HOMOTOPY PROPERTIES OF DIGITAL SIMPLE CLOSED CURVES

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Abstract:

In this paper we discuss some properties of digital simple closed curves and prove that a digital simple closed curve of more than four points is not contractible.

Keywords:
Digital image, Digital topology, Digital simple closed curve

1 Introduction

A digital image is a set X of lattice points that model a continuous object Y, where Y is a subset of a Euclidean space. Digital topology is concerned with developing a mathematical theory of such discrete objects so that digital images have topological properties that mirror those of the Euclidean objects they model; Applications of digital topology have been found shape description and in image processing operations such as thinning and skeletonization.

2 Preliminaries

Let Z be the set of integers. $Z^d$ is the set of lattice points in d-dimensional Euclidean space. Let $X \subset Z^d$ and let $k$ be some adjacency relation for the members of $X$. Then the pair $(X, k)$ is said to be binary digital image. For a positive integer $l$ with $1 \leq l \leq n$ and two distinct points $p = (p_1, p_2, ..., p_d)$, $q = (q_1, q_2, ..., q_d) \in Z^d$, $p$ and $q$ are $c_i$ adjacent if
there are atmost \( l \) indices \( i \) such that \( \|p_i - q_i\| = 1 \) and
for all other indices \( j \) such that \( \|p_j - q_j\| \neq 1 \), \( p_j = q_j \). \( C_i \) denotes the number of points \( q \in \mathbb{Z}^d \) that are \( C_i \)-adjacent to a given point \( p \in \mathbb{Z}^d \). Thus in \( \mathbb{Z} \) we have \( c_1 = 2 \), in \( \mathbb{Z}^2 \) we have \( c_1 = 4 \) and \( c_2 = 8 \) and so on.

2.1 Proposition (2)

Let \( X \subset \mathbb{Z}^{d_0} \) and \( Y \subset \mathbb{Z}^{d_1} \) be digital images with \( k_0 \) adjacency and \( k_1 \) adjacency respectively. Then the function \( f: X \to Y \) is \( (k_0, k_1) \) continuous if and only if for every \( k_0 \) adjacent points \( \{ x_0, x_1 \} \) of \( X \) either \( f(x_0) = f(x_1) \) or \( f(x_0) \) and \( f(x_1) \) are \( k_1 \) adjacent in \( Y \).

2.2 Definition [2]

Let \( X \subset \mathbb{Z}^{d_0} \) and \( Y \subset \mathbb{Z}^{d_1} \) be digital images with \( k_0 \) adjacency and \( k_1 \)-adjacency respectively. Two \( (k_0, k_1) \) continuous functions \( f, g: X \to Y \) are said to be digitally \( (k_0, k_1) \) homotopic in \( Y \) if there is a positive integer \( m \) and a function \( H: X \times [0, m] \to Y \) such that
- for all \( x \in X \), \( H(x, 0) = f(x) \) and \( H(x, m) = g(x) \)
- for all \( x \in X \) the induced function \( H_x: [0, m] \to Y \) defined by \( H_x(t) = H(x, t) \) for all \( t \in [0, m] \) is \( (2, k_1) \) continuous and
- for all \( t \in [0, m] \), the induced function \( H_t: X \to Y \) defined by \( H_t(x) = H(x, t) \) for all \( x \in X \) is \( (k_0, k_1) \) continuous.

2.3 Definition (4)

A digital simple closed \( k \)-curve \( X \) is required to satisfy the following. \( (X, k) \) is a digital image and the following property (SCC) is satisfied for some positive integer \( m \).

(SCC) There is a \( (2, k) \) continuous function \( f: [0, m-1] \to X \) such that
- \( f \) is one-to-one and onto and
• for all $t \in [0, m-1] \mathbb{Z}$, the set of $k$-neighbors of $f(t)$ in $f [0, m-1] \mathbb{Z}$ is $\{ f((t-1) \mod m), f((t+1) \mod m) \}$.

### 2.4 Definition(4)

If $S = \{ x_i \}_{0}^{m-1}$ where $x_i = f(i)$ for all $i \in [0, m-1] \mathbb{Z}$, then the points of $S$ are circularly ordered.

