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Eulerian integral associated with product of two multivariable I-functions and a class of polynomials of several variables

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ABSTRACT
The present paper is evaluated a new Eulerian integral associated with the product of two multivariable I-functions with general arguments and general class of polynomials. Several particular cases are given .

Keywords: Eulerian integral, multivariable I-function, generalized Lauricella function of several variables, multivariable H-function, generalized hypergeometric function, class of polynomials.

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1. Introduction

In this paper, we evaluate a new Eulerian integral of most general characters associated with the products of two multivariable I-functions defined by Nambisan et al [2] and general class of polynomials of several variables, but of greater order. Several particular cases are given.

The I-function is defined and represented in the following manner.

$$I(z_1,\cdots,z_r) = I^{0,n:m_1,n_1;\cdots;m_r,n_r}_{p,q:p_1,q_1;\cdots;p_r,q_r}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_{1,p} : \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_{1,q} : \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,p_r} \\ \\ (\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,q_r} \end{array}\right) \tag{1.1}$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.2}$$

where $\phi(s_1,\cdots,s_r),\theta_i(s_i), i=1,\cdots,r$ are given by :

$$\phi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{n}\Gamma^{A_j}\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)}{\prod_{j=n+1}^{p}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)\prod_{j=m+1}^{q}\Gamma^{B_j}\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_j\right)} \tag{1.3}$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{n_i} \Gamma^{C_j^{(i)}} \left(1 - c_j^{(i)} + \gamma_j^{(i)} s_i\right) \prod_{j=1}^{m_i} \Gamma^{D_j^{(i)}} \left(d_j^{(i)} - \delta_j^{(i)} s_i\right)}{\prod_{j=n_i+1}^{p_i} \Gamma^{C_j^{(i)}} \left(c_j^{(i)} - \gamma_j^{(i)} s_i\right) \prod_{j=m_i+1}^{q_i} \Gamma^{D_j^{(i)}} \left(1 - d_j^{(i)} - \delta_j^{(i)} s_i\right)} \tag{1.4}$$

For more details, see Nambisan et al [2].

Following the result of Braaksma [1] the I-function of r variables is analytic if :

$$U_i = \sum_{j=1}^{p} A_j \alpha_j^{(i)} - \sum_{j=1}^{q} B_j \beta_j^{(i)} + \sum_{j=1}^{p_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=1}^{q_i} D_j^{(i)} \delta_j^{(i)} \leqslant 0, i = 1, \cdots, r \tag{1.5}$$

The integral (2.1) converges absolutely if

$|arg(z_k)| < \frac{1}{2} \Delta_k \pi, k = 1, \cdots, r$ where

$$\Delta_k = -\sum_{j=n+1}^{p} A_j \alpha_j^{(k)} - \sum_{j=1}^{q} B_j \beta_j^{(k)} + \sum_{j=1}^{m_k} D_j^{(k)} \delta_j^{(k)} - \sum_{j=m_k+1}^{q_k} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n_k} C_j^{(k)} \gamma_j^{(k)} - \sum_{j=n_k+1}^{p_k} C_j^{(k)} \gamma_j^{(k)} > 0 \tag{1.6}$$

Consider the second multivariable I-function.

$$I(z'_1, \cdots, z'_s) = I_{p',q':p'_1,q'_1;\cdots;p'_s,q'_s}^{0,n':m'_1,n'_1;\cdots;m'_s,n'_s} \left(\begin{array}{c|c} \mathrm{z'}_1 & (\mathrm{a'}_j; \alpha'^{(1)}_j, \cdots, \alpha'^{(s)}_j; A'_j)_{1,p'} : \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z'}_s & (\mathrm{b'}_j; \beta'^{(1)}_j, \cdots, \beta'^{(s)}_j; B'_j)_{1,q'} : \end{array} \right.$$

$$\left. \begin{array}{l} (\mathrm{c'}_j^{(1)}, \gamma'^{(1)}_j; C'^{(1)}_j)_{1,p'_1}; \cdots; (c'^{(s)}_j, \gamma'^{(s)}_j; C'^{(s)}_j)_{1,p'_s} \\ \\ (\mathrm{d'}_j^{(1)}, \delta'^{(1)}_j; D'^{(1)}_j)_{1,q'_1}; \cdots; (d'^{(s)}_j, \delta'^{(s)}_j; D'^{(s)}_j)_{1,q'_s} \end{array} \right) \tag{1.7}$$

