2}$$

and the infinite series on the right side (2.1) is absolutely convergent, $R=ln+qv+pt+rw+k_1r+k_2q$ (2.3)

and $$\psi(w,v,u,t,e,k_1,k_2)=\frac{(-)^{t+w+k_2}(-v)_u(-t)_e(\alpha)_t l^n}{w!v!u!t!e!K_n k_1!k_2!}\frac{s^{w+k_1}F^{\gamma n-t}}{(1-\alpha-t)_e}(\alpha-\gamma n)_e$$

$$(-\beta-\delta n)_v g^{v+k_2}h^{\delta n-v-k_2}(v-\delta n)_{k_2}E^t\left(\frac{pe+rw+\lambda+qn}{l}\right)_n \tag{2.4}$$

where K_n is a sequence of constants.

By suitably specializing the parameters involving in (2.1), a general sequence of function reduced to generalized polynomials set studied by Raizada [6], a class of polynomials introduced by Fujiwara [2] and several others authors.

3. Integral representation of generalized hypergeometric function

The following generalized hypergeometric function in terms of multiple contour integrals is also required [10 ,page 39 eq .30]

$$\frac{\prod_{j=1}^{P}\Gamma(A_j)}{\prod_{j=1}^{Q}\Gamma(B_j)}\,{}_PF_Q\left[(A_P);(B_Q);-(x_1+\cdots+x_r)\right]$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\frac{\prod_{j=1}^{P}\Gamma(A_j+s_1+\cdots+s_r)}{\prod_{j=1}^{Q}\Gamma(B_j+s_1+\cdots+s_r)}\Gamma(-s_1)\cdots\Gamma(-s_r)x_1^{s_1}\cdots x_r^{s_r}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.1}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j+s_1+\cdots+s_r)$ are separated from those of $\Gamma(-s_j), j=1,\cdots,r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j=1,\cdots,r$

The equivalent form of Eulerian beta integral is :

$$\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}\mathrm{d}t=(b-a)^{\alpha+\beta}B(\alpha,\beta)(Re(\alpha)>0,Re(\beta)>0,a\neq b) \tag{3.2}$$

4. First integral

We note :

$$U = p_2, q_2; p_3, q_3; \cdots ; p_{r-1}, q_{r-1}; 0,0; 0,0; \cdots ; 0,0; 0,0; \cdots ; 0,0 \tag{4.1}$$

$$V = 0, n_2; 0, n_3; \cdots ; 0, n_{r-1}; 0,0; 0,0; \cdots ; 0,0; 0,0; \cdots ; 0,0 \tag{4.2}$$

$$X = m^{(1)}, n^{(1)}; \cdots ; m^{(r)}, n^{(r)}; 1,0; \cdots ; 1,0; 1,0; \cdots ; 1,0 \tag{4.3}$$

$$Y = p^{(1)}, q^{(1)}; \cdots ; p^{(r)}, q^{(r)}; 0,1; \cdots ; 0,1; 0,1; \cdots ; 0,1 \tag{3.4}$$

$$A = (a_{2k}; \alpha_{2k}^{(1)}, \alpha_{2k}^{(2)}); \cdots ; (a_{(r-1)k}; \alpha_{(r-1)k}^{(1)}, \alpha_{(r-1)k}^{(2)}, \cdots , \alpha_{(r-1)k}^{(r-1)}) \tag{3.5}$$

$$B = (b_{2k}; \beta_{2k}^{(1)}, \beta_{2k}^{(2)}); \cdots ; (b_{(r-1)k}; \beta_{(r-1)k}^{(1)}, \beta_{(r-1)k}^{(2)}, \cdots , \beta_{(r-1)k}^{(r-1)}) \tag{3.6}$$

$$\mathfrak{A} = (a_{rk}; \alpha_{rk}^{(1)}, \alpha_{rk}^{(2)}, \cdots , \alpha_{rk}^{(r)}, 0, \cdots , 0, 0 \cdots , 0) \tag{3.7}$$

