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Double integration of certain products of special functions with the Multivariable Aleph-function II

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ABSTRACT
The object of this paper is to establish three finite double integrals involving certain product of special functions, the multivariables Aleph-function and a general class of polynomials with general arguments. Our double integrals are quite general in character and a number of doubles integrals can be deduced as particular cases.

KEYWORDS :Multivariable Aleph-function, Aleph-function, finite double integrals, general class of polynomials, M-serie.

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1. Introduction and preliminaries.

The Aleph- function , introduced by Südland [9] et al , however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$\aleph(z) = \aleph^{M,N}_{P_i,Q_i,c_i;r}\left(z \,\middle|\, \begin{matrix} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r} \end{matrix} \right)$$

$$= \frac{1}{2\pi\omega}\int_L \Omega^{M,N}_{P_i,Q_i,c_i;r}(s) z^{-s} \mathrm{d}s \tag{1.1}$$

for all z different to 0 and

$$\Omega^{M,N}_{P_i,Q_i,c_i;r}(s) = \frac{\prod_{j=1}^{M}\Gamma(b_j + B_j s)\prod_{j=1}^{N}\Gamma(1 - a_j - A_j s)}{\sum_{i=1}^{r} c_i \prod_{j=N+1}^{P_i}\Gamma(a_{ji} + A_{ji}s)\prod_{j=M+1}^{Q_i}\Gamma(1 - b_{ji} - B_{ji}s)} \tag{1.2}$$

With :

$$|argz| < \frac{1}{2}\pi\Omega \quad \text{Where } \Omega = \sum_{j=1}^{M}\beta_j + \sum_{j=1}^{N}\alpha_j - c_i\left(\sum_{j=M+1}^{Q_i}\beta_{ji} + \sum_{j=N+1}^{P_i}\alpha_{ji}\right) > 0 \quad \text{with } i = 1, \cdots, r$$

For convergence conditions and other details of Aleph-function , see Südland et al [9].

Serie representation of Aleph-function is given by Chaurasia et al [1].

$$\aleph^{M,N}_{P_i,Q_i,c_i;r}(z) = \sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{B_G g!} z^{-s} \tag{1.3}$$

With $s = \eta_{G,g} = \dfrac{b_G + g}{B_G}, P_i < Q_i, |z| < 1$ and $\Omega^{M,N}_{P_i,Q_i,c_i;r}(s)$ is given in (1.2) (1.4)

The generalized polynomials defined by Srivastava [7], is given in the following manner :

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1,\cdots,y_s] = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} \frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}$$

$$A[N_1,K_1;\cdots;N_s,K_s]y_1^{K_1}\cdots y_s^{K_s} \tag{1.5}$$

Where $M_1,\cdots,M_s$ are arbitrary positive integers and the coefficients

In the present paper, we use the following notation

$$A_1 = \frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}A[N_1,K_1;\cdots;N_s,K_s] \tag{1.6}$$

The M-serie is defined, see Sharma [6].

$${}_{p'}M_{q'}^{\alpha}(y)=\sum_{s'=0}^{\infty}\frac{[(a_{p'})]_{s'}}{[(b_{q'})]_{s'}}\frac{y^{s'}}{\Gamma(\alpha s'+1)} \tag{1.7}$$

Here $\alpha\in\mathbb{C}, Re(\alpha)>0.$ $[(a_{p'})]_{s'}=(a_1)_{s'}\cdots(a_{p'})_{s'}$; $[(b_{q'})]_{s'}=(b_1)_{s'}\cdots(b_{q'})_{s'}.$
The serie (1.9) converge if $p^{'}\leqslant q^{'} and\, |y|<1.$

The Aleph-function of several variables generalize the multivariable h-function defined by H.M. Srivastava and R. Panda [8] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We have : $\aleph(z_1,\cdots,z_r) = \aleph_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}\left(\begin{array}{c|} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\right.$

$$\begin{array}{l} [(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}] \quad ,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]: \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad ,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}]: \end{array}$$

$$\left.\begin{array}{l} [(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;\ ;[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;;[(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.8}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.9}$$

and $$\theta_k(s_k)=\frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.10}$$

where $j=1$ to r and $k=1$ to r

Suppose , as usual , that the parameters

$a_j, j=1,\cdots,p; b_j, j=1,\cdots,q;$

$c_j^{(k)}, j=1,\cdots,n_k; c_{ji^{(k)}}^{(k)}, j=n_k+1,\cdots,p_{i^{(k)}};$

$d_j^{(k)}, j=1,\cdots,m_k; d_{ji^{(k)}}^{(k)}, j=m_k+1,\cdots,q_{i^{(k)}};$

with $k=1\cdots,r, i=1,\cdots,R$, $i^{(k)}=1,\cdots,R^{(k)}$

are complex numbers , and the $\alpha's, \beta's, \gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}+\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}+\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}-\sum_{j=1}^{m_k}\delta_j^{(k)}$$

$$-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}\leqslant 0 \tag{1.11}$$

The reals numbers τ_i are positives for $i=1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)}=1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma-i\infty$ to $\sigma+i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)$ with $j=1$ to m_k are separated from those of $\Gamma(1-a_j+\sum_{i=1}^{r}\alpha_j^{(k)}s_k)$ with $j=1$ to n and $\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)$ with $j=1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}-\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}>0, \text{ with } k=1\cdots,r, i=1,\cdots,R \text{ , } i^{(k)}=1,\cdots,R^{(k)} \tag{1.12}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.
We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,), max(\,|z_1|,\cdots,|z_r|\,)\rightarrow 0$

$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,), min(\,|z_1|,\cdots,|z_r|\,)\rightarrow\infty$

where, with $k=1,\cdots,r:\alpha_k=min[Re(d_j^{(k)}/\delta_j^{(k)})], j=1,\cdots,m_k$ and

$$\beta_k=max[Re((c_j^{(k)}-1)/\gamma_j^{(k)})], j=1,\cdots,n_k$$

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R \ ; \ V = m_1, n_1; \cdots ; m_r, n_r \tag{1.13}$$

$$\mathrm{W} = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i(1)}; R^{(1)}, \cdots , p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i(r)}; R^{(r)} \tag{1.14}$$

$$A = \{(a_j; \alpha_j^{(1)}, \cdots , \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots , \alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.15}$$

$$B = \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots , \beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.16}$$

$$C = \{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\}, \tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}, \cdots , \{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.17}$$

$$D = \{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}, \cdots, \{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.18}$$

The multivariable Aleph-function write :

$$\aleph(z_1, \cdots , z_r) = \aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A} : \mathrm{C} \\ \cdot & \\ \cdot & \cdots \\ \cdot & \\ \mathrm{z}_r & \mathrm{B} : \mathrm{D} \end{array}\right) \tag{1.19}$$

2. Finite double integrals

in this section, we evaluate three double finites integrals.

