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BEST PROXIMITY POINT THEOREMS IN METRIC SPACES

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Abstract. The purpose of this paper is to present a best proximity point theorems through rational expression with MT-function and introduced a new concept of generalized MT-cyclic rational contraction in what we call MT-cyclic rational contraction. Some best proximity point theorems for a mapping satisfy these conditions have been established in metric spaces. Our results generalize and improve some main results in Dass and Gupta [7] and references therein. We also give some examples to support our work.

1. Introduction:

A fundamental result in fixed point theory is the Banach Contraction Principle. One kind of a generalization of the Banach Contraction Principle is the notation of cyclical maps [15]. Fixed point theory is an important tool for solving equations $Tx = x$ for mappings T defined on subsets of metric spaces or normed spaces. Because a non-self mapping $T : A \to B$ does not necessarily have a fixed point, one often attempts to find an element x which is in some sense closest to Tx. Best proximity point theorems are relevant in this perspective. The notation of best proximity point is introduced in [6]. This definition is more general than the notation of cyclical maps, in sense that if the sets intersect, then every best proximity point is a fixed point. A sufficient condition for the uniqueness of the best proximity points in uniformly convex Banach spaces is given in [6]. Let A and B be non-empty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$. Then T is called cyclic map if $T(A) \subset B$ and $T(B) \subset A$. A point $x \in A \cup B$ is called a best proximity point if $d(x, Tx) = d(A, B)$, where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.

A mapping T of X into itself is called a contraction if there exists a positive real number $\alpha < 1$ with the property

$$d(Tx, Ty) \leq \alpha d(x, y) \tag{1.1}$$

for all $x, y \in X$. The well known Banach's [3] contraction mapping theorem may be stated as follows: Every contraction mapping of complete metric spaces X into itself has a unique fixed point. Many mathematicians worked on this principal. Banach's contraction principle has played an important role in the development of various results connected with fixed point and approximation theory.

The condition (1.1) entails $A \cap B$ being nonempty. Eldred and Veeramani [6] modified the condition (1.1) for the case $A \cap B = \phi$ as follows:

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha)d(A, B), \tag{1.2}$$

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for some $\alpha \in (0,1)$ and for all $x \in A$ and $y \in B$ where $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$.

The mapping T satisfying condition (1.2) is called a cyclic contraction. Eldred and Veeramani ([6], 3.10) obtained a unique best proximity point for the mapping T in a uniformly convex Banach space setting. Subsequently, a number of extensions and generalizations of their results appeared in [1, 4, 14, 13] and many others. In 1975, Dass and Gupta [7] proved the following fixed point theorem.

Theorem 1.1. *Let (X,d) be a complete metric space and $T : X \to X$ a mapping such that there exist, $\beta \geq 0$ with $\alpha + \beta < 1$ satisfying*

$$d(Tx,Ty) \leq \alpha\Big[\frac{d(y,Ty)[1+d(x,Tx)]}{[1+d(x,y)]}\Big] + \beta d(x,y) \tag{1.3}$$

for all $x, y \in X$. Then T has a unique fixed point in X.

The aim of this paper is to give a best proximity point theorem which is a variant of Theorem 1.3 in the setting of complete metric spaces.

2. Preliminaries :

In this section, we first define in what follows the MT-function which will be used throughout the paper to get new best proximity point theorems.

Definition 2.1. *(See [8]) A function $\psi : [0,\infty) \to [0,1)$ is said to be an MT-function if $\lim_{s\to t^+} \psi(s) < 1$ for all $t \in [0,1)$, (Mizoguchi-Takahashi.s condition).*

It is obvious that if $\psi : [0,\infty) \to [0,1)$ is a non-decreasing function or a non-increasing function, then ψ is an MT-function. So, the set of MT-functions is a rich class, but it is worth to mention that there exist functions which are not MT-function.

Example 2.2. *(See [10]) Let $\psi : [0,\infty) \to [0,1)$ be defined by*

$$\psi(t) = \begin{cases} \frac{sint}{t} & t \in (0, \frac{\pi}{2}] \\ 0 & otherwise \end{cases}$$

Since $\lim_{s\to t^+} \psi(s) = 1$, ψ is not an MT-function.

Very recently, Du [10] first proved some characterizations of MT-functions.