$$\mathfrak{B} = (b_{rk}; \beta_{rk}^{(1)}, \beta_{rk}^{(2)}, \cdots , \beta_{rk}^{(r)}, 0, \cdots , 0, 0, \cdots , 0) \tag{3.8}$$

$$\mathfrak{A}_1 = (a_k^{(1)}, \alpha_k^{(1)})_{1,p^{(1)}}; \cdots ; (a_k^{(r)}, \alpha_k^{(r)})_{1,p^{(r)}}; \ (1,0); \cdots ; (1,0); (1.0); \cdots ; (1.0) \tag{3.9}$$

$$\mathfrak{B}_1 = (b_k^{(1)}, \beta_k^{(1)})_{1,p^{(1)}}; \cdots ; (b_k^{(r)}, \beta_k^{(r)})_{1,p^{(r)}}; (0,1); \cdots ; (0,1); (0,1); \cdots ; (0,1) \tag{3.10}$$

$$A^* = [1+\sigma_i'; \rho_i^{(1,1)}, \cdots , \rho_i^{(1,r)}, \tau_i^{(1,1)}, \cdots , \tau_i^{(1,l)}, 1, 0, \cdots , 0]_{1,s}, \cdots ,$$

$$[1+\sigma_i^{(T)}; \rho_i^{(T,1)}, \cdots , \rho_i^{(T,r)}, \tau_i^{(T,1)}, \cdots , \tau_i^{(T,l)}, 1, 0, \cdots , 0]_{1,s}, [1-A_j; 0, \cdots , 0, 1, \cdots , 1, 0, \cdots , 0]_{1,P},$$

$$[1-\alpha_i; \delta_i', \cdots , \delta_i^{(r)}, \mu_i', \cdots , \mu_i^{(l)}, 1, \cdots , 1, 0, \cdots , 0]_{1,s},$$

$$[1-\beta_i; \eta_i', \cdots , \eta_i^{(r)}, \theta_i', \cdots , \theta_i^{(l)}, 0, \cdots , 0, 1, \cdots , 1]_{1,s} \tag{3.11}$$

$$B^* = [1+\sigma_i'; \rho_i^{(1,1)}, \cdots , \rho_i^{(1,r)}, \tau_i^{(1,1)}, \cdots , \tau_i^{(1,l)}, 0, \cdots , 0]_{1,s}, \cdots ,$$

$$[1+\sigma_i^{(T)}; \rho_i^{(T,1)}, \cdots , \rho_i^{(T,r)}, \tau_i^{(T,1)}, \cdots , \tau_i^{(T,l)}, 0, \cdots , 0]_{1,s}, [1-B_j; 0, \cdots , 0, 1, \cdots , 1, 0, \cdots , 0]_{1,Q},$$

$$[1-\alpha_i-\beta_i; (\delta_i'+\eta_i'), \cdots , (\delta_i^{(r)}+\eta_i^{(r)}), (\mu_i'+\theta_i'), \cdots , (\mu_i^{(l)}+\theta_i^{(l)}), 1, \cdots , 1]_{1,s} \tag{3.12}$$

We have the following multiple Eulerian integral and we obtain the I-function of $(r+l+T)-$variables

$$\int_{u_1}^{v_1}\cdots\int_{u_s}^{v_s}\prod_{i=1}^{s}\left[(x_i-u_i)^{\alpha_i-1}(v_i-x_i)^{\beta_i-1}\prod_{j=1}^{T}(u_i^{(j)}x_i+v_i^{(j)})^{\sigma_i^{(j)}}\right]$$

$$I_{U_r:p_r,q_r;W_r}^{V_r;0,n_r;X_r}\left(\begin{array}{c} \mathrm{z}_1\prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{\delta_i'}(v_i-x_i)^{\eta_i'}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{\delta_i^{(r)}}(v_i-x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,r)}}}\right] \end{array}\right)$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{k=1}^{l}g_k\prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{u_i^{(k)}}(v_i-x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,k)}}}\right]\right]\mathrm{d}x_1\cdots\mathrm{d}x_s$$