$$= \frac{1}{(2\pi\omega)^s} \int_{L'_1} \cdots \int_{L'_s} \psi(t_1, \cdots, t_s) \prod_{i=1}^{s} \xi_i(t_i) z_i^{t_i} \mathrm{d}t_1 \cdots \mathrm{d}t_s \tag{1.8}$$

where $\psi(t_1, \cdots, t_s), \xi_i(s_i), i = 1, \cdots, s$ are given by :

$$\psi(t_1, \cdots, t_s) = \frac{\prod_{j=1}^{n'} \Gamma^{A'_j} \left(1 - a'_j + \sum_{i=1}^{s} \alpha'^{(i)}_j t_j\right)}{\prod_{j=n'+1}^{p} {}'\Gamma^{A'_j} \left(a'_j - \sum_{i=1}^{s} \alpha'^{(i)}_j t_j\right) \prod_{j=m+1}^{q'} \Gamma^{B'_j} \left(1 - b'_j + \sum_{i=1}^{s} \beta'^{(i)}_j t_j\right)} \tag{1.9}$$

$$\xi_i(s_i) = \frac{\prod_{j=1}^{n'_i} \Gamma^{C'^{(i)}_j} \left(1 - c'^{(i)}_j + \gamma'^{(i)}_j t_i\right) \prod_{j=1}^{m'_i} \Gamma^{D'^{(i)}_j} \left(d'^{(i)}_j - \delta'^{(i)}_j t_i\right)}{\prod_{j=n'_i+1}^{p'_i} \Gamma^{C'^{(i)}_j} \left(c'^{(i)}_j - \gamma'^{(i)}_j t_i\right) \prod_{j=m'_i+1}^{q'_i} \Gamma^{D'^{(i)}_j} \left(1 - d'^{(i)}_j - \delta'^{(i)}_j t_i\right)} \tag{1.10}$$

For more details, see Nambisan et al [2].

Following the result of Braaksma [1] the I-function of r variables is analytic if :

$$U_i' = \sum_{j=1}^{p'} A_j'\alpha_j'^{(i)} - \sum_{j=1}^{q'} B_j'\beta_j'^{(i)} + \sum_{j=1}^{p_i'} C_j'^{(i)}\gamma_j'^{(i)} - \sum_{j=1}^{q_i'} D_j'^{(i)}\delta_j'^{(i)} \leqslant 0, i = 1, \cdots, s \tag{1.11}$$

The integral (2.1) converges absolutely if

where$|arg(z_k')| < \frac{1}{2}\Delta_k'\pi, k = 1, \cdots, s$

$$\Delta_k' = -\sum_{j=n'+1}^{p'} A_j'\alpha_j'^{(k)} - \sum_{j=1}^{q'} B_j'\beta_j'^{(k)} + \sum_{j=1}^{m_k'} D_j'^{(k)}\delta_j'^{(k)} - \sum_{j=m_k'+1}^{q_k'} D_j'^{(k)}\delta_j'^{(k)} + \sum_{j=1}^{n_k'} C_j'^{(k)}\gamma_j'^{(k)} - \sum_{j=n_k'+1}^{p_k'} C_j'^{(k)}\gamma_j'^{(k)} > 0 \tag{1.12}$$

Srivastava and Garg introduced and defined a general class of multivariable polynomials [4] as follows

$$S_L^{h_1,\cdots,h_u}[z_1,\cdots,z_u] = \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L} (-L)_{h_1R_1+\cdots+h_uR_u} B(E;R_1,\cdots,R_u)\frac{z_1^{R_1}\cdots z_u^{R_u}}{R_1!\cdots R_u!} \tag{1.13}$$