Integral 1

$$\int_0^{\pi/2}\int_0^1 e^{wi(\beta+\gamma)\theta}(sin\theta)^{\beta-1}(cos\theta)^{\gamma-1}x^{\rho-1}(1-x)^{\sigma-1}[ax+b(1-x)]^{-\rho-\sigma}$$

$${}_{p''}M_{q''}^{\alpha}(e^{wi(f+g)\theta}(sin\theta)^f(cos\theta)^g(ax)^{f'}[b(1-x)]^{g'}[ax+b(1-x)]^{-f'-g'})$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}(e^{wi(u+v)\theta}(sin\theta)^u(cos\theta)^v(ax)^{u'}[b(1-x)]^{v'}[ax+b(1-x)]^{-u'-v'})$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{e}^{wi(p_1'+q_1')\theta}(sin\theta)^{p_1'}(cos\theta)^{q_1'}(ax)^{s_1'}[b(1-x)]^{t_1'}[ax+b(1-x)]^{-s_1'-t_1'} \\ \cdots \\ \cdots \\ \mathrm{e}^{wi(p_s'+q_s')\theta}(sin\theta)^{p_s'}(cos\theta)^{q_s'}(ax)^{s_s'}[b(1-x)]^{t_s'}[ax+b(1-x)]^{s_s'-t_s'} \end{array}\right)$$

$$\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} \mathrm{z}_1 e^{wi(h_1+k_1)\theta}(sin\theta)^{h_1}(cos\theta)^{k_1}(ax)^{h_1'}[b(1-x)]^{k_1'}[ax+b(1-x)]^{-h_1'-k_1'} \\ \cdots \\ \cdots \\ \mathrm{z}_r e^{wi(h_r+k_r)\theta}(sin\theta)^{h_r}(cos\theta)^{k_r}(ax)^{h_r'}[b(1-x)]^{k_r'}[ax+b(1-x)]^{-h_r'-k_r'} \end{array}\right)\mathrm{d}x\mathrm{d}\theta$$

$$=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{l=0}^{\infty} \frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})[(a_{p''})]_l}{B_G g! \qquad [(b_{q''})]_l}\frac{1}{\Gamma(\alpha l+1)}A_1$$

$$exp(i\pi w(\beta+lf+u\eta_{G,g}+p'_1K_1+\cdots+p'_sK_s)/2)a^{-\rho}b^{-\sigma}\aleph^{0,\mathfrak{n}+4:V}_{U_{42}:W}\left(\begin{array}{c}\frac{z_1e^{i\pi wh_1/2}}{a^{h'_1}b^{k'_1}}\\ \cdot\\ \cdot\\ \frac{z_re^{i\pi wh_r/2}}{a^{h'_r}b^{k'_r}}\end{array}\right|$$

$$(1\text{-}\rho-f'l-u'\eta_{G,g}-\textstyle\sum_{i=1}^s s'_iK_i:h'_1,\cdots,h'_r),(1-\sigma-g'l-v'\eta_{G,g}-\sum_{i=1}^s t'_iK_i:k'_1,\cdots,k'_r)$$
$$\cdot\ \cdot\ \cdot$$
$$(1\text{-}\rho-\sigma-(f'+g')l-(u'+v')\eta_{G,g}-\textstyle\sum_{i=1}^s(s'_i+t'_i)K_i::h'_1+k'_1,\cdots,h'_r+k'_r)$$

$$(1\text{-}\beta-fl-u\eta_{G,g}-\textstyle\sum_{i=1}^s p'_iK_i:h_1,\cdots,h_r),(1-\gamma-gl-v\eta_{G,g}-\sum_{i=1}^s q'_iK_i;k_1,\cdots,k_r)$$
$$\cdot\ \cdot\ \cdot$$
$$(1\text{-}\beta-\gamma-(f+g)l-(u+v)\eta_{G,g}-\textstyle\sum_{i=1}^s(p'_i+q'_i)K_i:h_1+k_1,\cdots,h_r+k_r)$$

$$\left.\begin{array}{c}\mathrm{A}:\mathrm{C}\\ \cdot\ \cdot\ \cdot\\ \mathrm{B}:\mathrm{D}\end{array}\right)a^{-(lf'+u'\eta_{G,g}+s'_1K_1+\cdots+s'_sK_s)}\ b^{-(lg'+v'\eta_{G,g}+t'_1K_1+\cdots+t'_sK_s)} \quad (2.1)$$

Where $U_{42}=p_i+4,q_i+2,\tau_i;R$ and

Provided that

a) $p'_i,q'_i,s'_i,t'_i>0,i=1,\cdots,s;h_i,k_i,h'_i,k'_i>0,i=1,\cdots,r;p''<q''$ and $|ab|<1$

b) $Re[\rho+u'\min_{1\leqslant j\leqslant M}\dfrac{b_i}{B_i}+\sum_{i=1}^r h'_i\min_{1\leqslant j\leqslant m_i}\dfrac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

c) $Re[\sigma+v'\min_{1\leqslant j\leqslant M}\dfrac{b_i}{B_i}+\sum_{i=1}^r k'_i\min_{1\leqslant j\leqslant m_i}\dfrac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

d) $Re[\beta+u\min_{1\leqslant j\leqslant M}\dfrac{b_i}{B_i}+\sum_{i=1}^r h_i\min_{1\leqslant j\leqslant m_i}\dfrac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

e) $Re[\gamma+v\min_{1\leqslant j\leqslant M}\dfrac{b_i}{B_i}+\sum_{i=1}^r k_i\min_{1\leqslant j\leqslant m_i}\dfrac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

f) $|argz_k|<\dfrac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.11)

a,b are such that $ax+b(1-x)\neq 0$

Integral 2

$$\int_0^{\pi/2}\int_0^1 e^{wi(\beta+\gamma)\theta}(sin\theta)^{\beta-1}(cos\theta)^{\gamma-1}x^{f-1}(1-x)^{g-1}[ax+b(1-x)]^{-f-g}F(c,d;f;\frac{ax}{ax+b(1-x)}\Big)$$

$${}_{p''}M^{\alpha}_{q''}(e^{wi(f+g)\theta}(sin\theta)^f(cos\theta)^g(1-x)^{g'}[ax+b(1-x)]^{-f'-g'})$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}(e^{wi(u+v)\theta}(sin\theta)^u(cos\theta)^v(1-x)^{v'}[ax+b(1-x)]^{-u'-v'})$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{e}^{wi(p_1'+q_1')\theta}(sin\theta)^{p_1'}(cos\theta)^{q_1'}x^{s_1'}[b(1-x)]^{t_1'}[ax+b(1-x)]^{-s_1'-t_1'} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{e}^{wi(p_s'+q_s')\theta}(sin\theta)^{p_s'}(cos\theta)^{q_s'}x^{s_s'}[b(1-x)]^{t_s'}[ax+b(1-x)]^{s_s'-t_s'} \end{array}\right)$$