$$=\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{s}\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right]$$

$$I_{U:p_r+sT+P+2s;q_r+sT+Q+s;Y}^{V;p_r+sT+P+2s;X}\left(\begin{array}{c|c} \mathrm{z}_1w_1 & \mathrm{A}\ ;\mathrm{A}^*,\mathfrak{A};\mathfrak{A}_1\\ \cdots & \cdot\\ \cdots & \cdot\\ \mathrm{z}_rw_r & \cdot\\ \mathrm{g}_1W_1 & \cdot\\ \cdots & \cdot\\ \cdots & \cdot\\ \mathrm{g}_lW_l & \cdot\\ \mathrm{G}_1 & \cdot\\ \cdots & \cdot\\ \cdots & \cdot\\ \mathrm{G}_T & \mathrm{B}\ ;\mathrm{B}^*,\mathfrak{B};\mathfrak{B}_1 \end{array}\right) \tag{3.13}$$

Where

$$w_m=\prod_{i=1}^{s}\left[(v_i-u_i)^{\delta_i^{(m)}+\eta_i^{(m)}}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\rho_i^{(j,m)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\rho_i^{(j,m)}}\right],m=1,\cdots,r \tag{3.14}$$

$$W_k = \prod_{i=1}^{s}\left[(v_i-u_i)^{\mu_i^{(k)}+\theta_i^{(k)}}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\tau_i^{(j,k)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\tau_i^{(j,k)}}\right], k=1,\cdots,l \tag{3.15}$$

$$G_j = \prod_{i=1}^{s}\left[\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right], j=1,\cdots,W \tag{3.16}$$

$$G_j = -\prod_{i=1}^{s}\left[\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right], j=W+1,\cdots,T \tag{3.17}$$

Provided that :

(A) $W\in[0,T]; u_i, v_i\in\mathbb{R}; i=1,\cdots,r$

(B) $min\{\delta_i^{(t)},\eta_i^{(t)},\rho_i^{(j,t)},\mu_i^{(k)},\theta_i^{(k)},\tau_i^{(j,k)}\}\geqslant 0; j=1,\cdots,T; i=1,\cdots,s; k=1,\cdots,l; t=1,\cdots,r$

(C) $\sigma_i^{(j)}\in\mathbb{R}, U_i^{(j)}, V_i^{(j)}\in\mathbb{C}, z_t, g_k\in\mathbb{C}; j=1,\cdots,s; t=1,\cdots,r; k=1,\cdots,l; t=1,\cdots,r$

(D) $max\left[\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right]<1, i=1,\cdots,s; j=1,\cdots,W$ and

$$max\left[\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right]<1, i=1,\cdots,s; j=W+1,\cdots,T$$

(E) $$\Omega_i=\sum_{j=1}^{n^{(i)}}\alpha_j^{(i)}-\sum_{j=n^{(i)}+1}^{p^{(i)}}\alpha_j^{(i)}+\sum_{j=1}^{m^{(i)}}\beta_j^{(i)}-\sum_{j=m^{(i)}+1}^{q^{(i)}}\beta_j^{(i)}+\left(\sum_{j=1}^{n_2}\alpha_{2j}^{(i)}-\sum_{j=n_2+1}^{p_2}\alpha_{2j}^{(i)}\right)+\cdots+$$

$$\left(\sum_{j=1}^{n_s}\alpha_{sj}^{(i)}-\sum_{j=n_s+1}^{p_s}\alpha_{sj}^{(i)}\right)-\left(\sum_{j=1}^{q_2}\beta_{2j}^{(i)}+\sum_{j=1}^{q_3}\beta_{3j}^{(i)}+\cdots+\sum_{k=1}^{q_s}\beta_{sj}^{(i)}\right)-\delta_i^{(k)}-\eta_i^{(k)}-\sum_{j=1}^{T}\rho_i^{(j,k)}>0$$