2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [5 ,page 39 eq .30]

$$\frac{\prod_{j=1}^{P}\Gamma(A_j)}{\prod_{j=1}^{Q}\Gamma(B_j)}\,{}_PF_Q\left[(A_P);(B_Q);-(x_1+\cdots+x_r)\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\frac{\prod_{j=1}^{P}\Gamma(A_j+s_1+\cdots+s_r)}{\prod_{j=1}^{Q}\Gamma(B_j+s_1+\cdots+s_r)}\Gamma(-s_1)\cdots\Gamma(-s_r)x_1^{s_1}\cdots x_r^{s_r}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.1}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j+s_1+\cdots+s_r)$ are separated from those of $\Gamma(-s_j), j = 1,\cdots,r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j = 1,\cdots,r$

The Lauricella function $F_D^{(k)}$ is defined as

$$F_D^{(k)}[a,b_1,\cdots,b_k;c;x_1,\cdots,x_k] = \frac{\Gamma(c)}{\Gamma(a)\prod_{j=1}^{k}\Gamma(b_j)}\frac{1}{(2\pi\omega)^k}\int_{L_1}\cdots\int_{L_k}\frac{\Gamma\left(a+\sum_{j=1}^{k}\zeta_j\right)\Gamma(b_1+\zeta_1),\cdots,\Gamma(b_k+\zeta_k)}{\Gamma\left(c+\sum_{j=1}^{k}\zeta_j\right)}$$

$$\prod_{j=1}^{k}\Gamma(-\zeta_j)(-x_j)^{\zeta_i}\,\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_k \tag{2.2}$$

where $max\left[|arg(-x_1)|,\cdots,|arg(-x_k)|\right] < \pi, c \neq 0,-1,-2,\cdots$.

In order to evaluate a number of integrals of multivariable I-function, we first establish the formula

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}$$

$$\times F_D^{(k)}\left[\alpha,-\sigma_1,\cdots,-\sigma_k;\alpha+\beta;-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right] \tag{2.3}$$

where $a,b\in\mathbb{R}(a<b),\alpha,\beta,f_i,g_i,\sigma_i\in\mathbb{C},(i=1,\cdots,k)$; $min(Re(\alpha),Re(\beta))>0$ and

$$\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1$$

$F_D^{(k)}$ is a Lauricella's function of k-variables, see Srivastava et al ([6], page60)

The formula (2.2) can be establish by expanding $\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$ by means of the formula :

$$(1-z)^{-\alpha}=\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}z^r(|z|<1) \tag{2.4}$$

integrating term by term with the help of the integral given by Saigo and Saxena [3, page 93, eq.(3.2)] and applying the definition of the Lauricella function $F_D^{(k)}$ [5 page 454].
, page 454].

3.Eulerian integral

In this section , we note :

$$X=m_1,n_1;\cdots;m_r,n_r;m'_1,n'_1;\cdots;m'_1,n'_s;1,0;\cdots;1,0;1,0;\cdots;1,0 \tag{3.1}$$

$$Y=p_1,q_1;\cdots;p_r,q_r;p'_1,q'_1;\cdots;p'_s,q'_s;0,1;\cdots;0,1;0,1;\cdots;0,1 \tag{3.2}$$

$$A=(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)},0,\cdots,0,0,\cdots,0,0,\cdots,0;A_j)_{1,p} \tag{3.3}$$

$$B=(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)},0\cdots,0,0\cdots,0,0\cdots,0;B_j)_{1,q} \tag{3.4}$$

$$A'=(a'_j;0,\cdots,0,\alpha_j'^{(1)},\cdots,\alpha_j'^{(s)},0,\cdots,0,0,\cdots,0;A'_j)_{1,p'} \tag{3.5}$$

$$B'=(b'_j;0,\cdots,0,\beta_j'^{(1)},\cdots,\beta_j'^{(s)},0,\cdots,0,0,\cdots,0;B'_j)_{1,q'} \tag{3.6}$$

$$\mathrm{C}=(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,p_r};(c_j'^{(1)},\gamma_j'^{(1)};C_j'^{(1)})_{1,p'_1};\cdots;(c_j'^{(r)},\gamma_j^{(s)};C_j'^{(s)})_{1,p'_s}$$