Theorem 2.3. *Let $\psi : [0,\infty) \to [0,1)$ be a function. Then the following statements are equivalent.*
(a) psi is an MT-function,
(b) For each $t \in [0,\infty)$, there exists $r_t^{(1)}$ and $\varepsilon_t^{(1)} > 0$ such that $\psi(s) < r_t^{(1)}$ for all $s \in (t, t+\varepsilon_t^{(1)})$
(c) For each $t \in [0,\infty)$, there exists $r_t^{(2)}$ and $\varepsilon_t^{(2)} > 0$ such that $\psi(s) < r_t^{(2)}$ for all $s \in (t, t+\varepsilon_t^{(2)})$
(d) For each $t \in [0,\infty)$, there exists $r_t^{(3)}$ and $\varepsilon_t^{(3)} > 0$ such that $\psi(s) < r_t^{(3)}$ for all $s \in (t, t+\varepsilon_t^{(3)})$
(e) For each $t \in [0,\infty)$, there exists $r_t^{(4)}$ and $\varepsilon_t^{(4)} > 0$ such that $\psi(s) < r_t^{(4)}$ for all $s \in (t, t+\varepsilon_t^{(4)})$
(f) For any non-increasing sequence $\{x_n\}_{n\in N}$ in $[0,\infty)$, we have $0 \leq \sup_{n\in N} \psi(x_n) < L$.

(g) ψ is a function of contractive factor [9]*; that is, for any strictly decreasing sequence $\{x_n\}_{n\in N}$ in $[0,\infty)$, we have $0 \le \sup_{n\in N} \psi(x_n) < 1$.*

Definition 2.4. [11] *Let A and B be nonempty subsets of a metric space (X,d). If a map $T : A\cup B \to A\cup B$ satisfies:*
(a) $T(A) \subset B$ and $T(B) \subset A$;
(b) there exists an MT-function $\psi : [0,\infty) \to [0,1)$ such that

$$d(Tx,Ty) \le \psi(d(x,y))d(x,y) + (1-\psi(d(x,y)))d(A,B), \tag{2.1}$$

for all $x \in A$ and $y \in B$. Then T is called an MT-cyclic contraction with respect to ψ on $A\cap B$.

Motivated by the definition of MT-cyclic contraction and MT-functions, we introduce the following concept of generalized MT-cyclic rational contractions.

Definition 2.5. *Let A and B be nonempty subsets of a metric space (X,d). If a map $T : A\cup B \to A\cup B$ satisfies:*
(a) $T(A) \subset B$ and $T(B) \subset A$;
(b) there exists an MT-function $\psi : [0,\infty) \to [0,1)$ such that

$$d(Tx,Ty) \le \psi(d(x,y))\Big[d(x,y) + \frac{d(y,Ty)[1+d(x,Tx)]}{[1+d(x,y)]}\Big] + (1-\psi(d(x,y)))d(A,B), \tag{2.2}$$

for all $x \in A$ and $y \in B$. Then T is called an generalized MT-cyclic rational contraction with respect to ψ on $A\cap B$.

3. Generalized MT-Cyclic Rational Contraction

In this section, first, we establish the following existence theorem for generalized MT-cyclic rational contraction, which is one of the main results in this paper.

Theorem 3.1. *Let A and B be two nonempty, closed subsets of a metric space (X,d) and the mappings $T : A\cup B \to T : A\cup B$ satisfy condition (2.2), then there exists a sequence $\{x_n\}$ such that*

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = d(A,B).$$

Proof. Let $x_0 \in A\cup B$ be given. Define a sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for all $n \in N$. Since T is satisfy generalized MT-cyclic rational contraction with MT-function $\psi : [0,\infty) \to [0,1)$, we obtain from (2.2), we obtain

$$\begin{aligned}
d(x_1,x_2) &= d(Tx_0,Tx_1)\\
&\le \psi(d(x_0,x_1))\Big[d(x_0,x_1) + \frac{d(x_1,Tx_1)[1+d(x_0,Tx_0)]}{[1+d(x_0,x_1)]}\Big]\\
&\quad + (1-\psi(d(x_0,x_1)))d(A,B)\\
&\le \psi(d(x_0,x_1))\Big[d(x_0,x_1) + \frac{d(x_1,x_2)[1+d(x_0,x_1)]}{[1+d(x_0,x_1)]}\Big]\\
&\quad + (1-\psi(d(x_0,x_1)))d(A,B)\\
&\le \psi(d(x_0,x_1))\Big[d(x_0,x_1) + d(x_1,x_2)\Big] + (1-\psi(d(x_0,x_1)))d(A,B),
\end{aligned}$$

which implies that,

$$d(x_1,x_2)(1-\psi(d(x_0,x_1))) \leq \psi(d(x_0,x_1))d(x_0,x_1) + (1-\psi(d(x_0,x_1)))d(A,B). \tag{3.1}$$