$(i=1,\cdots,s; k=1,\cdots,r)$

(F) $Re\left[\alpha_i+\sum_{t=1}^{r}\delta_i^{(t)}\min_{1\leqslant j\leqslant m_i}\frac{b_j^{(t)}}{\beta_j^{(t)}}\right]>0; Re\left[\beta_i+\sum_{t=1}^{r}\eta_i^{(t)}\min_{1\leqslant j\leqslant m_i}\frac{b_j^{(t)}}{\beta_j^{(t)}}\right]>0; i=1,\cdots,s$

(G) $\left|arg\left(z_i\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\tau_i^{(j,k)}}\right)\right|<\frac{1}{2}\Omega_i\pi$

(H) $P\leqslant Q+1$. The equality holds, when , in addition,

either $P>Q$ and $\sum_{k=1}^{l}\left|g_k\left(\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\tau_i^{(j,k)}}\right)\right|^{\frac{1}{Q-P}}<1 \qquad (u_i\leqslant x_i\leqslant v_i; i=1,\cdots,s)$

or $P \leqslant Q$ and $\max_{1 \leqslant k \leqslant l} \left[\left| \left(g_k \prod_{j=1}^{T} (U_i^{(j)} x_i + V_i^{(j)})^{-\tau_i^{(j,k)}} \right) \right| \right] < 1 \qquad (u_i \leqslant x_i \leqslant v_i; i = 1, \cdots, s)$

(I) $a_{ij}, b_{ik}, \in \mathbb{C}\ (i = 1, \cdots, r; j = 1, \cdots, p_i; k = 1, \cdots, q_i); a_j^{(i)}, b_j^{(k)} \in \mathbb{C}$

$(i = 1, \cdots, r; j = 1, \cdots, p^{(i)}; k = 1, \cdots, q^{(i)})$

$\alpha_{ij}^{(k)}, \beta_{ij}^{(k)} \in \mathbb{R}_+$ ($(i = 1, \cdots, r, j = 1, \cdots, p_i, k = 1, \cdots, r)$; $\alpha_j^{(i)}, \beta_i^{(i)} \in \mathbb{R}_+$ $(i = 1, \cdots, r; j = 1, \cdots, p_i)$

Proof

To establish the formula (3.13), we first use contour integral representation with the help of (1.2) for the multivariable I-function occuring in its left-hand side and use the contour integral representation with the help of (3.1) for the generalized hypergeometric function ${}_PF_Q(.)$. We write.

$$\prod_{j=1}^{T} (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} = \prod_{j=1}^{W} (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} \prod_{j=W+1}^{T} (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} \tag{3.18}$$

where $K_i^{(j)} = \eta_i^{(j)} - \sum_{t=1}^{r} \rho_i^{(j,t)} \zeta_t - \sum_{k=1}^{l} \tau_i^{(j,k)} \zeta_k$ (3.19)

and express the factor occuring in R.H.S. Of (2.22) in terms of following Mellin-Barnes contour integral with the help of the result [9, page 18, eq.(2.6.4)]

$$\prod_{j=1}^{W} (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} = \prod_{j=1}^{W} \left[\frac{(U_i^{(j)} u_i + V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})} \right] \frac{1}{(2\pi\omega)^W} \int_{L_1'} \cdots \int_{L_W'} \prod_{j=1}^{W} \Big[\Gamma(-\zeta_j') \Gamma(-K_i^{(j)} + \zeta_j')$$

$$\prod_{j=1}^{W} \left[\frac{(U_i^{(j)} (x_i - u_i)}{(u_i U_i^{(j)} + V_i^{(j)})} \right]^{\zeta_j'} \Bigg] \mathrm{d}\zeta_1' \cdots \mathrm{d}\zeta_W' \tag{3.20}$$