$$(1,0;1);\cdots;(1,0;1);(1,0;1);\cdots;(1,0;1) \tag{3.7}$$

$$D=(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,q_r};(\mathrm{d}_j'^{(1)},\delta_j'^{(1)};D_j'^{(1)})_{1,q'_1};\cdots;(d_j'^{(s)},\delta_j'^{(s)};D_j'^{(s)})_{1,q'_s};$$

$$(0,1;1);\cdots;(0,1;1);(0,1;1);\cdots;(0,1;1) \tag{3.8}$$

$$K_1=(1-\alpha-\sum_{i=1}^{u}R_i(\mu_i+\mu_i');\mu_1,\cdots,\mu_r,\mu_1',\cdots,\mu_s',1,\cdots,1,\upsilon_1,\cdots,\upsilon_l;1) \tag{3.9}$$

$$K_2=(1-\beta-\sum_{i=1}^{u}R_i(\rho_i+\rho_i');\rho_1,\cdots,\rho_r,\rho_1',\cdots,\rho_s',0,\cdots,0,\tau_1,\cdots,\tau_l;1) \tag{3.10}$$

$$K_P=[1-A_j;0,\cdots,0,0,\cdots,0,0,\cdots,0,1,\cdots,1;1]_{1,P} \tag{3.11}$$

$$K_j=[1+\sigma_j-\sum_{i=1}^{u}R_i(\lambda_i^{(j)}+\lambda_i'^{(j)});\lambda_j^{(1)},\cdots,\lambda_j^{(r)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(s)},0\cdots,\underset{\text{j}}{1},\cdots,0,\zeta_j',\cdots,\zeta_j^{(l)};1]_{1,k} \tag{3.12}$$

$$L_1=(1-\alpha-\beta-\sum_{i=1}^{u}R_i(\mu_i+\mu_i'+\rho_i+\rho_i');\mu_1+\rho_1,\cdots,\mu_r+\rho_r,\mu_1'+\rho_1',\cdots,\mu_s'+\rho_s',$$

$$1,\cdots,1,\upsilon_1+\tau_1,\cdots,\upsilon_l+\tau_l;1) \tag{3.13}$$

$$L_Q=[1-B_j;0,\cdots,0,0,\cdots,0,0,\cdots,0,1\cdots,1;1]_{1,Q} \tag{3.14}$$

$$L_j=[1+\sigma_j-\sum_{i=1}^{u}R_i(\lambda_i^{(j)}+\lambda_i'^{(j)});\lambda_j^{(1)},\cdots,\lambda_j^{(r)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(s)},0\cdots,0,\zeta_j',\cdots,\zeta_j^{(l)};1]_{1,k} \tag{3.15}$$

Let $\mathfrak{A}=A,A;\mathfrak{B}=B,B';A,B,A'$ and B' are defined by (3.3), (3.4), (3.5) and (3.6), respectively

We the following generalized Eulerian integral :

$$\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_u}\left(\begin{array}{c}\mathrm{x}_1(t-a)^{\mu_1+\mu_1'}(b-t)^{\rho_1+\rho_1'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{x}_u(t-a)^{\mu_u+\mu_u'}(b-t)^{\rho_u+\rho_u'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(u)}-\lambda_j'^{(u)}}\end{array}\right)$$

$$I\left(\begin{array}{c}\mathrm{z}_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}}\end{array}\right)$$

$$I\left(\begin{array}{c} z'_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda'^{(1)}_j} \\ . \\ . \\ . \\ z'_s(t-a)^{\mu'_s}(b-t)^{\rho'_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda'^{(s)}_j} \end{array}\right)$$

$$ {}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{l} z''_i(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta^{(i)}_j}\right]\mathrm{d}t=$$

$$(b-a)^{\alpha+\beta+\sum_1^u R_i(\mu_i+\mu'_i+\rho_i+\rho'_i)}\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{k}(af_j+g_j)^{\sigma_i-\sum_{i=1}^{u}R_i(\lambda^{(i)}_j+\lambda'^{(i)}_j)}$$

$$\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}(-L)_{h_1R_1+\cdots+h_uR_u}B(E;R_1,\cdots,R_u)\frac{x_1^{R_1}\cdots x_u^{R_u}}{R_1!\cdots R_u!}$$