### 3 Homotopy properties of digital simple closed curves

#### Proposition (3.1)

Let $S_{a}$ be a digital simple closed $K_{a}$-curve, $a \in \{0, 1\}$ Let $f : S_{0} \to S_{1}$ be a $(k_{0}, k_{1})$ continuous function. If $|S_{0}| = |S_{1}|$, then the following are equivalent.

- (a) $f$ is one to one
- (b) $f$ is onto
- (c) $f$ is a $(k_{0}, k_{1})$ isomorphism

**Proof:**

Since $S_{0}$ is a finite set (a) $\implies$ (b)

(c) follows from (a) & (b) [definition of isomorphism]

(b) $\implies$ (c)

Let $S_{a} = \{ x_{a, i} \}_{0}^{n-1}$ where the points $S_{a}$ are circularly ordered, $a \in \{0, 1\}$. Let $x_{1, u} \in S_{1}$ and let $x_{0, v} = f^{-1}(x_{1, u})$. Then the $k_{1}$ neighbors of $x_{1, u}$ in $S_{1}$ are $x_{1, (u-1) \mod n}$ and $x_{1, (u+1) \mod n}$. The $k_{0}$ neighbors of $x_{0, v}$ in $S_{0}$ are $x_{0, (v-1) \mod n}$ and $x_{0, (v+1) \mod n}$. Since $f$ is a continuous bijection, choice of $x_{0, v}$ implies

$$f(\{ x_{0, (v-1) \mod n}, x_{0, (v+1) \mod n} \}) = \{ x_{1, (u-1) \mod n}, x_{1, (u+1) \mod n} \}.$$  

Thus, $f^{-1}(\{ x_{1, (u-1) \mod n}, x_{1, (u+1) \mod n} \}) = \{ x_{0, (v-1) \mod n}, x_{0, (v+1) \mod n} \}$.

Since $u$ is arbitrary $f^{-1}$ is $(k_{1}, k_{0})$ continuous, so $f$ is a $(k_{0}, k_{1})$ isomorphism.

#### Theorem 3.2.

Let $S$ be a simple closed $k$ curve and let $H : S \times [0, m] \mathbb{Z} \to S$ be a $(k, k)$ homotopy between an isomorphism $H_{0}$ and $H_{m} = f$, where $f(S) \neq S$, then $|S| = 4$. 
Proof

Let $S = \{x_i\}_{i=0}^{n-1}$ where the points of $S$ are circularly ordered. There exists $\omega \in [1, m] \mathbb{Z}$ such that $\omega = \min\{t \in [0, m] \mathbb{Z} \mid H_t(S) \neq S\}$.

Without loss of generality, $x_1 \notin H_\omega(S)$. Then the induced function $H_{\omega-1}$ is a bijection, so there exists $x_u \in S$ such that $H(x_u, \omega-1) = x_1$. By proposition 3.1, $H_{\omega-1}(\{x_{(u-1) \mod n}, x_{(u+1) \mod n}\}) = \{x_0, x_2\}$ and the continuity property of Homotopy implies $H(x_u, \omega) \in \{x_0, x_2\}$. Without loss of generality, $H(x_{(u-1) \mod n}, \omega) = x_0 \quad \ldots \quad (1)$ and $H(x_u, \omega) = x_2 \quad \ldots \quad (2)$

Suppose $n > 4$. Equ. (2) implies $H(x_{(u-1) \mod n}, \omega) \in \{x_1, x_2, x_3\}$ but this is impossible since

1. $H(x_{(u-1) \mod n}, \omega) \notin x_1$
2. $H(x_{(u-1) \mod n}, \omega) \notin \{x_2, x_3\}$ from equ (1) because $n > 4$ implies neither $x_2$ nor $x_3$ is $k$-adjacent to $x_0$.

The contradiction arose from the assumption that $n > 4$. Therefore we must have $n \leq 4$. Since a digital simple closed curve is assumed to have at least 4 points, we must have $n = 4$

4 Conclusion:

We have shown that digital simple closed curves of more than 4 points are not contractible.

5 References