From (3.1), we obtain that

$$d(x_1,x_2) - d(A,B) \leq \frac{\psi(d(x_0,x_1))}{1-\psi(d(x_0,x_1))}[d(x_0,x_1) - d(A,B)], \tag{3.2}$$

Similarly, we obtain from (2.2),

$$\begin{aligned} d(x_2,x_3) &= d(Tx_1,Tx_2) \\ &\leq \psi(d(x_1,x_2))\Big[d(x_1,x_2) + \frac{d(x_2,Tx_2)[1+d(x_1,Tx_1)]}{[1+d(x_1,x_2)]}\Big] \\ &\quad + (1-\psi(d(x_1,x_2)))d(A,B) \\ &\leq \psi(d(x_1,x_2))\Big[d(x_1,x_2) + \frac{d(x_2,x_3)[1+d(x_1,x_2)]}{[1+d(x_1,x_2)]}\Big] \\ &\quad + (1-\psi(d(x_1,x_2)))d(A,B) \\ &\leq \psi(d(x_1,x_2))\Big[d(x_1,x_2) + d(x_2,x_3)\Big] + (1-\psi(d(x_1,x_2)))d(A,B), \end{aligned}$$

which implies that,

$$d(x_2,x_3)(1-\psi(d(x_1,x_2))) \leq \psi(d(x_1,x_2))d(x_1,x_2) + (1-\psi(d(x_1,x_2)))d(A,B). \tag{3.3}$$

From (3.3), we obtain that

$$d(x_2,x_3) - d(A,B) \leq \frac{\psi(d(x_1,x_2))}{1-\psi(d(x_1,x_2))}[d(x_1,x_2) - d(A,B)]. \tag{3.4}$$

By this way if we continue. Hence inductively, we concluded that from (3.2) and (3.4), we have

$$d(x_n,x_{n+1}) - d(A,B) \leq \frac{\psi(d(x_{n-1},x_n)))}{1-\psi(d(x_{n-1},x_n))}[d(x_{n-1},x_n) - d(A,B)]. \tag{3.5}$$

Since $\psi(t) < 1$ for all $t \in [0,\infty)$, where $\frac{\psi(t)}{1-\psi(t)} < 1$ for all $t \in [0,\infty)$. By (3.5), we obtain

$$d(x_n,x_{n+1}) - d(A,B) \leq d(x_{n-1},x_n) - d(A,B),$$

which implies that $d(x_n,x_{n+1}) < d(x_{n-1},x_n)$ for all $n \in N$. So the sequence $\{d(x_n,x_{n+1})\}$ is strictly non-increasing in $[0,\infty)$. Since ψ is an MT-function By Theorem 2.3, we get

$$0 \leq \sup_{n\in N} \psi(d(x_n,x_{n+1})) < 1.$$

Suppose $k = \sup_{n\in N} \psi(d(x_n,x_{n+1}))$. Theforefore $0 \leq k < 1$, since $\psi(d(x_n,x_{n+1})) \leq k$, we get

$$1-\psi(d(x_n,x_{n+1})) \geq 1-k.$$

Then

$$\frac{\psi(d(x_n,x_{n+1}))}{1-\psi(d(x_n,x_{n+1}))} \leq \frac{k}{1-k}$$

for all $n \in N$. Hence

$$0 \leq \sup_{n\in N} \frac{\psi(d(x_n,x_{n+1}))}{1-\psi(d(x_n,x_{n+1}))} \leq \frac{k}{1-k} < 1.$$

Assume that

$$\alpha = \sup_{n\in N} \frac{\psi(d(x_n, x_{n+1}))}{1-\psi(d(x_n, x_{n+1}))}.$$