and

$$\prod_{j=W+1}^{T} (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} = \prod_{j=W+1}^{T} \left[\frac{(U_i^{(j)} v_i + V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})} \right] \frac{1}{(2\pi\omega)^{T-W}} \int_{L_{W+1}'} \cdots \int_{L_T'} \prod_{j=W+1}^{T} \Big[\Gamma(-\zeta_j') \Gamma(-K_i^{(j)} + \zeta_j')$$

$$\prod_{j=W+1}^{T} \left[-\frac{(U_i^{(j)} (v_i - x_i)}{(v_i U_i^{(j)} + V_i^{(j)})} \right]^{\zeta_j'} \Bigg] \mathrm{d}\zeta_{W+1}' \cdots \mathrm{d}\zeta_T' \tag{3.21}$$

We apply the Fubini's theorem for multiple integral. Finally evaluating the innermost **x**-integral with the help of (3.2) and reinterpreting the multiple Mellin-Barnes contour integral in terms of multivariable I-function of $(r + l + T)-$ variables, we obtain the formula (3.13)

5. Second formula

We note :

$$U = p_2, q_2; p_3, q_3; \cdots; p_{r-1}, q_{r-1}; 0, 0; \cdots; 0, 0 \tag{5.1}$$

$$V = 0, n_2; 0, n_3; \cdots ; 0, n_{r-1}; 0, 0; \cdots ; 0, 0 \tag{5.2}$$

$$X = m^{(1)}, n^{(1)}; \cdots ; m^{(r)}, n^{(r)}; 1, 0; \cdots ; 1, 0 \tag{5.3}$$

$$Y = p^{(1)}, q^{(1)}; \cdots ; p^{(r)}, q^{(r)};; 0, 1; \cdots ; 0, 1 \tag{5.4}$$

$$A = (a_{2k}; \alpha_{2k}^{(1)}, \alpha_{2k}^{(2)}); \cdots ; (a_{(r-1)k}; \alpha_{(r-1)k}^{(1)}, \alpha_{(r-1)k}^{(2)}, \cdots , \alpha_{(r-1)k}^{(r-1)}) \tag{5.5}$$

$$B = (b_{2k}; \beta_{2k}^{(1)}, \beta_{2k}^{(2)}); \cdots ; (b_{(r-1)k}; \beta_{(r-1)k}^{(1)}, \beta_{(r-1)k}^{(2)}, \cdots , \beta_{(r-1)k}^{(r-1)}) \tag{5.6}$$

$$\mathfrak{A} = (a_{rk}; \alpha_{rk}^{(1)}, \alpha_{rk}^{(2)}, \cdots , \alpha_{rk}^{(r)}, 0, \cdots , 0) \tag{5.7}$$

$$\mathfrak{B} = (b_{rk}; \beta_{rk}^{(1)}, \beta_{rk}^{(2)}, \cdots , \beta_{rk}^{(r)}, 0, \cdots , 0) \tag{5.8}$$

$$\mathfrak{A}_1 = (a_k^{(1)}, \alpha_k^{(1)})_{1,p^{(1)}}; \cdots ; (a_k^{(r)}, \alpha_k^{(r)})_{1,p^{(r)}};\ (1,0); \cdots ; (1,0) \tag{5.9}$$

$$\mathfrak{B}_1 = (b_k^{(1)}, \beta_k^{(1)})_{1,p^{(1)}}; \cdots ; (b_k^{(r)}, \beta_k^{(r)})_{1,p^{(r)}}; (0,1); \cdots ; (0,1) \tag{5.10}$$

$$A^* = [1 + \upsilon_i' + \mu_i' K - \theta_i' R; \rho_i^{(1,1)}, \cdots , \rho_i^{(1,r)}, 1, 0, \cdots , 0]_{1,s}, \cdots ,$$

$$[1 + \upsilon_i^{(T)} + \mu^{(T)} K - \theta_i^{(K)} R; \rho_i^{(T,1)}, \cdots , \rho_i^{(T,r)}, 1, 0, \cdots , 0]_{1,s},$$

