INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES AND APPLICATIONS
Volume 5 • Number 1 • January-June 2015

GROUP OF HOMEOMORPHISMS ON MINKOWSKI SPACE WITH TIME TOPOLOGY

Gunjan Agrawal and Nisha Godani

Department of Mathematics, Faculty of Science, Dayalbagh Educational Institute (Deemed University)
Dayalbagh, Agra, India
E-mail: dei.gunjan@yahoo.com and nishagodani.dei@gmail.com

Abstract: *Zeeman [13] introduced time topology on Minkowski space as the finest topology that induces Euclidean topology on every timelike straight line and conjectured that the homeomorphism group of Minkowski space with time topology is equal to the group generated by the Lorentz group, translations and dilatations. During the exploration of a significant topology on spacetime, Göbel [4] proved this conjecture for strongly causal spacetimes but the proof is quite complicated. In the present paper, a simpler proof is provided.*

Keywords : *Minkowski space, Lorentz group, Timelike curve, Time cone, Time topology, Homeomorphism group*

1. INTRODUCTION

Einstein developed the theory of General Relativity and modeled spacetime mathematically as a connected, Hausdorff, paracompact, C^∞, real 4-dimensional time orientable manifold without boundary with a C^∞ Lorentz metric and associated pseudo Riemannian connection. Minkowski spacetime is the spacetime of Einstein's theory of Special Relativity.

The Euclidean topology on the Minkowski spacetime was considered to be unsuitable as it does not incorporate the causal structure of spacetime and its homeomorphism group is too large to be of any physical significance. In 1967, Zeeman [13] provided a breakthrough in this direction by introducing the notion of fine topology using the Lorentz metric and proved its homeomorphism group G to be isomorphic to the group generated by the Lorentz group, translations and dilatations. In the same paper, Zeeman also defined some other non-Euclidean topologies such as t-topology, s-topology, space topology, time topology etc. and conjectured these to possess G as their group of homeomorphisms. Nanda[8, 9] proved this conjecture for t, s and space topologies. Continuing the topological study of spacetime, in 1976, Göbel [3] defined and studied general relativistic analogue of fine topology on manifolds while Hawking et.al.[5] proposed and studied path topology on strongly causal spacetime and Low[6] proved its non-simple connectedness. Agrawal and Shrivastava [1, 2] explored the path topology on Minkowski space.

Further, Göbel [4] introducing infinitely many topologies on spacetime, obtained a physically relevant topology with all the required physical properties. In the same paper, Göbel [4] also proved Zeeman's conjecture for the homeomorphism group of Minkowski space with time topology. The present paper is aimed at proving the Zeeman's conjecture in the context of time topology using a simpler approach similar to one used in [7] for the computation of homeomorphism group of Minkowski space with path topology.

The paper begins with the necessary notation and preliminaries in Section 2. In Section 3, Lorentz transformations, translations and dilatations are proved to be homeomorphisms on Minkowski space with time topology. Further, studying properties of homeomorphisms on Minkowski space with time topology, its group of homeomorphism has been calculated in Section 4. Finally, Section 5 concludes the paper.

2. NOTATION AND PRELIMINARIES

Let the set of natural and real numbers be denoted by N and R respectively. For $n \in N$ and $n > 1$, the n-dimensional real vector space R^n with the bilinear form $g: R^n \times R^n \to R$ such that g is symmetric, nondegenerate and there exists a basis $\{e_0, e_1 \dots e_{n-1}\}$ for R^n with $g(e_i, e_j) = 1$ if $i = j = 0$, -1 if $i = j = 1, \dots, n-1$ and 0 otherwise is called the n-dimensional Minkowski space, denoted by M and the bilinear form g is called the Lorentz inner product. Also g induces an indefinite characteristic quadratic form Q on M defined as $Q(x) = g(x, x)$, $x \in M$. According as $Q(x)$ is positive, zero, or negative, $x \in M$ is called timelike, lightlike or spacelike. The sets $C^T(x) = \{y \in M: y = x \ or \ Q(y - x) > 0\}$, $C^L(x) = \{y \in M: y = x \ or \ Q(y - x) = 0\}$ and $C^S(x) = \{y \in M: y = x \ or \ Q(y - x) < 0\}$ are respectively called the *time cone*, *light cone* and *space cone* at x. Let the coordinates of $x \in M$ with respect to the basis $\{e_0, e_1 \dots e_{n-1}\}$ be denoted by x^i, where $i = 0, 1, \dots, n-1$, x^0 is the time coordinate and $x^1, x^2, \dots x^{n-1}$ are the space coordinates. Then the sets $C^{T+}(x) = \{y \in C^T(x): y^0 > x^0\}$ and $C^{T-}(x) = \{y \in C^T(x): y^0 < x^0\}$ are called the future and past timecones at $x \in M$. The elements of $C^{T+}(x)$ and $C^{T-}(x)$ are called future and past directed timelike vectors respectively [7].