Then $\alpha \in [0,1)$. From (3.5), we have

$$\begin{aligned} d(x_n, x_{n+1}) - d(A,B) &\leq \frac{\psi(d(x_{n-1}, x_n)))}{1-\psi(d(x_{n-1}, x_n))}[d(x_{n-1}, x_n) - d(A,B)] \\ &\leq \alpha[d(x_{n-1}, x_n) - d(A,B)] \\ &\leq \alpha^2[d(x_{n-1}, x_n) - d(A,B)] \\ &\cdots \\ &\leq \alpha^n[d(x_0, x_1) - d(A,B)]. \end{aligned} \tag{3.6}$$

Since $\alpha \in [0,1)$ and taking $n \to \infty$ in (3.6), we have $\lim_{n\to\infty} \alpha^n = 0$, So that, we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A,B).$$

This completes the proof. □

Theorem 3.2. *Let A and B be two nonempty, closed subsets of a metric space (X,d) and the mappings $T : A\cup B \to T : A\cup B$ satisfy condition (2.2), let $x_0 \in A$ and defined $x_n = Tx_{n-1}$. Suppose that the sequence $\{x_{2n}\}$ has a subsequence converging to some element x in A. Then the sequence $\{x_n\}$ is bounded.*

Proof. It follows from Theorem 3.1, that $\{d(x_{2n-1}, x_{2n})\}$ is convergent and hence it is bounded. Suppose the sequence $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ converges to some element x in A. Since $T : A\cup B \to A\cup B$ satisfy generalized MT-cyclic rational contraction, we get

$$\begin{aligned} d(A,B) &\leq d(x_{2n_k}, Tx_0) = d(Tx_{2n_k-1}, Tx_0) \\ &\leq \psi(d(x_{2n_k-1}, x_0))\Big[d(x_{2n_k-1}, x_0) + \frac{d(x_0, Tx_0)[1+d(x_{2n_k-1}, Tx_{2n_k-1})]}{[1+d(x_{2n_k-1}, x_0)]}\Big] \\ &\quad + (1-\psi(d(x_{2n_k-1}, x_0)))d(A,B) \\ &\leq \psi(d(x_{2n_k-1}, x_0))\Big[d(x_{2n_k-1}, x_0) + \frac{d(x_0, Tx_0)[1+d(x_{2n_k-1}, x_{2n_k})]}{[1+d(x_{2n_k-1}, x_{2n_k}1) + d(x_0, x_{2n_k})]}\Big] \\ &\quad + (1-\psi(d(x_{2n_k-1}, x_0)))d(A,B) \\ &\leq \psi(d(x_{2n_k-1}, x_0))\Big[d(x_{2n_k-1}, x_0) + \frac{d(x_0, Tx_0)[1+d(x_{2n_k-1}, x_{2n_k})]}{[1+d(x_{2n_k-1}, x_{2n_k}1)]}\Big] \\ &\quad + (1-\psi(d(x_{2n_k-1}, x_0)))d(A,B) \\ &\leq \psi(d(x_{2n_k-1}, x_0))\Big[d(x_{2n_k-1}, x_0) + d(x_0, Tx_0)\Big] \\ &\quad + (1-\psi(d(x_{2n_k-1}, x_0)))d(A,B). \end{aligned}$$

Since the sequence $\{x_{2n_k}\}$ converges to x in A and from Theorem 3.1, that $\{d(x_{2n-1}, x_{2n})\}$ converges to $d(A,B)$. So that last inequality implies that

$$\begin{aligned} d(A,B) &\leq d(x_{2n_k}, Tx_0) \\ &\leq \psi(d(x_{2n_k-1}, x_0))d(x_0, Tx_0) + d(A,B). \end{aligned}$$

Since $0 \leq \psi(d(x_{2n_k-1}, x_0)) < 1$. Therefore, the sequence $\{x_{2n}\}$ is bounded. Hence the sequence $\{x_n\}$ is also bounded. This completes the proof. □

Theorem 3.3. *Let A and B be two nonempty, closed subsets of a metric space (X, d) and the mappings $T : A \cup B \to T : A \cup B$ satisfy condition (2.2), let $x_0 \in A$ and defined $x_n = Tx_{n-1}$. Suppose that the sequence $\{x_{2n}\}$ has a subsequence converging to some element x in A. Then, x is a best proximity point of T.*