$$[1 - \alpha_i - \epsilon_i K - \zeta_i R; \delta_i', \cdots , \delta_i^{(r)}, 1, \cdots , 1, 0, \cdots , 0]_{1,s},$$

$$[1 - \beta_i - \sigma_i K - \lambda_i R; \eta_i', \cdots , \eta_i^{(r)}, 0, \cdots , 0, 1, \cdots , 1]_{1,s} \tag{5.11}$$

$$B^* = [1 + \upsilon_i' + \mu_i' K - \theta_i' R; \rho_i^{(1,1)}, \cdots , \rho_i^{(1,r)}, 0, \cdots , 0]_{1,s}, \cdots ,$$

$$[1 + \upsilon_i^{(T)} + \mu^{(T)} K - \theta_i^{(K)} R; \rho_i^{(T,1)}, \cdots , \rho_i^{(T,r)}, 0, \cdots , 0]_{1,s},$$

$$[1 - \alpha_i - \beta_i - (\epsilon_i + \sigma_i) K - (\zeta_i + \lambda_i) R; (\delta_i' + \eta_i'), \cdots , (\delta_i^{(r)} + \eta_i^{(r)}), 1, \cdots , 1]_{1,s} \tag{5.12}$$

In this section we will note $R_n^{\alpha,\beta}[x; E, F, g, h; p, q; \gamma; \delta; e^{-sx^r}]$ by $R_n^{\alpha,\beta}(x)$

We have the following multiple Eulerian integral

$$\int_{u_1}^{v_1} \cdots \int_{u_s}^{v_s} \prod_{i=1}^{s} \left[(x_i - u_i)^{\alpha_i - 1} (v_i - x_i)^{\beta_i - 1} \prod_{j=1}^{T} (u_i^{(j)} x_i + v_i^{(j)})^{\sigma_i^{(j)}} \right]$$

$$I_{U_r:p_r,q_r;W_r}^{V_r;0,n_r;X_r}\left(\begin{array}{c} \mathrm{z}_1\prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{\delta'_i}(v_i-x_i)^{\eta'_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,1)}}}\right] \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r\prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{\delta_i^{(r)}}(v_i-x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,r)}}}\right] \end{array}\right) R_n^{\alpha,\beta}\left[\prod_{j=1}^{s}\left[\frac{(x_i-u_i)^{\zeta_i}(v_i-x_i)^{\lambda_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\theta_i^{(j)}}}\right]\right]$$

$$S_N^M\left[\prod_{j=1}^{s}(x_i-u_i)^{\epsilon_i}(v_i-x_i)^{\sigma_i}\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\mu_i^{(j)}}\right]\mathrm{d}x_1\cdots\mathrm{d}x_r$$

$$=\prod_{j=1}^{s}\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\upsilon_i^{(j)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\upsilon_i^{(j)}}\right]\sum_{K=0}^{[N/M]}\sum_{w,v,u,t,e,k_1,k_2}$$

$$\psi'(w,v,u,t,e,k_1,k_2)I_{U:p_r+sT+2s;q_r+sT+s;Y}^{V;p_r+sT+2s;X}\left(\begin{array}{c|c} \mathrm{z}_1w_1 & \mathrm{A}\ ;\mathrm{A}^*,\mathfrak{A};\mathfrak{A}_1 \\ \cdot\ \cdot\ \cdot & \cdot \\ \cdot\ \cdot\ \cdot & \cdot \\ \mathrm{z}_rw_r & \cdot \\ & \cdot \\ \mathrm{G}_1 & \cdot \\ \cdot\ \cdot\ \cdot & \cdot \\ \cdot\ \cdot\ \cdot & \cdot \\ \mathrm{G}_T & \mathrm{B}\ ;\mathrm{B}^*,\mathfrak{B};\mathfrak{B}_1 \end{array}\right) \tag{5.13}$$