$$I^{0,n+n'+l+k+2;X}_{p+p'+l+k+2,q+q'+l+k+1;Y}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \\ \cdots & \\ \cdots & \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \mathfrak{A}\ ,\mathrm{K}_1,K_2,K_P,K_j:C \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(1)}_j}} & . \\ \cdots & . \\ \cdots & . \\ \frac{z'_s(b-a)^{\mu'_s+\rho'_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(s)}_j}} & . \\ \frac{(b-a)f_1}{af_1+g_1} & . \\ \cdots & . \\ \cdots & . \\ \frac{(b-a)f_k}{af_k+g_k} & \\ \frac{z''_1(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(1)}}} & \mathfrak{B}\ ,\mathrm{L}_1,L_j,L_Q,:D \\ \cdots & \\ \cdots & \\ \frac{z''_l(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(l)}}} & \end{array}\right) \tag{3.14}$$

We obtain the I-function of $r+s+k+l$ variables.

Provided that

(A) $a,b\in\mathbb{R}(a<b);\mu_i,\rho_i,\mu'_j,\rho'_j\lambda_v^{(i)};\lambda_v'^{(i)}\in\mathbb{R}^+$, $f_i,g_j,\tau_v,\sigma_j\in\mathbb{C}$ $(i=1,\cdots,r;j=1,\cdots;s;v=1,\cdots,k)$

(B) $m_j,n_j,p_j,q_j(j=1,\cdots,r),n,p,q\in\mathbb{N}^*$ **;**$\delta_j^{(i)}\in\mathbb{R}_+(j=1,\cdots,q_i;i=1,\cdots,r)$

$\alpha_j^{(i)}\in\mathbb{R}_+(j=1,\cdots,p;i=1,\cdots,r),\beta_j^{(i)}\in\mathbb{R}_+(j=1,\cdots,q;i=1,\cdots,r),\gamma_j^{(i)}\in\mathbb{R}_+(j=1,\cdots,p_i;i=1,\cdots,r)$

$a_j(j=1,\cdots,p),b_j(j=1,\cdots,q),c_j^{(i)}(j=1,\cdots,p_i,i=1,\cdots,r),d_j^{(i)}(j=1,\cdots,q_i,i=1,\cdots,r)\in\mathbb{C}$

The exposants $A_j(j=1,\cdots,p),B_j(j=1,\cdots,q),C_j^{(i)}(j=1,\cdots,p_i;i=1,\cdots,r),D_j^{(i)}(j=1,\cdots,q_i;i=1,\cdots,r)$

of various gamma function involved in (1.3) and (1.4) may take non integer values.

$m'_j,n'_j,p'_j,q'_j(j=1,\cdots,s),n',p',q'\in\mathbb{N}^*$ **;**$\delta_j'^{(i)}\in\mathbb{R}_+(j=1,\cdots,q'_i;i=1,\cdots,s)$

$\alpha_j'^{(i)}\in\mathbb{R}_+(j=1,\cdots,p';i=1,\cdots,s),\beta_j'^{(i)}\in\mathbb{R}_+(j=1,\cdots,q';i=1,\cdots,r),\gamma_j'^{(i)}\in\mathbb{R}_+(j=1,\cdots,p'_i;i=1,\cdots,s)$

$a'_j(j=1,\cdots,p'),b'_j(j=1,\cdots,q'),c_j'^{(i)}(j=1,\cdots,p'_i,i=1,\cdots,s),d_j'^{(i)}(j=1,\cdots,q'_i,i=1,\cdots,s)\in\mathbb{C}$

The exposants

$A'_j(j=1,\cdots,p'),B'_j(j=1,\cdots,q'),C_j'^{(i)}(j=1,\cdots,p'_i;i=1,\cdots,s),D_j'^{(i)}(j=1,\cdots,q'_i;i=1,\cdots,s)$

of various gamma function involved in (1.9) and (1.10) may take non integer values.

