ON L-FUZZY $\omega$-BASICALLY DISCONNECTED SPACES

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Abstract

In this paper L-fuzzy $\omega$-closed and L-fuzzy $\omega$-open sets are introduced. Also a new class of L-fuzzy topological space called L-fuzzy $\omega$-basically disconnected space is introduced. Several characterizations and some interesting properties are also given.

Keywords L-fuzzy $\omega$-closed, L-fuzzy $\omega$-open set, L-fuzzy $\omega$-basically disconnected space, L-fuzzy $\omega^*$-continuous map, L-fuzzy $\omega^*$-irresolute, strong $F_\omega$ L-fuzzy $\omega^*$-continuous map, lower (upper) L-fuzzy $\omega^*$-continuous map.

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1. Introduction

The fuzzy concept has invaded almost all branches of Mathematics since the introduction of the concept by Zadeh[14]. Fuzzy sets have applications in many fields such as information [11] and control [12]. The theory of fuzzy topological spaces was introduced and developed by Chang [3] and since then various important notions in classical topology have been extended to fuzzy topological spaces. Rodabaugh [7] discussed normality and the L-fuzzy unit interval. He [8] also studied fuzzy addition in the L-fuzzy real line. Hoeche [6] studied the characterizations of L-topologies by L-valued neighbourhoods. An L-fuzzy normal spaces and Tietze extension theorem were discussed by Tomash Kubiak [14]. The concept of $\omega$-open set was studied in [9]. The purpose of this paper is to introduce L-fuzzy $\omega$-closed, L-fuzzy $\omega$-open sets and a new class of L-fuzzy topological spaces called L-fuzzy $\omega$-basically disconnected space. In this connection several characterizations and some interesting properties are also given.

2. Preliminaries

Definition [1] Let $(X, T)$ be a fuzzy topological space and $\lambda$ be a fuzzy set in $(X, T)$, $\lambda$ is called a fuzzy $G_\delta$-set if $\lambda = \bigwedge_{i=1}^{\infty} \lambda_i$ where each $\lambda_i \in T$, $i \in I$.

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Definition [2] Throughout this paper $(L, \leq, \cdot)$ stands for an infinitely distributive lattice with an order reversing involution. Such a lattice being complete has a least element 0 and a greatest element 1. Let $X$ be a non-empty set. An L-fuzzy set in $X$ is an element of the set $L^X$ of all functions from $X$ to $L$.

Definition [4] The L-fuzzy real line $R(L)$ is the set of all monotone decreasing elements $\lambda \in L^R$ satisfying $\bigvee \{ \lambda(t) \mid t \in R \} = 1$ and $\bigwedge \{ \lambda(t) \mid t \in R \} = 0$, after the identification of $\lambda, \mu \in L^R$ if $\lambda(t) = \mu(t)$ for all $t \in R$ where $\lambda(t) = \bigwedge \{ \lambda(s) \mid s < t \}$ and $\lambda(t) = \bigvee \{ \lambda(s) \mid s > t \}$. The natural L-fuzzy topology on $R(L)$ is generated from the subbases $\{ L_t, R_t \mid t \in R \}$, where $L(\lambda) \equiv \lambda(t^-)$ and $R(\lambda) \equiv \lambda(t^+)$.

The L-fuzzy unit interval $I(L) [5]$ is a subset of $R(L)$ such that $[\lambda \quad \in I(L) \text{ if }\lambda(t) = 1 \text{ for } t < 0 \text{ and } \lambda(t) = 0 \text{ for } t > 1$. It is equipped with the subspace L-fuzzy topology.
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**Definition 14** If \( A \in L^X \) is crisp, then \((A,T_A)\) is an L-fuzzy topological space called a crisp subspace of \((X,T)\), where \(T_A=\{U/A \mid U \in T\}\) is called the subspace L-fuzzy topology.

**Definition 9** A subset of a topological space \((X,T)\) is called \(\omega\)-closed in \((X,T)\) if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open in \((X,T)\). A subset \(A\) is called \(\omega\)-open in \((X,T)\) if its complement, \(A^C\) is \(\omega\)-closed.

**Definition 13**
Let \((X,T)\) be any fuzzy topological space. \((X,T)\) is called fuzzy basically disconnected if the closure of every fuzzy open \(F\) set is fuzzy open.