Let $\ll$ denote the partial ordering on M. Then for $x, y \in M$, x chronologically precedes y, denoted by $x \ll y$, if $y - x$ is a future-directed timelike vector or equivalently if $y \in C^{T+}(x)$. A map $F: M \to M$ is said to be a *causal automorphism*, if it is one-one, onto and both F and F^{-1} preserve $\ll$, i.e. $x \ll y$ iff $F(x) \ll F(y)$. The causal automorphisms form a group which is called as *causality group*.

Let J denote an open, closed or half-open interval in R, M^E denote n-dimensional Minkowski space with Euclidean topology and $\alpha : J \to M^E$ be continuous. Then α is called a *future timelike curve* (or *past timelike curve* respectively) at $t_0 \in J$, if there exists an open connected set U in J containing t_0 such that $t_1 < t_0 < t_2$ implies $\alpha(t_1) \ll \alpha(t_0) \ll \alpha(t_2)$ (or $\alpha(t_2) \ll \alpha(t_0) \ll \alpha(t_1)$ respectively), for every $t_1, t_2 \in U$. Further, $\alpha: J \to M$ is called a *future timelike curve* if it is future timelike at every $t_0 \in J$. Similarly, a *past timelike curve* is defined. Thus, a curve is called a *timelike curve* if it is either a future timelike curve or a past timelike curve [7].

The *time topology* on M is the finest topology that induces Euclidean topology on every timelike straight line [13]. Let M^T denote the n-dimensional Minkowski space with time topology. The collection of sets of the form $C^{T+}(p) \cap C^{T-}(q)$ for all $p, q \in M$ forms a basis for Alexandrov topology on M [5]. Let M^A denote the n-dimensional Minkowski space with

Alexandrov topology. On Minkowski space, Alexandrov topology coincides with Euclidean topology [10].

For $A \subseteq M$, A^E and A^T denote the subspaces of M^E and M^T respectively and I denote a closed interval in R.

3. LORENTZ TRANSFORMATION

In the present section describing Lorentz transformations, translations and dilatations on M, it has been proved that these maps are homeomorphisms when M is considered with time topology.

A Lorentz transformation is a linear map $L: M \to M$ such that $g(L(x), L(y)) = g(x, y)$ for all $x, y \in M$. A Lorentz transformation L is called an *orthochronous Lorentz transformation* or a *nonorthochronous Lorentz transformation* according as $g(x, Lx) > 0$ or < 0. An orthochronous Lorentz transformation preserves the orientation of all timelike vectors. It is known that L is a Lorentz transformation iff it preserves the quadratic form, i.e. $Q(Lv) = Q(v)$, for all $v \in M$ which implies that $v \in M$ is lightlike, timelike or spacelike iff Lv is lightlike, timelike or spacelike. Further, a Lorentz transformation is a bijective map that maps orthonormal basis to orthonormal basis and the determinant of the associated matrix is ± 1 [7]. Also, the eigenvalues corresponding to non-lightlike eigenvectors are ± 1 and the product of the eigenvalues corresponding to two linearly independent lightlike eigenvectors is 1 [11]. The collection of Lorentz transformations forms a group with respect to composition and is called the *Lorentz group* and the subgroup consisting of orthochronous Lorentz transformations is called the *orthochronous Lorentz group* [7]. A map $T: M \to M$ is called *translation* if $T(x) = x + c$ for all $x \in M$ and for some fixed $c \in M$ and a map $D: M \to M$ is called *dilatation* if $D(x) = kx$ for some positive real number k. The group generated by the orthochronous Lorentz group, translations and dilatations is equal to the causality group [12].