(C) $$\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1$$

(D) $$Re\big[\alpha+\sum_{i=1}^{r}\mu_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}+\sum_{i=1}^{s}\mu'_i\min_{1\leqslant j\leqslant m'_i}\frac{d_j'^{(i)}}{\delta_j'^{(i)}}\big]>0$$

$$Re\big[\beta+\sum_{i=1}^{r}\rho_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}+\sum_{i=1}^{s}\rho'_i\min_{1\leqslant j\leqslant m'_i}\frac{d_j'^{(i)}}{\delta_j'^{(i)}}\big]>0$$

(E) $$U_i=\sum_{j=1}^{p}A_j\alpha_j^{(i)}-\sum_{j=1}^{q}B_j\beta_j^{(i)}+\sum_{j=1}^{p_i}C_j^{(i)}\gamma_j^{(i)}-\sum_{j=1}^{q_i}D_j^{(i)}\delta_j^{(i)}\leqslant 0,i=1,\cdots,r$$

$$U'_i=\sum_{j=1}^{p'}A'_j\alpha_j'^{(i)}-\sum_{j=1}^{q'}B'_j\beta_j'^{(i)}+\sum_{j=1}^{p'_i}C_j'^{(i)}\gamma_j'^{(i)}-\sum_{j=1}^{q'_i}D_j'^{(i)}\delta_j'^{(i)}\leqslant 0,i=1,\cdots,s$$

(F) $$\Delta_k=-\sum_{j=n+1}^{p}A_j\alpha_j^{(k)}-\sum_{j=1}^{q}B_j\beta_j^{(k)}+\sum_{j=1}^{m_k}D_j^{(k)}\delta_j^{(k)}-\sum_{j=m_k+1}^{q_k}D_j^{(k)}\delta_j^{(k)}+\sum_{j=1}^{n_k}C_j^{(k)}\gamma_j^{(k)}-\sum_{j=n_k+1}^{p_k}C_j^{(k)}\gamma_j^{(k)}$$

$$-\mu_i-\rho_i-\sum_{l=1}^{k}\lambda_j^{(i)}>0\quad(i=1,\cdots,r)$$

$$\Delta'_k=-\sum_{j=n'+1}^{p'}A'_j\alpha_j'^{(k)}-\sum_{j=1}^{q'}B'_j\beta_j'^{(k)}+\sum_{j=1}^{m'_k}D_j'^{(k)}\delta_j'^{(k)}-\sum_{j=m'_k+1}^{q'_k}D_j'^{(k)}\delta_j'^{(k)}+\sum_{j=1}^{n'_k}C_j'^{(k)}\gamma_j'^{(k)}-\sum_{j=n'_k+1}^{p'_k}C_j'^{(k)}\gamma_j'^{(k)}$$

$$-\mu_i' - \rho_i' - \sum_{l=1}^{k} \lambda_j'^{(i)} > 0 \quad (i = 1, \cdots, s)$$

(H) $\left| arg \left(z_i \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(i)}} \right) \right| < \frac{1}{2} \Delta_i \pi \quad (a \leqslant t \leqslant b; i = 1, \cdots, r)$

$$\left| arg \left(z_i' \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j'^{(i)}} \right) \right| < \frac{1}{2} \Delta_i' \pi \quad (a \leqslant t \leqslant b; i = 1, \cdots, s)$$

(G) $P \leqslant Q + 1$. The equality holds, when , in addition,

either $P > Q$ and $\left| z_i \left(\prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(i)}} \right) \right|^{\frac{1}{Q-P}}$ <1 $\qquad (a \leqslant t \leqslant b)$

or $P \leqslant Q$ and $\max_{1 \leqslant i \leqslant k} \left[\left| \left(z_i' \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(i)}} \right) \right| \right] < 1 \qquad (a \leqslant t \leqslant b)$

Proof

First expressing the class of polynomial $S_L^{h_1, \cdots, h_u}[.]$ in serie with the help of (1.13), expressing the I-function of r variables by the Mellin-Barnes contour integral with the help of the equation (1.2), the I-function of s variables by the Mellin-Barnes contour integral with the help of the equation (1.8), the generalized hypergeometric function ${}_PF_Q(.)$ in Mellin-Barnes contour integral with the help of (2.1). Now collect the power of $(f_j t + g_j)$ with $j = 1, \cdots, k$ and use the equations (2.1) and (2.2) and we obtain $k-$Mellin-Barnes contour integral. Changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) and interpreting $(r+s+k+l)-$Mellin-barnes contour integral in multivariable I-function defined by Nambisan et al [2], we obtain the desired result.