**Definition 10** An L-fuzzy set \(\lambda\) of an L-fuzzy topological space \((X,T)\) is called L-fuzzy \(\omega\)-closed in \((X,T)\) if \(\text{L-cl}(\lambda) \leq \mu\) whenever \(\lambda \leq \mu\) and \(\mu\) is L-fuzzy semi-open in \((X,T)\). The complement of L-fuzzy \(\omega\)-closed set is L-fuzzy \(\omega\)-open.

**Definition 10** Let \((X,T)\) be an L-fuzzy topological space. For any L-fuzzy set \(\lambda\) in \((X,T)\), L-fuzzy \(\omega\)-closure of \(\lambda\) (briefly, L-cl(\(\lambda\))) is defined as
\[
\text{L-\text{cl}}(\lambda) = \bigwedge \{ \mu : \mu \geq \lambda \text{ and } \mu \text{ is L-fuzzy } \omega\text{-closed} \}.
\]

### 3. Characterizations and properties of L-fuzzy \(\omega\)-basically disconnected spaces.

In this section a new class of set called L-fuzzy \(\omega\)-closed set and thereby a new class of space called L-fuzzy \(\omega\)-basically disconnected space is introduced. Some interesting properties and characterizations are also discussed.

**Definition 3.1** Let \((X,T)\) be any L-fuzzy topological space and \(\lambda\) be any L-fuzzy set in \((X,T)\). \(\lambda\) is called
\[
\text{(a)} \quad \text{an L-fuzzy } G_\omega \text{ set if } \lambda = \bigwedge_{i=1}^\infty \lambda_i \text{ where each } \lambda_i \text{ is L-fuzzy open.}
\]
\[
\text{(b)} \quad \text{an L-fuzzy } F_\omega \text{ set if } \lambda = \bigvee_{i=1}^\infty \lambda_i \text{ where each } (1-\lambda_i) \text{ is L-fuzzy open.}
\]

**Definition 3.2** Let \(\lambda\) be any L-fuzzy set in the L-fuzzy topological space \((X,T)\). Then we define
\[
\text{L-int}(\lambda) = \{ \mu : \mu \leq \lambda \text{ and } \mu \text{ is L-fuzzy open} \} \quad \text{and} \quad \text{L-cl}(\lambda) = \{ \mu : \mu \geq \lambda \text{ and } \mu \text{ is L-fuzzy closed} \}.
\]

**Definition 3.3** Let \(\lambda\) be any L-fuzzy set in the L-fuzzy topological space \((X,T)\). \(\lambda\) is called L-fuzzy semi-open if \(\lambda \subseteq \text{L-cl}( \text{L-int}(\lambda))\).

**Definition 3.4** An L-fuzzy set \(\lambda\) of an L-fuzzy topological space \((X,T)\) is called L-fuzzy \(\omega\)-closed in \((X,T)\) if \(L-\text{ox cl}(\lambda) \leq \mu\) whenever \(\lambda \leq \mu\) and \(\mu\) is L-fuzzy semi-open in \((X,T)\). The complement of L-fuzzy \(\omega\)-closed set is L-fuzzy \(\omega\)-open.

**Note 3.1**
\(**a\)** Let \((X,T)\) be an L-fuzzy topological space. An L-fuzzy set \(\lambda\) in \((X,T)\) which is both L-fuzzy \(\omega\)-open and L-fuzzy \(F_\omega\) is denoted by L-fuzzy \(\omega\)-open \(F_\omega\).

\(**b\)** Let \((X,T)\) be an L-fuzzy topological space. An L-fuzzy set \(\lambda\) in \((X,T)\) which is both L-fuzzy \(\omega\)-closed and L-fuzzy \(G_\omega\) is denoted by L-fuzzy \(\omega\)-closed \(G_\omega\).

**Notation 3.1** An L-fuzzy set \(\lambda\) which is both L-fuzzy \(\omega\)-open \(F_\omega\) and L-fuzzy \(\omega\)-closed \(G_\omega\) is denoted by L-fuzzy \(\omega\)-COGF.