$$\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} \mathrm{z}_1e^{wi(h_1+k_1)\theta}(sin\theta)^{h_1}(cos\theta)^{k_1}[b(1-x)]^{k_1'}[ax+b(1-x)]^{-h_1'-k_1'} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{z}_re^{wi(h_r+k_r)\theta}(sin\theta)^{h_r}(cos\theta)^{k_r}[b(1-x)]^{k_r'}[ax+b(1-x)]^{-h_r'-k_r'} \end{array}\right)\mathrm{d}x\mathrm{d}\theta$$

$$=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{l=0}^{\infty}\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_Gg!}A_1\frac{[(a_{p''})]_l}{[(b_{q''})]_l}\frac{1}{\Gamma(\alpha l+1)}\Gamma(f)exp(i\pi w\sum_{k=1}^{s}p_k'K_k/2)$$

$$exp(i\pi w(\beta+lf+u\eta_{G,g})/2)\aleph_{U_{53}:W}^{0,\mathfrak{n}+4:V}\left(\begin{array}{c} \frac{z_1e^{i\pi wh_1/2}}{b^{k_1'}} \\ \cdot \\ \cdot \\ \frac{z_re^{i\pi wh_r/2}}{b^{k_r'}} \end{array}\middle| \begin{array}{c} (1\text{-g-}\sigma-g'l-v'\eta_{G,g}-\sum_{i=1}^s t_i'K_i:k_1',\cdots,k_r'), \\ \cdot\cdot\cdot \\ (1\text{-f-g+c-}\sigma-g'l-v'\eta_{G,g}-\sum_{i=1}^s t_i'K_i:k_i',\cdots,k_r'), \end{array}\right.$$

$$\begin{array}{c} (1\text{-}\beta-fl-u\eta_{G,g}-\sum_{i=1}^s p_i'K_i:h_1,\cdots,h_r),(1-\gamma-gl-v\eta_{G,g}-\sum_{i=1}^s q_i'K_i:k_1,\cdots,k_r) \\ \cdot\cdot\cdot \\ (1\text{-}\beta-\gamma-(f+g)l-(u+v)\eta_{G,g}-\sum_{i=1}^s(p_i'+q_i')K_i:h_1+k_1,\cdots,h_r+k_r) \end{array}$$

$$\left.\begin{array}{c} (1\text{+c+d-f-g-}\sigma-g'l-v'\eta_{G,g}-\sum_{i=1}^s t_i'K_i:k_1',\cdots,k_r'),\mathrm{A}:\mathrm{C} \\ \cdot\cdot\cdot \qquad\qquad \cdot\cdot\cdot \\ (1\text{-f-g+d-}\sigma-g'l-v'\eta_{G,g}-\sum_{i=1}^s t_i'K_i:k_1',\cdots,k_r'),\ \mathrm{B}:\mathrm{D} \end{array}\right)b^{-(g+lg'+v'\eta_{G,g}+\sum_{k=1}^s t_k'K_k)} \quad (2.2)$$

Where $U_{53}=p_i+5,q_i+3,\tau_i;R$

Provided that

a) $p_i',q_i',s_i',t_i'>0,i=1,\cdots,s;h_i,k_i,h_i',k_i'>0,i=1,\cdots,r;p''<q''$ and $|b|<1,Re(f)>0$

b) $Re[g+v'\min_{1\leqslant j\leqslant M}\frac{b_i}{B_i}+\sum_{i=1}^r k_i'\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

c) $Re[\beta+u\min_{1\leqslant j\leqslant M}\frac{b_i}{B_i}+\sum_{i=1}^r h_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

d) $Re[\gamma+v\min_{1\leqslant j\leqslant M}\frac{b_i}{B_i}+\sum_{i=1}^r k_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

e) $|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.11)

a, b are such that $ax + b(1-x) \neq 0$

Integral 3

$$\int_0^{\pi/2}\int_0^1 e^{wi(\beta+\gamma)\theta}(sin\theta)^{\beta-1}(cos\theta)^{\gamma-1}x^{\rho-1}(1-x)^{\sigma-1}{}_{p''}M_{q''}^{\alpha}(e^{wi(f+g)\theta}(sin\theta)^f(cos\theta)^g x^{f'}(1-x)^{g'})$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix} \mathrm{e}^{wi(p_1'+q_1')\theta}(sin\theta)^{p_1'}(cos\theta)^{q_1'}x^{s_1'}(1-x)^{t_1'} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{e}^{wi(p_s'+q_s')\theta}(sin\theta)^{p_s'}(cos\theta)^{q_s'}x^{s_s'}(1-x)^{t_s'} \end{pmatrix}\aleph_{P_i,Q_i,c_i;r'}^{M,N}(\ e^{wi(u+v)\theta}(sin\theta)^u(cos\theta)^v x^{u'}(1-x)^{v'})$$

$$\aleph_{U:W}^{0,\mathfrak{n}:V}\begin{pmatrix} \mathrm{z}_1 e^{wi(h_1+k_1)\theta}(sin\theta)^{h_1}(cos\theta)^{k_1}x^{h_1'}(1-x)^{k_1'} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{z}_r e^{wi(h_r+k_r)\theta}(sin\theta)^{h_r}(cos\theta)^{k_r}x^{h_r'}(1-x)^{k_r'} \end{pmatrix}\mathrm{d}x\mathrm{d}\theta$$

$$=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{l=0}^{\infty}A_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}\frac{[(a_{p''})]_l}{[(b_{q''})]_l}\frac{1}{\Gamma(\alpha l+1)}$$

$$exp(i\pi w(\beta + lf + u\eta_{G,g} + \sum_{k=1}^{s} t_k' K_k)/2)\ \aleph_{U_{42}:W}^{0,\mathfrak{n}+4:V}\left(\begin{array}{c} \mathrm{z}_1 e^{i\pi w h_1/2} \\ \cdot \\ \cdot \\ \mathrm{z}_r e^{i\pi w h_r/2} \end{array}\right|$$

$$\begin{array}{c}(1\text{-}\rho - f'l - u'\eta_{G,g} - \sum_{i=1}^{s} s_i'K_i : h_1', \cdots, h_r'), (1-\sigma - g'l - v'\eta_{G,g} - \sum_{i=1}^{s} t_i'K_i : k_1', \cdots, k_r') \\ \cdot\ \cdot\ \cdot \\ (1\text{-}\rho - \sigma - (f'+g')l - (u'+v')\eta_{G,g} - \sum_{i=1}^{s}(s_i'+t_i')K_i :: h_1'+k_1', \cdots, h_r'+k_r')\end{array}$$

$$\begin{array}{c}(1\text{-}\beta - fl - u\eta_{G,g} - \sum_{i=1}^{s} p_i'K_i : h_1, \cdots, h_r), (1-\gamma - gl - v\eta_{G,g} - \sum_{i=1}^{s} q_i'K_i ; k_1, \cdots, k_r) \\ \cdot\ \cdot\ \cdot \\ (1\text{-}\beta - \gamma - (f+g)l - (u+v)\eta_{G,g} - \sum_{i=1}^{s}(p_i'+q_i')K_i : h_1+k_1, \cdots, h_r+k_r)\end{array}$$

$$\left.\begin{array}{c}\mathrm{A} : \mathrm{C} \\ \cdot\ \cdot\ \cdot \\ \mathrm{B} : \mathrm{D}\end{array}\right) \tag{2.3}$$