Lemma 3.1: Let M be the n-dimensional Minkowski space. Then a Lorentz transformation on M maps a timelike straight line onto a timelike straight line.

Proof: Let L be a Lorentz transformation on M and τ be a timelike straight line in M and $P \in \tau$. Then there exists a timelike vector, say v, parallel to τ such that $x = P + tv$, for every $x \in \tau$ and for some $t \in R$. Since L is linear and L maps timelike vector to timelike vector, therefore $L(x) = L(P) + tL(v)$ and it defines a straight line parallel to the timelike vector $L(v)$. This proves that $L(\tau)$ is a timelike straight line.

Proposition 3.2: Let M be the n-dimensional Minkowski space and L be a Lorentz transformation on M. Then (i) L is a homeomorphism on M^A (ii) L is a homeomorphism on M^E and (iii) L is a homeomorphism on M^T.

Proof: (i) In view of the fact that L is bijective [pp. 12; 8], it is sufficient to prove that both L and L^{-1} are open maps on M^A. Let $x \in M$. Since L preserves quadratic form [7], image of timelike vector is a timelike vector. For the fact that L either preserves or reverses the orientation of all timelike vectors, either $L(C^{T+}(x)) = C^{T+}(L(x))$ or $L(C^{T+}(x)) = C^{T-}(L(x))$. Let $L(C^{T+}(x)) = C^{T+}(L(x))$ and $y \in C^{T+}(x)$. Then $L(y) \in C^{T+}(L(x))$ and $L(C^{T-}(y)) = C^{T-}(L(y))$. Thus $L(C^{T+}(x) \cap C^{T-}(y)) = C^{T+}(L(x)) \cap C^{T-}(L(y))$. It is known that the collection of all sets of the form $C^{T+}(a) \cap C^{T-}(b)$, where $a, b \in M$, forms a basis for M^A [5], therefore L is an open map on M^A . A similar argument proves L^{-1} to be an open map on M^A.

(ii) It follows from Proposition 3.2 (i) and the fact that for Minkowski space, Alexandrov topology coincides with Euclidean topology [pp. 34; 11].

(iii) Since L is bijective [pp. 12; 8], it is sufficient to prove that both L and L^{-1} are open maps on M^T. Let G be an open set in M^T and τ be a timelike straight line. Then $G \cap \tau$ is open in τ^E Thus there exists an open set H in M^E such that $G \cap \tau = H \cap \tau$ and hence $L(G) \cap L(\tau) = L(H) \cap L(\tau)$. By Proposition 3.2 (ii), $L(H)$ is open in M^E and by Lemma 3.1, $L(\tau)$ is a timelike straight line, therefore $L(H) \cap L(\tau)$ is open in $L(\tau)^E$, that is, $L(G) \cap L(\tau)$ is open in $L(\tau)^E$. Hence, $L(G)$ is open in M^T as time topology is the finest topology that induces Euclidean topology on every timelike straight line. This proves that L is an open map on M^T. A similar argument proves L^{-1} to be an open map on M^T.

Proposition 3.3: Let M be the n-dimensional Minkowski space and T be a translation on M. Then T is a homeomorphism on M^T.

Proof: Similar to that of the proof of Proposition 3.2.

Proposition 3.4: Let M be the n-dimensional Minkowski space and D be a dilatation on M. Then D is a homeomorphism on M^T

Proof: Similar to that of the proof of Proposition 3.2.

4. HOMEOMORPHISM GROUP

In this section, studying the properties of a homeomorphism on M^T, the homeomorphism group of M^T has been determined.

Proposition 4.1: Let M be the n-dimensional Minkowski space and h be a homeomorphism from M^T to M^T. Then $h(C^T(x)) = C^T(h(x))$ for $x \in M$.