**Definition 3.5** Let \((X,T)\) be an L-fuzzy topological space. For any L-fuzzy set \(\lambda\) in \((X,T)\), L-fuzzy \(\omega^*-\) closure of \(\lambda\) (briefly, \(L\omega^*-\text{cl}(\lambda)\)) is defined as
\[
L\omega^*-\text{cl}(\lambda) = \bigwedge \{ \mu : \mu \geq \lambda \text{ and } \mu \text{ is L-fuzzy } \omega^*\text{-closed} \}.
\]

**Definition 3.6** Let \((X,T)\) be an L-fuzzy topological space. For any L-fuzzy set \(\lambda\) in \((X,T)\), L-fuzzy \(\omega^*-\) interior of \(\lambda\) (briefly, \(L\omega^*-\text{int}(\lambda)\)) is defined as \(L\omega^*-\text{int}(\lambda) = \bigvee \{ \mu : \mu \leq \lambda \text{ and } \mu \text{ is L-fuzzy } \omega^*\text{-open} \} \).
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Remark 3.1 Let $(X, T)$ be an L-fuzzy topological space. For any L-fuzzy set $\lambda$ in $(X, T)$

a) $1 - L\tilde{\omega}^\ast$-int $(\lambda) = L\tilde{\omega}^\ast$-cl $(1 - \lambda)$.

b) $1 - L\tilde{\omega}^\ast$-cl $(\lambda) = L\tilde{\omega}^\ast$-int $(1 - \lambda)$.

Definition 3.7 Let $(X, T)$ and $(Y, S)$ be any two L-fuzzy topological spaces. A mapping $f : (X, T) \to (Y, S)$ is called L-fuzzy $\tilde{\omega}^\ast$-continuous if $f^{-1}(\lambda)$ is L-fuzzy $\tilde{\omega}$-closed $G_\delta$ in $(X, T)$ for every L-fuzzy closed and L-fuzzy $G_\delta$ set $\lambda$ in $(Y, S)$.

Definition 3.8 Let $(X, T)$ and $(Y, S)$ be any two L-fuzzy topological spaces. A mapping $f : (X, T) \to (Y, S)$ is called L-fuzzy $\tilde{\omega}^\ast$-irresolute if the inverse image of every L-fuzzy $\tilde{\omega}$-open $F_\sigma$ set in $(Y, S)$ is L-fuzzy $\tilde{\omega}$-open $F_\sigma$ in $(X, T)$.

Definition 3.9 Let $(X, T)$ and $(Y, S)$ be any two L-fuzzy topological spaces. A mapping $f : (X, T) \to (Y, S)$ is said to be L-fuzzy $\tilde{\omega}^\ast$-open if the image of every L-fuzzy $\tilde{\omega}$-open $F_\sigma$ set in $(X, T)$ is L-fuzzy $\tilde{\omega}$-open $F_\sigma$ in $(Y, S)$.

Proposition 3.11 Let $(X, T)$ and $(Y, S)$ be any two L-fuzzy topological spaces. Then $f : (X, T) \to (Y, S)$ is L-fuzzy $\tilde{\omega}^\ast$-irresolute iff $f (L\tilde{\omega}^\ast$-$cl (\lambda)) \subseteq L\tilde{\omega}^\ast$-$cl (f (\lambda))$, for every L-fuzzy set $\lambda$ in $(Y, S)$.

Proposition 3.2 Let $(X, T)$ and $(Y, S)$ be any two L-fuzzy topological spaces and let $f : (X, T) \to (Y, S)$ be an L-fuzzy $\tilde{\omega}^\ast$-open surjective function. Then $f^{-1}(L\tilde{\omega}^\ast$-$cl (\lambda)) \subseteq L\tilde{\omega}^\ast$-$cl (f^{-1}(\lambda))$, for each L-fuzzy set $\lambda$ in $(Y, S)$.

Definition 3.10 Let $(X, T)$ be any L-fuzzy topological space. $(X, T)$ is called L-fuzzy $\tilde{\omega}$-basically disconnected if the L-fuzzy $\tilde{\omega}$-closure of every L-fuzzy $\tilde{\omega}$-open $F_\sigma$ set is L-fuzzy $\tilde{\omega}$-open $F_\sigma$.

Proposition 3.3 For an L-fuzzy topological space $(X, T)$ the following statements are equivalent:

(a) $(X, T)$ is an L-fuzzy $\tilde{\omega}$-basically disconnected space.

(b) For each L-fuzzy $\tilde{\omega}$-closed $G_\delta$ set $\lambda$, $L\tilde{\omega}^\ast$-$int (\lambda)$ is L-fuzzy $\tilde{\omega}$-closed $G_\delta$.

(c) For each L-fuzzy $\tilde{\omega}$-open $F_\sigma$ set $\lambda$, $L\tilde{\omega}^\ast$-$cl (\lambda) + L\tilde{\omega}^\ast$-$cl (1 - L\tilde{\omega}^\ast$-$cl (\lambda)) = 1$.