Where $U_{42} = p_i + 4, q_i + 2, \tau_i; R$

Provided that

a) $p'_i, q'_i, s'_i, t'_i > 0, i = 1, \cdots, s; h_i, k_i, h'_i, k'_i > 0, i = 1, \cdots, r; p'' < q''$

b) $Re[\rho + u' \min_{1 \leqslant j \leqslant M} \frac{b_i}{B_i} + \sum_{i=1}^{r} h'_i \min_{1 \leqslant j \leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}] > 0$

c) $Re[\sigma + v' \min_{1 \leqslant j \leqslant M} \frac{b_i}{B_i} + \sum_{i=1}^{r} k'_i \min_{1 \leqslant j \leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}] > 0$

d) $Re[\beta + u \min_{1 \leqslant j \leqslant M} \frac{b_i}{B_i} + \sum_{i=1}^{r} h_i \min_{1 \leqslant j \leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}] > 0$

e) $Re[\gamma + v \min_{1 \leqslant j \leqslant M} \frac{b_i}{B_i} + \sum_{i=1}^{r} k_i \min_{1 \leqslant j \leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}] > 0$

f) $|arg z_k| < \frac{1}{2} A_i^{(k)} \pi$, where $A_i^{(k)}$ is given in (1.11)

Proof of (2.1)

To derive (2.1), express the general class of polynomial in series with the help of (1.5), the M-function in serie with the help (1.6), the Aleph-function of one variable in serie with the help (1.3) and the multivariable Aleph-function as given in (1.7).Now interchange the order of summations and integrations (which is permissible under the coonditions stated above), evaluate the inner θ-integral and x-integral with the help of known results given [2] and [3], we arrive at the desired result. We use the similar methods to prove (2.2) and (2.3) by making use of known results [3] and [4].

3. Particular cases

Taking $N_i \rightarrow 0$, $i = 1, \cdots, r$, the result in (2.3) reduces to the following result.

$$\int_0^{\pi/2} \int_0^1 e^{wi(\beta+\gamma)\theta} (sin\theta)^{\beta-1} (cos\theta)^{\gamma-1} x^{\rho-1} (1-x)^{\sigma-1} {}_{p''}M_{q''}^{\alpha} (e^{wi(f+g)\theta} (sin\theta)^f (cos\theta)^g x^{f'} (1-x)^{g'})$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N} (e^{wi(u+v)\theta} (sin\theta)^u (cos\theta)^v x^{u'} (1-x)^{v'})$$

$$\aleph_{U:W}^{0,\mathfrak{n}:V} \left(\begin{array}{c} \mathrm{z}_1 e^{wi(h_1+k_1)\theta} (sin\theta)^{h_1} (cos\theta)^{k_1} x^{h'_1} (1-x)^{k'_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{z}_r e^{wi(h_r+k_r)\theta} (sin\theta)^{h_r} (cos\theta)^{k_r} x^{h'_r} (1-x)^{k'_r} \end{array} \right) \mathrm{d}x \mathrm{d}\theta$$

$$= \sum_{G=1}^{M} \sum_{g=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N} (\eta_{G,g})}{B_G g!} \frac{[(a_{p''})]_l}{[(b_{q''})]_l} \frac{1}{\Gamma(\alpha l + 1)} exp(i\pi w(\beta + lf + u\eta_{G,g})/2)$$

$$\aleph_{U_{42}:W}^{0,\mathfrak{n}+4:V} \left(\begin{array}{c} \mathrm{z}_1 e^{i\pi w h_1/2} \\ \cdot \\ \cdot \\ \mathrm{z}_r e^{i\pi w h_r/2} \end{array} \middle| \begin{array}{c} (1\text{-}\rho - f'l - u'\eta_{G,g} : h'_1, \cdots, h'_r), (1 - \sigma - g'l - v'\eta_{G,g} : k'_1, \cdots, k'_r) \\ \cdot\ \cdot\ \cdot \\ (1\text{-}\rho - \sigma - (f'+g')l - (u'+v')\eta_{G,g} : h'_1 + k'_1, \cdots, h'_r + k'_r) \end{array} \right)$$

$$(1\text{-}\beta - fl - u\eta_{G,g} - pK_1 - p'K_2 : h_1, \cdots, h_r), (1 - \gamma - gl - v\eta_{G,g} - qK_1 - q'K_2 : k_1, \cdots, k_r)$$
$$\cdots$$
$$(1\text{-}\beta - \gamma - (f+g)l - (u+v)\eta_{G,g} - (p+q)K_1 - (p'+q')K_2 : h_1 + k_1, \cdots, h_r + k_r)$$

$$\left. \begin{matrix} \mathrm{A} : \mathrm{C} \\ \cdots \\ \mathrm{B} : \mathrm{D} \end{matrix} \right) \tag{3.1}$$

Where $U_{42} = p_i + 4, q_i + 2, \tau_i; R$

Provided that

a) $h_i, k_i, h_i', k_i' > 0, i = 1, \cdots, r; p'' < q''$

b) $Re[\rho + u' \min\limits_{1 \leqslant j \leqslant M} \dfrac{b_i}{B_i} + \sum\limits_{i=1}^{r} h_i' \min\limits_{1 \leqslant j \leqslant m_i} \dfrac{d_j^{(i)}}{\delta_j^{(i)}}] > 0$

c) $Re[\sigma + v' \min\limits_{1 \leqslant j \leqslant M} \dfrac{b_i}{B_i} + \sum\limits_{i=1}^{r} k_i' \min\limits_{1 \leqslant j \leqslant m_i} \dfrac{d_j^{(i)}}{\delta_j^{(i)}}] > 0$

d) $Re[\beta + u \min\limits_{1 \leqslant j \leqslant M} \dfrac{b_i}{B_i} + \sum\limits_{i=1}^{r} h_i \min\limits_{1 \leqslant j \leqslant m_i} \dfrac{d_j^{(i)}}{\delta_j^{(i)}}] > 0$

e) $Re[\gamma + v \min\limits_{1 \leqslant j \leqslant M} \dfrac{b_i}{B_i} + \sum\limits_{i=1}^{r} k_i \min\limits_{1 \leqslant j \leqslant m_i} \dfrac{d_j^{(i)}}{\delta_j^{(i)}}] > 0$

f) $|argz_k| < \dfrac{1}{2} A_i^{(k)} \pi$, where $A_i^{(k)}$ is given in (1.11)

The special cases of the double integrals involving the Hermite polynomials, the Laguerre polynomials and the Jacobi polynomials can be obtained by similar methods.