Proof. Suppose the sequence $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ converges to some element x in A. In light of the fact that the sequence $\{x_{2n_k}\}$ has a subsequence of $\{x_{2n}\}$ converging to some element x in A. So, it results from Theorem (3.1) that $d(x_{2n_k}, x_{2n_k-1}) \to d(A, B)$. Since T is a generalized MT-cyclic rational contraction, it follows that

$$\begin{aligned}
d(A,B) &\leq d(x_{2n_k}, Tx) \\
&= d(Tx_{2n_k-1}, Tx) \\
&\leq \psi(d(x_{2n_k-1}, x))\Big[d(x_{2n_k-1}, x) + \frac{d(x,Tx)[1 + d(x_{2n_k-1}, Tx_{2n_k-1})]}{[1 + d(x_{2n_k-1}, x)]}\Big] \\
&\quad + (1 - \psi(d(x_{2n_k-1}, x)))d(A,B) \\
&\leq \psi(d(x_{2n_k-1}, x))\Big[d(x_{2n_k-1}, x) + \frac{d(x,Tx)[1 + d(x_{2n_k-1}, x_{2n_k})]}{[1 + d(x_{2n_k-1}, x)]}\Big] \\
&\quad + (1 - \psi(d(x_{2n_k-1}, x)))d(A,B) \\
&\leq \psi(d(x_{2n_k-1}, x))\Big[d(x_{2n_k-1}, x_{2n_k}) + d(x, x_{2n_k}) \\
&\quad + \frac{d(x,Tx)[1 + d(x_{2n_k-1}, x_{2n_k})]}{[1 + d(x_{2n_k-1}, x_{2n_k}) + d(x, x_{2n_k})]}\Big] + (1 - \psi(d(x_{2n_k-1}, x)))d(A,B).
\end{aligned}$$

Taking $n \to \infty$ above inequality, then we obtain

$$\begin{aligned}
d(A,B) &\leq d(x, Tx) \\
&\leq \psi(d(x_{2n_k-1}, x))\Big[d(A,B) + 0) + \frac{d(x,Tx) \cdot [1 + d(A,B)]}{1 + d(A,B) + 0}\Big] \\
&\quad + (1 - \psi(d(x_{2n_k-1}, x)))d(A,B) \\
&\leq \psi(d(x_{2n_k-1}, x))[d(A,B) + d(x,Tx)] + (1 - \psi(d(x_{2n_k-1}, x)))d(A,B).
\end{aligned}$$

Suppose $\lambda = \sup_{n \in N} \psi(d(x_{2n_k-1}, x))$. Therefore $0 \leq \lambda < 1$, hence, we obtain

$$\begin{aligned}
d(A,B) &\leq d(x, Tx) \\
&\leq \lambda[d(A,B) + d(x,Tx)] + (1 - \lambda)d(A,B) \\
&\leq \frac{\lambda}{(1-\lambda)} d(A,B) + d(A,B),
\end{aligned}$$

since $0 \leq \lambda < 1$. So that the last inequality implies that

$$d(x, Tx) = d(A,B),$$

that is x is a best proximity point of T. This completes the proof. □

The example below shows that Theorem 3.1 is indeed an extension of results due to several authors.

Example 3.4. *Consider the complete metric space $X = R$ with the usual metrics. Suppose that $A = [1, \infty)$, $B = (-\infty, -1]$ and define $T : A \cup B \to A \cup B$ by $Tx = -2x$ for all $x \in A \cup B$. If $\psi : [1, \infty) \to [0, 1)$ is defined by $\psi(t) = \frac{t}{2(1+t)}$ for all $t \geq 0$. Then T is a generalized MT-rational cyclic contraction which is neither*

a cyclic $\phi-$contraction nor a generalized MT-cyclic contraction. Now, notice that $d(A,B) = 4$ and $T(A) \subset B, T(B) \subset A$. Putting $x = 2$ and $y = -2$, then we obtain $d(x,Tx) = d(A,B) = 4$ and $dy,Ty) = d(A,B) = 4$. Then it is easy to say that T is a generalized MT-cyclic rational contraction map. It is clear that $x = 2$ and $y = -2$ are best proximity point of T.

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