(d) For every pair of L-fuzzy $\tilde{\omega}$-open $F_\sigma$ sets $\lambda$ and $\mu$ such that $L\tilde{\omega}^\ast$-$cl (\lambda) + \mu = 1$, we have $L\tilde{\omega}^\ast$-$cl (\lambda) + L\tilde{\omega}^\ast$-$cl (\mu) = 1$.

Proposition 3.4 Let $(X, T)$ be any L-fuzzy $\omega$-basically disconnected space and $(Y, S)$ be any L-fuzzy topological space. Let $f : (X, T) \to (Y, S)$ be L-fuzzy $\tilde{\omega}$-irresolute, L-fuzzy $\tilde{\omega}$-open and surjective function. Then $(Y, S)$ is L-fuzzy $\tilde{\omega}$-basically disconnected.

Definition 3.11 Let $\{ (X_\alpha, T_\alpha) / \alpha \in \Delta \}$ be a family of disjoint L-fuzzy topological spaces. Let $X = \bigcup_{\alpha \in \Delta} X_\alpha$.

Define $T = \{ \lambda \in L^X / \lambda / X_\alpha$ is L-fuzzy $\tilde{\omega}$-open $F_\sigma$ in $(X_\alpha, T_\alpha) \}$. Then $(X, T)$ is an L-fuzzy topological space called the L-fuzzy topological sum of $\{ (X_\alpha, T_\alpha) / \alpha \in \Delta \}$.

Proposition 3.5 Let $\{ (X_\alpha, T_\alpha) / \alpha \in \Delta \}$ be a family of disjoint L-fuzzy $\tilde{\omega}$-basically disconnected spaces and let $(X, T)$ be their L-fuzzy topological sum. Then $(X, T)$ is L-fuzzy $\tilde{\omega}$-basically disconnected.

Definition 3.12 Let $(X, T)$ be an L-fuzzy topological space. A mapping $f : X \to R(L)$ is called lower (resp. upper) L-fuzzy $\tilde{\omega}$-continuous if $f^{-1} (R_t)$ (resp. $f^{-1} (L_t)$) is L-fuzzy $\tilde{\omega}$-open $F_\sigma$ (resp. L-fuzzy $\tilde{\omega}$-closed $G_\delta$) for each $t \in R$.

Proposition 3.6 Let $(X, T)$ be an L-fuzzy topological space. Then $(X, T)$ is L-fuzzy $\tilde{\omega}$-basically disconnected iff for all L-fuzzy $\tilde{\omega}$-open $F_\sigma$ set $\lambda$ and an L-fuzzy $\tilde{\omega}$-closed $G_\delta$ set $\mu$ such that $\lambda \leq \mu$, $L\tilde{\omega}^\ast$-$cl (\lambda) \leq L\tilde{\omega}^\ast$-$int (\mu)$. 

Remark 3.2 Let $(X, T)$ be an L-fuzzy $\tilde{\omega}$-basically disconnected space. Let $\{ \lambda_i, 1 - \mu_i / i \in N \}$ be a collection such that $\lambda_i$'s are L-fuzzy $\tilde{\omega}$-open $F_\sigma$ and $\mu_i$'s are L-fuzzy $\tilde{\omega}$-closed $G_\delta$ and let
\( \lambda, \mu \) are L-fuzzy \( \tilde{o} \)-coGF. If \( \lambda_i \leq \lambda \leq \mu_i \) and \( \lambda_i \leq \mu \leq \mu_i \) for all \( i, j \in \mathbb{N} \), then there exists an L-fuzzy \( \tilde{o} \)-coGF set \( \gamma \) such that \( \tilde{L}\tilde{o}^*\-cl ( \lambda_i ) \leq \gamma \leq \tilde{L}\tilde{o}^*\-int ( \mu_i ) \), for all \( i, j \in \mathbb{N} \).