4. Multivariable I-function

If $\tau_i = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = 1$, the Aleph-function of several variables degenere to the I-function of several variables.

The following finite double integrals have been derived in this section for multivariable I-function defined by Sharma et al [5]. In these section, we note

$$B_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \text{ with } k = 1 \cdots, r, i = 1, \cdots, R \text{ , } i^{(k)} = 1, \cdots, R^{(k)}$$

Integral 1

$$\int_0^{\pi/2}\int_0^1 e^{wi(\beta+\gamma)\theta}(sin\theta)^{\beta-1}(cos\theta)^{\gamma-1}x^{\rho-1}(1-x)^{\sigma-1}[ax+b(1-x)]^{-\rho-\sigma}$$

$${}_{p''}M^{\alpha}_{q''}(e^{wi(f+g)\theta}(sin\theta)^f(cos\theta)^g(ax)^{f'}[b(1-x)]^{g'}[ax+b(1-x)]^{-f'-g'})$$

$$\aleph^{M,N}_{P_i,Q_i,c_i;r'}(e^{wi(u+v)\theta}(sin\theta)^u(cos\theta)^v(ax)^{u'}[b(1-x)]^{v'}[ax+b(1-x)]^{-u'-v'})$$

$$S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left(\begin{array}{c} \mathrm{e}^{wi(p'_1+q'_1)\theta}(sin\theta)^{p'_1}(cos\theta)^{q'_1}(ax)^{s'_1}[b(1-x)]^{t'_1}[ax+b(1-x)]^{-s'_1-t'_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{e}^{wi(p'_s+q'_s)\theta}(sin\theta)^{p'_s}(cos\theta)^{q'_s}(ax)^{s'_s}[b(1-x)]^{t'_s}[ax+b(1-x)]^{s'_s-t'_s} \end{array}\right)$$

$$I\left(\begin{array}{c} \mathrm{z}_1 e^{wi(h_1+k_1)\theta}(sin\theta)^{h_1}(cos\theta)^{k_1}(ax)^{h'_1}[b(1-x)]^{k'_1}[ax+b(1-x)]^{-h'_1-k'_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{z}_r e^{wi(h_r+k_r)\theta}(sin\theta)^{h_r}(cos\theta)^{k_r}(ax)^{h'_r}[b(1-x)]^{k'_r}[ax+b(1-x)]^{-h'_r-k'_r} \end{array}\right)\mathrm{d}x\mathrm{d}\theta$$

$$=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{l=0}^{\infty}A_1\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(\eta_{G,g})}{B_G g!}exp(i\pi w(\beta+lf+u\eta_{G,g})/2)$$

$$exp(i\pi w(p'_1K_1+\cdots+p'_sK_s)/2)\,\frac{[(a_{p''})]_l}{[(b_{q''})]_l}\,\frac{1}{\Gamma(\alpha l+1)}\,a^{-\rho}b^{-\sigma}\,I^{0,\mathfrak{n}+4:V}_{U_{42}:W}\left(\begin{array}{c} \frac{z_1 e^{i\pi w h_1/2}}{a^{h'_1}b^{k'_1}} \\ \cdot \\ \cdot \\ \frac{z_r e^{i\pi w h_r/2}}{a^{h'_r}b^{k'_r}} \end{array}\right|$$

$$\begin{array}{c}(1\text{-}\rho-f'l-u'\eta_{G,g}-\sum_{i=1}^s s'_iK_i:h'_1,\cdots,h'_r),(1-\sigma-g'l-v'\eta_{G,g}-\sum_{i=1}^s t'_iK_i:k'_1,\cdots,k'_r)\\ \cdot\ \cdot\ \cdot \\ (1\text{-}\rho-\sigma-(f'+g')l-(u'+v')\eta_{G,g}-\sum_{i=1}^s(s'_i+t'_i)K_i::h'_1+k'_1,\cdots,h'_r+k'_r)\end{array}$$

$$\begin{array}{c}(1\text{-}\beta-fl-u\eta_{G,g}-\sum_{i=1}^s p'_iK_i:h_1,\cdots,h_r),(1-\gamma-gl-v\eta_{G,g}-\sum_{i=1}^s q'_iK_i;k_1,\cdots,k_r)\\ \cdot\ \cdot\ \cdot \\ (1\text{-}\beta-\gamma-(f+g)l-(u+v)\eta_{G,g}-\sum_{i=1}^s(p'_i+q'_i)K_i:h_1+k_1,\cdots,h_r+k_r)\end{array}$$

$$\left.\begin{array}{c}\mathrm{A'}:\mathrm{C'}\\ \cdot\ \cdot\ \cdot \\ \mathrm{B'}:\mathrm{D'}\end{array}\right)\ a^{-(lf'+u'\eta_{G,g}+s'_1K_1+\cdots+s'_sK_s)}\ b^{-(lg'+v'\eta_{G,g}+t'_1K_1+\cdots+t'_sK_s)} \quad (4.1)$$

Where $U_{42}=p_i+4,q_i+2;R$

under the same conditions that (2.1) and $|argz_k|<\frac{1}{2}B_i^{(k)}\pi$

a, b are such that $ax + b(1-x) \neq 0$

Integral 2

$$\int_0^{\pi/2}\int_0^1 e^{wi(\beta+\gamma)\theta}(sin\theta)^{\beta-1}(cos\theta)^{\gamma-1}x^{f-1}(1-x)^{g-1}[ax+b(1-x)]^{-f-g}F(c,d;f;\frac{ax}{ax+b(1-x)})$$

$${}_{p''}M^{\alpha}_{q''}(e^{wi(f+g)\theta}(sin\theta)^{f}(cos\theta)^{g}(1-x)^{g'}[ax+b(1-x)]^{-f'-g'})$$

$$\aleph^{M,N}_{P_i,Q_i,c_i;r'}(e^{wi(u+v)\theta}(sin\theta)^{u}(cos\theta)^{v}(1-x)^{v'}[ax+b(1-x)]^{-u'-v'})$$

$$S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left(\begin{array}{c} \mathrm{e}^{wi(p'_1+q'_1)\theta}(sin\theta)^{p'_1}(cos\theta)^{q'_1}x^{s'_1}[b(1-x)]^{t'_1}[ax+b(1-x)]^{-s'_1-t'_1} \\ \cdots \\ \cdots \\ \mathrm{e}^{wi(p'_s+q'_s)\theta}(sin\theta)^{p'_s}(cos\theta)^{q'_s}x^{s'_s}[b(1-x)]^{t'_s}[ax+b(1-x)]^{s'_s-t'_s} \end{array}\right)$$