Proof: Let $y \in C^T(x)$ and $\gamma : I \to M$ be a curve such that $\gamma(I)$ is a line segment joining x to y. Then γ is a timelike curve joining x to y and as stated in the proof of Theorem 6.2 [pp. 852; 5], $ho\gamma$ is a timelike curve joining $h(x)$ to $h(y)$. It is known that for $a, b \in M$, $a \ll b$ iff there exists a timelike curve joining a to b [pp. 15; 11]. Thus $h(x) \ll h(y)$. This proves that $h\left(C^T(x)\right) \subseteq C^T(h(x))$. Using this containment for h^{-1}, the result follows.

Lemma 4.2: Let M be the n-dimensional Minkowski space and $x \in M$. Then $C^{T+}(x)$ and $C^{T-}(x)$ are path connected.

Proof: Let I_1 denote [0,1] and $\gamma : I_1 \to C^{T+}(x)$ defined as $\gamma(t) = (1-t)a + tb$, where $t \in I_1$ and $a, b \in C^{T+}(x)$, be denoted by γ_{ab}. Further, let $y, z \in C^{T+}(x)$. Then $z - y$ is a timelike, lightlike or a spacelike vector. If $z - y$ is a timelike vector, then $[\gamma_{yz}(I_1)]^E = [\gamma_{yz}(I_1)]^T$ and hence γ_{yz} is a path in $C^{T+}(x)$ from *y* to *z*. If $z - y$ is a lightlike or a spacelike vector, choose $w \in C^T(y) \cap C^T(z) \cap C^{T+}(x)$. Then *w*-*y* and *w*-*z* are timelike vectors and hence γ_{yw} and γ_{wz} are paths from *y* to *w* and *w* to *z* respectively in $C^{T+}(x)$ and the join of γ_{yw} and γ_{wz}, is a path from *y* to *z* in $C^{T+}(x)$. This proves that $C^{T+}(x)$ is path connected. A similar argument proves $C^{T-}(x)$ to be path connected.

Proposition 4.3: Let *M* be the *n*-dimensional Minkowski space and *h* be a homeomorphism from M^T to M^T. and $x \in M$. Then either (i) $h(C^{T+}(x)) = C^{T+}(h(x))$ and $h(C^{T-}(x)) = C^{T-}(h(x))$ or (ii) $h(C^{T+}(x)) = C^{T-}(h(x))$ and $h(C^{T-}(x)) = C^{T+}(h(x))$ for $x \in M$.

Proof: Let $p \in C^{T+}(x)$. Then by Proposition 4.1, $h(p)$ belongs to $C^{T+}(h(x))$ or $C^{T-}(h(x))$. Let $h(p) \in C^{T+}(h(x))$ and $q \in C^{T+}(x)$. We now assert that $h(q) \in C^{T+}(h(x))$. To prove the assertion, suppose to the contrary that $h(q) \notin C^{T+}(h(x))$. By Proposition 4.1, $h(q) \in C^{T-}(h(x))$. By Proposition 4.2, $C^{T+}(x)$ is connected, $h(C^{T+}(x))$ is connected. Since $h(C^{T+}(x)) \cap C^{T+}(h(x))$ and $h(C^{T+}(x)) \cap C^{T-}(h(x))$ are disjoint and both are nonempty and open in $h(C^{T+}(x))$, $h(C^{T+}(x))$ is disconnected, a contradiction. This proves that $h(C^{T+}(x)) \subseteq C^{T+}(h(x))$. A similar argument proves that $h(C^{T-}(x)) \subseteq C^{T-}(h(x))$. Further, by Proposition 4.1, $h\big(C^{T+}(x)\big) = C^{T+}(h(x))$ and $h\big(C^{T-}(x)\big) = C^{T-}(h(x))$. This proves (i). If $h(p) \in C^{T-}(h(x))$, a similar argument proves that (ii) holds.

Proposition 4.4: Let *M* be the *n*-dimensional Minkowski space and *h* be a homeomorphism from M^T to M^T Then *h* either preserves the order $\ll$ or reverses the order $\ll$.