4. Particular cases

a) If $B(L; R_1, \cdots, R_u) = \frac{\prod_{j=1}^{A} (a_j)_{R_1\theta_j' + \cdots + R_u\theta_j^{(u)}} \prod_{j=1}^{B'} (b_j')_{R_1\phi_j'} \cdots \prod_{j=1}^{B^{(u)}} (b_j^{(u)})_{R_u\phi_j^{(u)}}}{\prod_{j=1}^{C} (c_j)_{m_1\psi_j' + \cdots + m_u\psi_j^{(u)}} \prod_{j=1}^{D'} (d_j')_{R_1\delta_j'} \cdots \prod_{j=1}^{D^{(u)}} (d_j^{(u)})_{R_u\delta_j^{(u)}}}$ (4.1)

then the general class of multivariable polynomial $S_L^{R_1, \cdots, R_u}[x_1, \cdots, x_u]$ reduces to generalized Lauricella function defined by Srivastava et al [6].

$$F_{C:D';\cdots;D^{(s)}}^{1+A:B';\cdots;B^{(s)}} \left(\begin{array}{c} \mathrm{x}_1 \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{x}_u \end{array} \middle| \begin{array}{c} [(\text{-L});\mathrm{R}_1, \cdots, R_u][(a); \theta', \cdots, \theta^{(u)}] : [(b'); \phi']; \cdots ; [(b^{(u)}), \phi^{(u)}] \\ [(\mathrm{c});\psi', \cdots, \psi^{(u)}] : [(d'); \delta']; \cdots ; [(d^{(u)}), \delta^{(u)}] \end{array} \right) \quad (4.2)$$

and we have the formula

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^k (f_jt+g_j)^{\sigma_j}$$

$$F^{1+A:B';\cdots;B^{(t)}}_{C:D';\cdots;D^{(t)}}\left(\begin{array}{c} \mathrm{x}_1(t-a)^{\mu_1+\mu'_1}(b-t)^{\rho_1+\rho'_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}} \\ . \\ . \\ . \\ \mathrm{x}_u(t-a)^{\mu_u+\mu'_u}(b-t)^{\rho_u+\rho'_u}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(u)}-\lambda_j'^{(u)}} \end{array}\right.$$

$$\left.\begin{array}{c} [(\text{-L});\mathrm{R}_1,\cdots,R_u][(a);\theta',\cdots,\theta^{(u)}]:[(b');\phi'];\cdots;[(b^{(u)}),\phi^{(u)}] \\ [(\mathrm{c});\psi',\cdots,\psi^{(u)}]:[(d');\delta'];\cdots;[(d^{(u)}),\delta^{(u)}] \end{array}\right)$$

$$I\left(\begin{array}{c} \mathrm{z}_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(1)}} \\ . \\ . \\ . \\ \mathrm{z}_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$$I\left(\begin{array}{c} \mathrm{z'}_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ . \\ . \\ . \\ \mathrm{z'}_s(t-a)^{\mu'_s}(b-t)^{\rho'_s}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{array}\right)$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^l z''_i(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^k(f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t=$$

$$(b-a)^{\alpha+\beta+\sum_1^u R_i(\mu_i+\mu'_i+\rho_i+\rho'_i)}\frac{\prod_{j=1}^Q\Gamma(B_j)}{\prod_{j=1}^P\Gamma(A_j)}\prod_{j=1}^k(af_j+g_j)^{\sigma_j+\sum_{i=1}^u R_i(\lambda_j^{(i)}+\lambda_j'^{(i)})}$$

$$\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}(-L)_{h_1R_1+\cdots+h_uR_u}B(E;R_1,\cdots,R_u)\frac{x_1^{R_1}\cdots x_u^{R_u}}{R_1!\cdots R_u!}$$