$$I\left(\begin{array}{c} \mathrm{z}_1 e^{wi(h_1+k_1)\theta}(sin\theta)^{h_1}(cos\theta)^{k_1}[b(1-x)]^{k'_1}[ax+b(1-x)]^{-h'_1-k'_1} \\ \cdots \\ \cdots \\ \mathrm{z}_r e^{wi(h_r+k_r)\theta}(sin\theta)^{h_r}(cos\theta)^{k_r}[b(1-x)]^{k'_r}[ax+b(1-x)]^{-h'_r-k'_r} \end{array}\right)\mathrm{d}x\mathrm{d}\theta$$

$$=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{l=0}^{\infty}A_1\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(\eta_{G,g})[(a_{p''})]_l}{B_G g!\qquad\qquad [(b_{q''})]_l}\frac{1}{\Gamma(\alpha l+1)}\Gamma(f)exp(i\pi w(\beta+lf)/2)$$

$$exp(i\pi w(u\eta_{G,g}+\sum_{k=1}^{r}p_kK_k)/2)I^{0,\mathfrak{n}+4:V}_{U_{53}:W}\left(\begin{array}{c} \frac{z_1 e^{i\pi w h_1/2}}{b^{k'_1}} \\ \cdot \\ \cdot \\ \frac{z_r e^{i\pi w h_r/2}}{b^{k'_r}} \end{array}\middle|\begin{array}{c} (\text{1-g-}\sigma-g'l-v'\eta_{G,g}-\sum_{i=1}^{s}t'_iK_i:k'_1,\cdots,k'_r), \\ \cdots \\ (\text{1-f-g+c-}\sigma-g'l-v'\eta_{G,g}-\sum_{i=1}^{s}t'_iK_i:k'_i,\cdots,k'_r), \end{array}\right.$$

$$\begin{array}{c}(\text{1-}\beta-fl-u\eta_{G,g}-\sum_{i=1}^{s}p'_sK_s:h_1,\cdots,h_r),(1-\gamma-gl-v\eta_{G,g}-\sum_{i=1}^{s}q'_sK_s:k_1,\cdots,k_r), \\ \cdots \\ (\text{1-}\beta-\gamma-(f+g)l-(u+v)\eta_{G,g}-\sum_{i=1}^{s}(p'_s+q'_s)K_s:h_1+k_1,\cdots,h_r+k_r),\end{array}$$

$$\left.\begin{array}{c}(\text{1+c+d-f-g-}\sigma-g'l-v'\eta_{G,g}-\sum_{i=1}^{s}t'_iK_i:k'_1,\cdots,k'_r),\ \mathrm{A}:\mathrm{C} \\ \cdots \qquad\qquad\qquad \cdots \\ (\text{1-f-g+d-}\sigma-g'l-v'\eta_{G,g}-\sum_{i=1}^{s}t'_iK_i:k'_1,\cdots,k'_r),\ \mathrm{B}:\mathrm{D}\end{array}\right)b^{-(g+lg'+v'\eta_{G,g}+\sum_{k=1}^{s}t'_kK_k)} \quad (4.2)$$

Where $U_{53}=p_i+5,q_i+3;R$

under the same conditions that (4.2) and $|argz_k| < \frac{1}{2}B_i^{(k)}\pi$

a, b are such that $ax + b(1-x) \neq 0$

Integral 3

$$\int_0^{\pi/2}\int_0^1 e^{wi(\beta+\gamma)\theta}(sin\theta)^{\beta-1}(cos\theta)^{\gamma-1}x^{\rho-1}(1-x)^{\sigma-1}{}_{p''}M_{q''}^{\alpha}(e^{wi(f+g)\theta}(sin\theta)^f(cos\theta)^g x^{f'}(1-x)^{g'})$$

$$I\left(\begin{array}{c} \mathrm{z}_1 e^{wi(h_1+k_1)\theta}(sin\theta)^{h_1}(cos\theta)^{k_1}x^{h_1'}(1-x)^{k_1'} \\ \cdots \\ \cdots \\ \mathrm{z}_r e^{wi(h_r+k_r)\theta}(sin\theta)^{h_r}(cos\theta)^{k_r}x^{h_r'}(1-x)^{k_r'} \end{array}\right)\aleph_{P_i,Q_i,c_i;r'}^{M,N}(e^{wi(u+v)\theta}(sin\theta)^u(cos\theta)^v x^{u'}(1-x)^{v'})$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{e}^{wi(p_1'+q_1')\theta}(sin\theta)^{p_1'}(cos\theta)^{q_1'}x^{s_1'}(1-x)^{t_1'} \\ \cdots \\ \cdots \\ \mathrm{e}^{wi(p_s'+q_s')\theta}(sin\theta)^{p_s'}(cos\theta)^{q_s'}x^{s_s'}(1-x)^{t_s'} \end{array}\right)\mathrm{d}x\mathrm{d}\theta$$

$$=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{l=0}^{\infty}A_1\ \frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}\frac{[(a_{p''})]_l}{[(b_{q''})]_l}\frac{1}{\Gamma(\alpha l+1)}$$

$$exp(i\pi w(\beta+lf+u\eta_{G,g}+p_1'K_1+\cdots+p_s'K_s)/2)\ I_{U_{42}:W}^{0,\mathfrak{n}+4:V}\left(\begin{array}{c} \mathrm{z}_1 e^{i\pi wh_1/2} \\ \cdot \\ \cdot \\ \mathrm{z}_r e^{i\pi wh_r/2} \end{array}\right|$$

$$\begin{array}{c}(1\text{-}\rho-f'l-u'\eta_{G,g}-\sum_{i=1}^s s_i'K_i:h_1',\cdots,h_r'),(1-\sigma-g'l-v'\eta_{G,g}-\sum_{i=1}^s t_i'K_i:k_1',\cdots,k_r') \\ \cdots \\ (1\text{-}\rho-\sigma-(f'+g')l-(u'+v')\eta_{G,g}-\sum_{i=1}^s(s_i'+t_i')K_i::h_1'+k_1',\cdots,h_r'+k_r')\end{array}$$

$$\begin{array}{c}(1\text{-}\beta-fl-u\eta_{G,g}-\sum_{i=1}^s p_i'K_i:h_1,\cdots,h_r),(1-\gamma-gl-v\eta_{G,g}-\sum_{i=1}^s q_i'K_i;k_1,\cdots,k_r) \\ \cdots \\ (1\text{-}\beta-\gamma-(f+g)l-(u+v)\eta_{G,g}-\sum_{i=1}^s(p_i'+q_i')K_i:h_1+k_1,\cdots,h_r+k_r)\end{array}$$

$$\left.\begin{array}{c}\mathrm{A'}:\mathrm{C'} \\ \cdots \\ \mathrm{B'}:\mathrm{D'}\end{array}\right) \tag{4.3}$$

Where $U_{42} = p_i + 4, q_i + 2; R$

under the same conditions that (2.3) and $|argz_k| < \frac{1}{2}B_i^{(k)}\pi$

5. Multivariable H-function

If $\tau_i = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = 1$and $R = R^{(1)} =, \cdots, R^{(r)} = 1$,the Aleph-function of several variables degenere to the H-function of several variables.