Proof: Let $x \in M$ and $y \in C^{T+}(x)$. Then $h(y) \in C^{T+}(h(x))$ or $h(y) \in C^{T-}(h(x))$, by Proposition 4.3. Let $h(y) \in C^{T+}(h(x))$. Then $h(x) \ll h(y)$. We now assert that $x' \ll y'$ implies $h(x') \ll h(y')$ for all $x'\ y' \in M$. Choose $y'' \in C^{T+}(x) \cap C^{T+}(x')$. Thus $x, x' \ll y''$ and $h(x) \ll h(y'')$. Considering timecone at y'', by Proposition 4.3 $h(x') \ll h(y'')$. Since $x' \ll y', y''$, therefore again by Proposition 4.3 and considering timecone at x', $h(x') \ll h(y')$ as required. If $h(y) \in C^{T-}(h(x))$, a similar argument proves that $h(y') \ll h(x')$. This completes the proof.

Proposition 4.5: Let *M* be the *n*-dimensional Minkowski space. Then a homeomorphism on M^T is a composition of a Lorentz transformation, dilatation and translation on *M*.

Proof: Let *h* be a homeomorphism on M^T and K be the group generated by the orthochronous Lorentz group, translations and dilatations on *M*. Then by Proposition 4.4, *h* preserves $\ll$ or reverses $\ll$. Let *h* preserves $\ll$. Then there exist $T, D, L: M \to M$, where T, D, L denote a translation, dilatation and orthochronous Lorentz transformation respectively such that $h = ToDoL$, by Theorem 1.6.2 [pp. 66; 8]. Hence $h \in K$. If h reverses $\ll$, hog preserves $\ll$, where $g: M \rightarrow M$ such that $g(x) = -x$. Again by Theorem 1.6.2 [pp. 66; 8], $hog = T'oD'oL'$ for some translation T', dilatation D' and orthochronous Lorentz transformation L'on M. Then $h = T'oD'oL'og$. Since $L'og$ preserves the Lorentz inner product, $h \in K$.

Proposition 4.6: Let M be the n-dimensional Minkowski space. Then the group of homeomorphisms on M^T is equal to the group generated by the Lorentz group, translations and dilatations on M.

Proof: The result follows from Propositions 3.2, 3.3, 3.4 and 4.5.

5. CONCLUSION

Göbel [4] obtained the group of homeomorphisms on Minkowski space with the physically significant time topology, as a consequence of a result on manifolds. In the present paper, this group has been studied using the basic mathematical tools defined on Minkowski space, thus providing a shorter and simpler route for a very desirable result.

REFERENCE

(i) G. Agrawal and S. Shrivastava, *t*-topology on the *n*-dimensional Minkowski space, J. Math. Phys., 50, 053515-1—6, 2009.

(ii) G. Agrawal and S. Shrivastava, Simple-connectedness of Minkowski space with *t*-topology, Proceedings of the Thirty Foruth National Systems Conference, Surathkal, Mangalore, 1-3, 2010.

(iii) R. Göbel, Zeeman topologies on space-times of general relativity theory, Comm. Math. Phys., 46, 289—307, 1976.

(iv) R. Göbel, The smooth path topology for curved space–time which incorporates the conformal structure and analytic Feynman tracks, J. Math. Phys., 17, 845-853, 1976.

(v) S. W. Hawking, A. R. King and P. J. McCarthy, A new topology for curved space-time which incorporates the causal differential and conformal structures, J. Math. Phys., 17, 174—181, 1976.

(vi) R. J. Low, Simple connectedness of spacetime in the path topology, Class. Quantum Grav., 27, 107001-1—4, 2010.

(vii) G. L. Naber, *The geometry of Minkowski spacetime* (Springer-Verlag, New York, 1992).

(viii) S. Nanda, Topology for Minkowski space, J. Math. Phys., 12, 394—401, 1971.

(ix) S. Nanda, Weaker versions of Zeeman's conjectures on topologies for Minkowski space, J. Math. Phys., 13, 12—15, 1972.

(x) R. Penrose, *Techniques of differential topology in relativity* (Society for Industrial and Applied Mathematics, Philadelphia, 1972).

(xi) M. A. J. Victoria and M. S. Caja, *An introduction to Lorentzian geometry and its applications* (Webs.um.es/majava/charlas/cursodeLorentz.pdf, 2010).

(xii) E. C. Zeeman, Causality implies the Lorentz group, J. Math. Phys., 5, 490-493, 1964.

(xiii) E. C. Zeeman, The topology of Minkowski space, Topology, 6, 161—170, 1967.