where $\psi'(w,v,u,t,e,k_1,k_2)$

$$=\frac{(-n)_{MK}A_{N,K}\psi(w,v,u,t,e,k_1,k_2,)\prod_{i=1}^{s}(v_i-u_i)^{(\epsilon_i+\sigma_i)K+(\zeta_i+\lambda_i)R}}{K!\prod_{i=1}^{s}\left[\prod_{j=1}^{W}(u_iU_i^{(j)}+V_i^{(j)})^{-\mu_i^{(j)}K+\theta_i^{(j)}R}\prod_{j=W+1}^{T}(u_iU_i^{(j)}+V_i^{(j)})^{-\mu_i^{(j)}K+\theta_i^{(j)}R}\right]} \tag{5.14}$$

$\psi(w,v,u,t,e,k_1,k_2)$ and R are given by (2.4) and (2.3) respectively.

$$w_l=\prod_{i=1}^{s}\left[(v_i-u_i)^{\delta_i^{(l)}+\eta_i^{(l)}}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\rho_i^{(j,l)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\rho_i^{(j,l)}}\right],l=1,\cdots,r \tag{5.15}$$

$$G_j=\prod_{i=1}^{s}\left[\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right],j=1,\cdots,W \tag{5.16}$$

$$G_j = -\prod_{i=1}^{s}\left[\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right], j = W+1,\cdots,T \tag{5.17}$$

Provided that :

(A) $W \in [0,T]; u_i, v_i \in \mathbb{R}; i = 1,\cdots,r$

(B) $min\{\zeta_i,\lambda_i,\theta_i^{(j)},\delta_i^{(l)},\eta_i^{(l)},\rho_i^{(j,l)}\} \geqslant 0; j = 1,\cdots,T; i = 1,\cdots,s; l = 1,\cdots,r$

(C) $Re(\alpha_i,\beta_i,v_i^{(j)}) > 0 (i = 1,\cdots,s; j = 1,\cdots,T); |\tau| < 1$

(D) $max\left[\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right] < 1, i = 1,\cdots,s; j = 1,\cdots,W$ and

$$max\left[\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right] < 1, i = 1,\cdots,s; j = W+1,\cdots,T$$

(E) $$\Omega_i = \sum_{j=1}^{n^{(i)}}\alpha_j^{(i)} - \sum_{j=n^{(i)}+1}^{p^{(i)}}\alpha_j^{(i)} + \sum_{j=1}^{m^{(i)}}\beta_j^{(i)} - \sum_{j=m^{(i)}+1}^{q^{(i)}}\beta_j^{(i)} + \left(\sum_{j=1}^{n_2}\alpha_{2j}^{(i)} - \sum_{j=n_2+1}^{p_2}\alpha_{2j}^{(i)}\right) + \cdots +$$

$$\left(\sum_{j=1}^{n_s}\alpha_{sj}^{(i)} - \sum_{j=n_s+1}^{p_s}\alpha_{sj}^{(i)}\right) - \left(\sum_{j=1}^{q_2}\beta_{2j}^{(i)} + \sum_{j=1}^{q_3}\beta_{3j}^{(i)} + \cdots + \sum_{k=1}^{q_s}\beta_{sj}^{(i)}\right) - \delta_i^{(k)} - \eta_i^{(k)} - \sum_{j=1}^{T}\rho_i^{(j,k)} > 0$$

$(i = 1,\cdots,s; k = 1,\cdots,r)$

(F) $Re\big[\alpha_i + \sum_{t=1}^{r}\delta_i^{(t)}\min_{1\leqslant j\leqslant m_i}\frac{b_j^{(t)}}{\beta_j^{(t)}}\big] > 0; Re\big[\beta_i + \sum_{t=1}^{r}\eta_i^{(t)}\min_{1\leqslant j\leqslant m_i}\frac{b_j^{(t)}}{\beta_j^{(t)}}\big] > 0; i = 1,\cdots,s$