The following finite double integrals have been derived in this section for multivariable H-function defined by Srivastava et al [8]. In these section, we note

$$A_i = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(i)} - \sum_{j=\mathfrak{n}+1}^{p} \alpha_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{n_i} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \gamma_j^{(i)} + \sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} \delta_j^{(i)} > 0\,,$$

Integral 1

$$\int_0^{\pi/2}\int_0^1 e^{wi(\beta+\gamma)\theta}(sin\theta)^{\beta-1}(cos\theta)^{\gamma-1}x^{\rho-1}(1-x)^{\sigma-1}[ax+b(1-x)]^{-\rho-\sigma}$$

$${}_{p''}M_{q''}^{\alpha}(e^{wi(f+g)\theta}(sin\theta)^f(cos\theta)^g(ax)^{f'}[b(1-x)]^{g'}[ax+b(1-x)]^{-f'-g'})$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}(e^{wi(u+v)\theta}(sin\theta)^u(cos\theta)^v(ax)^{u'}[b(1-x)]^{v'}[ax+b(1-x)]^{-u'-v'})$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{e}^{wi(p_1'+q_1')\theta}(sin\theta)^{p_1'}(cos\theta)^{q_1'}(ax)^{s_1'}[b(1-x)]^{t_1'}[ax+b(1-x)]^{-s_1'-t_1'} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{e}^{wi(p_s'+q_s')\theta}(sin\theta)^{p_s'}(cos\theta)^{q_s'}(ax)^{s_s'}[b(1-x)]^{t_s'}[ax+b(1-x)]^{s_s'-t_s'} \end{array}\right)$$

$$H\left(\begin{array}{c} \mathrm{z}_1 e^{wi(h_1+k_1)\theta}(sin\theta)^{h_1}(cos\theta)^{k_1}(ax)^{h_1'}[b(1-x)]^{k_1'}[ax+b(1-x)]^{-h_1'-k_1'} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{z}_r e^{wi(h_r+k_r)\theta}(sin\theta)^{h_r}(cos\theta)^{k_r}(ax)^{h_r'}[b(1-x)]^{k_r'}[ax+b(1-x)]^{-h_r'-k_r'} \end{array}\right)\mathrm{d}x\mathrm{d}\theta$$

$$=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{l=0}^{\infty} A_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})[(a_{p''})]_l}{B_G g!\qquad [(b_{q''})]_l}\frac{1}{\Gamma(\alpha l+1)}a^{-\rho}b^{-\sigma}$$

$$exp(i\pi w(p_1'K_1+\cdots+p_s'K_s')/2)H_{p+4,q+2:W}^{0,\mathfrak{n}+4:V}\left(\begin{array}{c} \frac{z_1 e^{i\pi w h_1/2}}{a^{h_1'}b^{k_1'}} \\ \cdot \\ \cdot \\ \frac{z_r e^{i\pi w h_r/2}}{a^{h_r'}b^{k_r'}} \end{array}\right|$$

$$(1\text{-}\rho-f'l-u'\eta_{G,g}-\textstyle\sum_{i=1}^s s_i'K_i : h_1',\cdots,h_r'),(1-\sigma-g'l-v'\eta_{G,g}-\sum_{i=1}^s t_i'K_i : k_1',\cdots,k_r')$$
$$\cdot\ \cdot\ \cdot$$
$$(1\text{-}\rho-\sigma-(f'+g')l-(u'+v')\eta_{G,g}-\textstyle\sum_{i=1}^s(s_i'+t_i')K_i :: h_1'+k_1',\cdots,h_r'+k_r')$$

$$(1\text{-}\beta - fl - u\eta_{G,g} - \sum_{i=1}^{s} p_i' K_i : h_1, \cdots, h_r), (1 - \gamma - gl - v\eta_{G,g} - \sum_{i=1}^{s} q_i' K_i ; k_1, \cdots, k_r)$$

$$\cdots$$

$$(1\text{-}\beta - \gamma - (f+g)l - (u+v)\eta_{G,g} - \sum_{i=1}^{s}(p_i' + q_i')K_i : h_1 + k_1, \cdots, h_r + k_r)$$

$$\left.\begin{matrix} \text{A'} : \text{C'} \\ \cdots \\ \text{B'} : \text{D'} \end{matrix}\right) \quad a^{-(lf' + u'\eta_{G,g} + s_1' K_1 + \cdots + s_s' K_s)} \; b^{-(lg' + v'\eta_{G,g} + t_1' K_1 + \cdots + t_s' K_s)} \tag{5.1}$$

under the same conditions that (2.1) and $|argz_i| < \frac{1}{2} A_i \pi \; i = 1, \cdots, r$

a, b are such that $ax + b(1-x) \neq 0$

Integral 2

$$\int_0^{\pi/2} \int_0^1 e^{wi(\beta+\gamma)\theta} (sin\theta)^{\beta-1} (cos\theta)^{\gamma-1} x^{f-1} (1-x)^{g-1} [ax + b(1-x)]^{-f-g} F(c, d; f; \frac{ax}{ax + b(1-x)})$$

$${}_{p''}M_{q''}^{\alpha}(e^{wi(f+g)\theta} (sin\theta)^f (cos\theta)^g (1-x)^{g'} [ax + b(1-x)]^{-f'-g'})$$

$$\aleph_{P_i, Q_i, c_i; r'}^{M,N}(e^{wi(u+v)\theta} (sin\theta)^u (cos\theta)^v (1-x)^{v'} [ax + b(1-x)]^{-u'-v'})$$

$$S_{N_1, \cdots, N_s}^{M_1, \cdots, M_s}\left(\begin{matrix} \mathrm{e}^{wi(p_1' + q_1')\theta} (sin\theta)^{p_1'} (cos\theta)^{q_1'} x^{s_1'} [b(1-x)]^{t_1'} [ax + b(1-x)]^{-s_1' - t_1'} \\ \cdots \\ \cdots \\ \mathrm{e}^{wi(p_s' + q_s')\theta} (sin\theta)^{p_s'} (cos\theta)^{q_s'} x^{s_s'} [b(1-x)]^{t_s'} [ax + b(1-x)]^{s_s' - t_s'} \end{matrix}\right)$$

$$H\left(\begin{matrix} \mathrm{z}_1 e^{wi(h_1 + k_1)\theta} (sin\theta)^{h_1} (cos\theta)^{k_1} [b(1-x)]^{k_1'} [ax + b(1-x)]^{-h_1' - k_1'} \\ \cdots \\ \cdots \\ \mathrm{z}_r e^{wi(h_r + k_r)\theta} (sin\theta)^{h_r} (cos\theta)^{k_r} [b(1-x)]^{k_r'} [ax + b(1-x)]^{-h_r' - k_r'} \end{matrix}\right) \mathrm{d}x \mathrm{d}\theta$$