$$I_{p+p'+l+k+2,q+q'+l+k+1;Y}^{0,n+n'+l+k+2;X}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \\ \cdots & \\ \cdots & \mathfrak{A}\ ,\mathrm{K}_1,K_2,K_P,K_j:C \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(1)}_j}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z'_s(b-a)^{\mu'_s+\rho'_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(s)}_j}} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{(b-a)f_k}{af_k+g_k} & \\ \frac{z''_1(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(1)}}} & \mathfrak{B},L_1,L_j,L_Q,:D \\ \cdots & \\ \cdots & \\ \frac{z''_l(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(l)}}} & \end{array}\right) \tag{4.3}$$

where $B(L;R_1,\cdots,R_u)=\dfrac{\prod_{j=1}^{A}(a_j)_{R_1\theta'_j+\cdots+R_u\theta_j^{(u)}}\prod_{j=1}^{B'}(b'_j)_{R_1\phi'_j}\cdots\prod_{j=1}^{B^{(u)}}(b_j^{(u)})_{R_u\phi_j^{(u)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi'_j+\cdots+m_u\psi_j^{(u)}}\prod_{j=1}^{D'}(d'_j)_{R_1\delta'_j}\cdots\prod_{j=1}^{D^{(u)}}(d_j^{(u)})_{R_u\delta_j^{(u)}}}$ (4.4)

b) If $A_j=B_j=C_j^{(i)}=D_j^{(i)}=A'_j=B'_j=C'^{(i)}_j=D'^{(i)}_j=1$, **the multivariable I-function defined by Nambisan et al [2] degenere in multivariable H-function defined by Srivastava et al [7]. We obtain**

$$\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_u}\left(\begin{array}{c} \mathrm{x}_1(t-a)^{\mu_1+\mu'_1}(b-t)^{\rho_1+\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}-\lambda'^{(1)}_j} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{x}_u(t-a)^{\mu_u+\mu'_u}(b-t)^{\rho_u+\rho'_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(u)}-\lambda'^{(u)}_j} \end{array}\right)$$

$$H\left(\begin{array}{c} \mathrm{z}_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$$H\left(\begin{array}{c} \text{z'}_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ . \\ . \\ . \\ \text{z'}_s(t-a)^{\mu'_s}(b-t)^{\rho'_s}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{array}\right)$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{l} z_i''(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t =$$

$$(b-a)^{\alpha+\beta+\sum_1^u R_i(\mu_i+\mu_i'+\rho_i+\rho_i')}\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j+\sum_{i=1}^u R_i(\lambda_j^{(i)}+\lambda_j'^{(i)})}$$

$$\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}(-L)_{h_1R_1+\cdots+h_uR_u}B(E;R_1,\cdots,R_u)\frac{x_1^{R_1}\cdots x_u^{R_u}}{R_1!\cdots R_u!}$$

$$H^{0,n+n'+l+k+2;X}_{p+p'+l+k+2,q+q'+l+k+1;Y}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(1)}}} & \\ \cdots & \\ \cdots & \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(r)}}} & \mathfrak{A}\ ,\mathrm{K}_1,K_2,K_P,K_j:C \\ \frac{z_1'(b-a)^{\mu_1'+\rho_1'}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(1)}}} & . \\ \cdots & . \\ \cdots & . \\ \frac{z_s'(b-a)^{\mu_s'+\rho_s'}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(s)}}} & . \\ \frac{(b-a)f_1}{af_1+g_1} & . \\ \cdots & . \\ \cdots & . \\ \frac{(b-a)f_k}{af_k+g_k} & \\ \frac{z_1''(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^k(af_j+g_j)^{\zeta_j^{(1)}}} & \mathfrak{B}\ ,\mathrm{L}_1,L_j,L_Q,:D \\ \cdots & \\ \cdots & \\ \frac{z_l''(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^k(af_j+g_j)^{\zeta_j^{(l)}}} & \end{array}\right) \quad \textbf{(4.5)}$$

under the same conditions (3.14) with $A_j = B_j = C_j^{(i)} = D_j^{(i)} = A'_j = B'_j = C'^{(i)}_j = D'^{(i)}_j = 1$

Remark

By the following similar procedure, the results of this document can be extented to product of any finite number of multivariable I-functions defined by Nambisan et al [2].

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of two multivariable I-functions defined by nambisan et al [2] with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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