**Proposition 3.7** Let \(( X, T )\) be an L-fuzzy \( \omega \)-basically disconnected space. Let \( \{ \lambda_i \}_{i \in \mathbb{N}} \) and \( \{ \mu_i \}_{i \in \mathbb{N}} \) be monotone increasing collections of L-fuzzy \( \tilde{o} \)-open \( F_0 \) sets and L-fuzzy \( \tilde{o} \)-closed \( G_0 \) sets of \(( X, T )\) and suppose that \( \lambda_i \leq \mu_i \) whenever \( \lambda_i < \mu_i \) (Q is the set of all rational numbers). Then there exists a monotone increasing collection \( \{ \gamma \}_{i \in \mathbb{N}} \) of L-fuzzy \( \tilde{o} \)-coGF sets of \(( X, T )\) such that \( \tilde{L}\tilde{o}^*\-cl ( \lambda_i ) \leq ( \gamma_i ) \) and \( \gamma_i \leq \tilde{L}\tilde{o}^*\-int ( \mu_i ) \) whenever \( \lambda_i < \mu_i \).

**Proposition 3.8** Let \(( X, T )\) be any L-fuzzy topological space; let \( \lambda \in L^X \) and let \( f : X \to R(L) \) be such that

\[
 f(x)(t) = \begin{cases} 
 1 & \text{if } t < 0 \\
 \chi(0) & \text{if } 0 \leq t \leq 1 \\
 0 & \text{if } t > 0
\end{cases}
\]

for all \( x \in X \). Then \( f \) is lower (resp. upper) L-fuzzy \( \tilde{o}^* \)-continuous iff \( \lambda \) is L-fuzzy \( \tilde{o} \)-open \( F_0 \) (resp. L-fuzzy \( \tilde{o} \)-open \( F_0 \)) / L-fuzzy \( \tilde{o} \)-closed \( G_0 \).

**Definition 3.13** The characteristic function of \( \lambda \in L^X \) is the map \( \chi : X \to I(L) \) defined by \( \chi(\lambda)(x) = (\lambda(x)) \), \( x \in X \).

**Proposition 3.9** Let \(( X, T )\) be an L-fuzzy topological space and \( \lambda \in L^X \). Then \( \chi \) is lower (resp.upper) L-fuzzy \( \tilde{o}^* \)-continuous iff \( \lambda \) is L-fuzzy \( \tilde{o} \)-open \( F_0 \) (resp. L-fuzzy \( \tilde{o} \)-open \( F_0 \)) / L-fuzzy \( \tilde{o} \)-closed \( G_0 \).

**Definition 3.14** Let \(( X, T )\) and \(( Y, S )\) be any two L-fuzzy topological spaces. A mapping \( f : ( X, T ) \to ( Y, S ) \) is called strong \( F_0 \) L-fuzzy \( \tilde{o}^* \)-continuous if \( f^{-1}(\lambda) \) is L-fuzzy \( \tilde{o} \)-coGF set of \(( X, T )\), for every L-fuzzy \( \tilde{o} \)-open \( F_0 \) set \( \lambda \) of \(( Y, S )\).

**Proposition 3.10** Let \(( X, T )\) be an L-fuzzy topological space. Then the following statements are equivalent:

(a) \(( X, T )\) is an L-fuzzy \( \tilde{o} \)-basically disconnected space.

(b) If \( g, h : X \to R(L) \) where \( g \) is lower L-fuzzy \( \tilde{o}^* \)-continuous, \( h \) is upper L-fuzzy \( \tilde{o}^* \)-continuous, then there exists \( f \in C_{F_0} \tilde{L}\tilde{o} \) such that \( g \leq f \leq h \). [\( C_{F_0} \tilde{L}\tilde{o} \) = collection of all strong \( F_0 \) L-fuzzy \( \tilde{o}^* \)-continuous function on \( X \) with values in \( R(L) \)].

(c) If \( \lambda \) is L-fuzzy \( \tilde{o} \)-closed \( G_0 \) and \( \mu \) is L-fuzzy \( \tilde{o} \)-open \( F_0 \) sets such that \( \mu \leq \lambda \), then there exists a strong \( F_0 \) L-fuzzy \( \tilde{o}^* \)-continuous function \( f : X \to I(L) \) such that \( \mu \leq (1-L\lambda)f \leq R_0f \leq \lambda \).

**Proposition 3.11** Let \(( X, T )\) be an L-fuzzy \( \tilde{o} \)-basically disconnected space and let \( A \subset X \) be such that \( \chi_A \) is L-fuzzy \( \tilde{o}^* \)-open. Let \( f : ( A, T/A ) \to ( I(L) \) be strong \( F_0 \) L-fuzzy \( \tilde{o}^* \)-continuous. Then \( f \) has a strong \( F_0 \) L-fuzzy \( \tilde{o}^* \)-continuous extension over \(( X, T )\).

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