$$= \sum_{G=1}^{M} \sum_{g=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \sum_{l=0}^{\infty} A_1 \frac{(-)^g \Omega_{P_i, Q_i, c_i, r'}^{M,N}(\eta_{G,g})}{B_G g!} \frac{[(a_{p''})]_l}{[(b_{q''})]_l} \frac{1}{\Gamma(\alpha l + 1)} b^{-g}$$

$$exp(i\pi w(\beta + lf + u\eta_{G,g} + p' K_1 + \cdots + p_s' K_s)/2)\Gamma(f)$$

$$H_{p+5, q+3:W}^{0, \mathfrak{n}+4:V}\left(\begin{matrix} \frac{z_1 e^{i\pi w h_1/2}}{b^{k_1'}} \\ \cdot \\ \cdot \\ \frac{z_r e^{i\pi w h_r/2}}{b^{k_r'}} \end{matrix}\ \middle|\ \begin{matrix} (1\text{-g-}\sigma - g'l - v'\eta_{G,g} - \sum_{i=1}^{s} t_i' K_i : k_1', \cdots, k_r'), \\ \cdots \\ (1\text{-f-g+c-}\sigma - g'l - v'\eta_{G,g} - \sum_{i=1}^{s} t_i' K_i : k_i', \cdots, k_r'), \end{matrix}\right.$$

$$(1\text{-}\beta - fl - u\eta_{G,g} - \textstyle\sum_{i=1}^{s} p'_s K_s : h_1, \cdots, h_r), (1-\gamma - gl - v\eta_{G,g} - \sum_{i=1}^{s} q'_s K_s : k_1, \cdots, k_r),$$

$$\cdots$$

$$(1\text{-}\beta - \gamma - (f+g)l - (u+v)\eta_{G,g} - \textstyle\sum_{i=1}^{s}(p'_s + q'_s)K_s : h_1 + k_1, \cdots, h_r + k_r),$$

$$\left.\begin{array}{c} (1\text{+c+d-f-g-}\sigma - g'l - v'\eta_{G,g} - \sum_{i=1}^{s} t'_i K_i : k'_1, \cdots, k'_r), \mathrm{A} : \mathrm{C} \\ \cdots \qquad \cdots \\ (1\text{-f-g+d-}\sigma - g'l - v'\eta_{G,g} - \sum_{i=1}^{s} t'_i K_i : k'_1, \cdots, k'_r), \quad \mathrm{B} : \mathrm{D} \end{array}\right) b^{-(lg' + v'\eta_{G,g} + \sum_{k=1}^{s} t'_k K_k)} \tag{5.2}$$

Under the same conditions that (2.2) with $|argz_i| < \dfrac{1}{2}A_i\pi \; i = 1, \cdots, r$

a, b are such that $ax + b(1-x) \neq 0$

Integral 3

$$\int_0^{\pi/2}\int_0^1 e^{wi(\beta+\gamma)\theta}(sin\theta)^{\beta-1}(cos\theta)^{\gamma-1}x^{\rho-1}(1-x)^{\sigma-1}{}_{p''}M^{\alpha}_{q''}(e^{wi(f+g)\theta}(sin\theta)^f(cos\theta)^g x^{f'}(1-x)^{g'})$$

$$\aleph^{M,N}_{P_i,Q_i,c_i;r'}(e^{wi(u+v)\theta}(sin\theta)^u(cos\theta)^v x^{u'}(1-x)^{v'}) H\left(\begin{array}{c} \mathrm{z}_1 e^{wi(h_1+k_1)\theta}(sin\theta)^{h_1}(cos\theta)^{k_1}x^{h'_1}(1-x)^{k'_1} \\ \cdots \\ \cdots \\ \mathrm{z}_r e^{wi(h_r+k_r)\theta}(sin\theta)^{h_r}(cos\theta)^{k_r}x^{h'_r}(1-x)^{k'_r} \end{array}\right)$$

$$S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left(\begin{array}{c} \mathrm{e}^{wi(p'_1+q'_1)\theta}(sin\theta)^{p'_1}(cos\theta)^{q'_1}x^{s'_1}(1-x)^{t'_1} \\ \cdots \\ \cdots \\ \mathrm{e}^{wi(p'_s+q'_s)\theta}(sin\theta)^{p'_s}(cos\theta)^{q'_s}x^{s'_s}(1-x)^{t'_s} \end{array}\right)\mathrm{d}x\mathrm{d}\theta$$

$$= \sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{l=0}^{\infty} A_1 \frac{(-)^g \Omega^{M,N}_{P_i,Q_i,c_i,r'}(\eta_{G,g})}{B_G g!}$$

$$exp(i\pi w(\beta + lf + u\eta_{G,g} + p'K_1 + \cdots + p'_s K_s)/2)\frac{[(a_{p''})]_l}{[(b_{q''})]_l}\frac{1}{\Gamma(\alpha l + 1)} H^{0,\mathfrak{n}+4:V}_{p+4:q+2:W}\left(\begin{array}{c} \mathrm{z}_1 e^{i\pi w h_1/2} \\ \cdot \\ \cdot \\ \mathrm{z}_r e^{i\pi w h_r/2} \end{array}\right|$$

$$(1\text{-}\rho - f'l - u'\eta_{G,g} - \textstyle\sum_{i=1}^{s} s'_i K_i : h'_1, \cdots, h'_r), (1-\sigma - g'l - v'\eta_{G,g} - \sum_{i=1}^{s} t'_i K_i : k'_1, \cdots, k'_r)$$

$$\cdots$$

$$(1\text{-}\rho - \sigma - (f'+g')l - (u'+v')\eta_{G,g} - \textstyle\sum_{i=1}^{s}(s'_i + t'_i)K_i : h'_1 + k'_1, \cdots, h'_r + k'_r)$$

$$(1\text{-}\beta - fl - u\eta_{G,g} - \textstyle\sum_{i=1}^{s} p'_i K_i : h_1, \cdots, h_r), (1-\gamma - gl - v\eta_{G,g} - \sum_{i=1}^{s} q'_i K_i ; k_1, \cdots, k_r)$$

$$\cdots$$

$$(1\text{-}\beta - \gamma - (f+g)l - (u+v)\eta_{G,g} - \textstyle\sum_{i=1}^{s}(p'_i + q'_i)K_i : h_1 + k_1, \cdots, h_r + k_r)$$

$$\begin{array}{l} A" : C" \\ \quad \cdots \\ B" : D" \end{array}\Bigg) \qquad (5.3)$$

under the same conditions that (2.3) with $|argz_i| < \frac{1}{2}A_i\pi$, $i = 1, \cdots, r$

6. Conclusion

The aleph-function of several variables presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions of several variables such as , multivariable I-function defined by Sharma et al [5], multivariable H-function , defined by Srivastava et al [8].

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