(G) $\left|arg\left(z_i\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\rho_i^{(j,k)}}\right)\right| < \frac{1}{2}\Omega_i\pi$

(H) $a_{ij}, b_{ik}, \in \mathbb{C}\ (i = 1,\cdots,r; j = 1,\cdots,p_i; k = 1,\cdots,q_i); a_j^{(i)}, b_j^{(k)} \in \mathbb{C}$

$(i = 1,\cdots,r; j = 1,\cdots,p^{(i)}; k = 1,\cdots,q^{(i)})$

$\alpha_{ij}^{(k)},\beta_{ij}^{(k)} \in \mathbb{R}_+$ ($(i = 1,\cdots,r, j = 1,\cdots,p_i, k = 1,\cdots,r)$; $\alpha_j^{(i)},\beta_i^{(i)} \in \mathbb{R}_+\ (i = 1,\cdots,r; j = 1,\cdots,p_i)$

(I) The series occuring on the right-hand side of (5.13) is absolutely and uniformly convergent

Proof

To establish the formula (5.13), we first use series representation (2.1) and (1.11) for $R_n^{\alpha,\beta}[.]$ and $S_N^M(.)$ respectively and contour integral representation with the help of (1.2) for the multivariable I-function defined by Prasad [5] occuring in its left-hand side. Changing the order of integration ans summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) . We have :

$$\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}=\prod_{j=1}^{W}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}\prod_{j=W+1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}} \tag{5.18}$$

where $K_i^{(j)}=v_i^{(j)}-\theta_i^{(j)}R-\sum_{l=1}^{r}\rho_i^{(j,l)}\psi_l; i=1,\cdots,s; j=1,\cdots,T$ (5.19)

and express the factors occuring in R.H.S. Of (5.13) in terms of following Mellin-Barnes contour integral , we obtain :

$$\prod_{j=1}^{W}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}=\prod_{j=1}^{W}\left[\frac{(U_i^{(j)}u_i+V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})}\right]\frac{1}{(2\pi\omega)^W}\int_{L_1'}\cdots\int_{L_W'}\prod_{j=1}^{W}\Big[\Gamma(-\zeta_j')\Gamma(-K_i^{(j)}+\zeta_j')$$

$$\prod_{j=1}^{W}\left[\frac{(U_i^{(j)}(x_i-u_i)}{(u_iU_i^{(j)}+V_i^{(j)})}\right]^{\zeta_j'}\Bigg]\mathrm{d}\zeta_1'\cdots\mathrm{d}\zeta_W' \tag{5.20}$$

and

$$\prod_{j=W+1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}=\prod_{j=W+1}^{T}\left[\frac{(U_i^{(j)}v_i+V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})}\right]\frac{1}{(2\pi\omega)^{T-W}}\int_{L_{W+1}'}\cdots\int_{L_T'}\prod_{j=W+1}^{T}\Big[\Gamma(-\zeta_j')\Gamma(-K_i^{(j)}+\zeta_j')$$

$$\prod_{j=W+1}^{T}\left[\frac{(U_i^{(j)}(x_i-v_i)}{(v_iU_i^{(j)}+V_i^{(j)})}\right]^{\zeta_j'}\Bigg]\mathrm{d}\zeta_{W+1}'\cdots\mathrm{d}\zeta_T' \tag{5.21}$$

We apply the Fubini's theorem for multiple integral. Finally evaluating the innermost **x-**integral with the help of (3.2) and reinterpreting the multiple Mellin-Barnes contour integral in terms of multivariable I-function of $(r+T)-$ variables, we obtain the formula (5.12)

Remarks

If $U_r=V_r=A=B=0$, the multivariable I-function degenere in multivariable H-function defined by Srivastava et al [11]. The formula (3.13) have been established by Goyal et al [4] and the formula (5.12) have been established by Garg [3].

6. Conclusion

In this paper we have evaluated two generalized multiple Eulerian integrals involving the multivariable I-functions defined by Prasad [5] with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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