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THE DOMINATION SUBDIVISION NUMBER OF TRANSFORMATION GRAPHS

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Abstract

Let $G = (V, E)$ be a simple undirected graph. A subset D of $V(G)$ is said to be dominating set if every vertex of $V(G)-D$ is adjacent to at least one vertex in D. The minimum cardinality taken over all minimal dominating sets of G is the domination number of G and is denoted by $\gamma(G)$. The minimum number of edges of G whose subdivision increase the domination number is called the domination subdivision number of G and is denoted by $sd_\gamma(G)$. The transformation graph of G is a simple graph with vertex set $V(G) \cup E(G)$ in which adjacency is defined as follows: (a) two elements in V(G) are adjacent if and only if they are not adjacent in G (b) two elements in $E(G)$ are adjacent if and only if they are adjacent in G and (c) an element of $V(G)$ and an element of $E(G)$ are adjacent if and only if they are not incident in G . It is denoted by G^{-+-}. We investigate the domination subdivision number of G^{-+-} . We characterize the extremal graphs for connected and disconnected graphs separately.

Keywords : transformation graph, domination number, domination subdivision number.
AMS Subject Classification : 05C69.

1 Introduction

Let $G = (V, E)$ be a simple undirected graph of order n and size m. If $v \in V(G)$, then the *neighbourhood* of v is the set $N(v)$ conisting of all vertices u which are adjacent to v. The *closed neighbourhood* is $N[v] = N(v) \cup \{v\}$. The *degree* of v in G is $|N(v)|$ and is denoted by $deg(v)$. The *maximum degree* of G is max $\{deg(v) : v \in V(G)\}$ and is denoted by $\Delta(G)$. The *diameter* of G is maximum distance between any two vertices of G and is denoted by $diam(G)$. A vertex that is adjacent to a pendent vertex is called a *support.* A support is said to be *strong* if it is adjacent to more than one pendant vertex. An edge $e = uv$ is said to be *subdivided* if it is deleted and replaced by a $u-v$ path of length two with a new internal vertex w. For $E' \subseteq E(G)$, $G\|E'$ denotes the graph obtained by subdividing all the edges of E' and $G\|\{e\}$ is simply denoted by $G\|e$. Terms not defined are used in the sense of [7].

A set $D \subseteq V(G)$ is a *dominating set* if every vertex in $V-D$ is adjacent to at least one vertex in D. The minimum cardinality taken over all dominating sets of G is called the *domination number* of G and is denoted by $\gamma(G)$. A set $D' \subseteq E(G)$ is an *edge dominating set* of G if every edge in $E-D'$ is adjacent to at least one edge in D'. The minimum cardinality taken over all edge dominating sets of G is called the *edge domination number* of G and is denoted by $\gamma'(G)$. An edge dominating set D' is said to be an *independent edge dominating set* if no two edges are adjacent in $\langle D' \rangle$. The minimum cardinality taken over all independent edge dominating sets of G is called the *independent edge domination number* of G and is denoted by $\gamma_i{}'(G)$. For domination parameters, we follow [9].

Note 1.1. *For any connected graph G, $\gamma_i'(G) = \gamma'(G)$*

The domination subdivision number of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the domination number of G and is denoted by $sd_\gamma(G)$. That is, $sd_\gamma(G) = min\{|E'| : \gamma(G\|E') > \gamma(G)\}$.

The *transformation graph* of G, denoted G^{-+-}, is a simple graph with vertex set $V(G) \cup E(G)$ in which adjacency is defined as follows: (a) two elements in $V(G)$ are adjacent if and only if they are not adjacent in G (b) two elements in $E(G)$ are adjacent if and only if they are adjacent in G and (c) one element in $V(G)$ and one element in $E(G)$ are adjacent if and only if they are not incident in G.

In [10], Arumugam and Velammal introduced the concept of the domination subdivision number and they proved that for any tree, $sd_\gamma(G) \leq 3$. Also, they conjectured that for any connected graph with at least three vertices, $sd_\gamma(G) \leq 3$. In [1], Bhattacharya and Vijayakumar disproved this conjecture. Moreover, Aram, Favaron and Sheikholeslami [3] characterized trees with domination subdivision number three. Benecke and Mynhardt [5] characterized trees with domination subdivision number one. In [8], Haynes et al. conjectured that $sd_\gamma(G) \leq \delta(G)+1$ but in [6], O. Favaron et al. disproved it and proved that for any claw-free graph G with $\delta(G) \geq 3$ $sd_\gamma(G) \leq \delta(G)+1$. Several authors [4, 8] studied the subdivision numbers for various domination parameters.

Wu and Meng [11] proved that G^{-+-} is connected if and only if G is not a star and $diam(G) \leq 3$. In [12], Xu and Wu studied the connectivity and independence number of G^{-+-}. They also obtained a necessary and sufficient condition for G^{-+-} to be hamiltonian. In [2], the domination number of G^{-+-} was investigated. Here we study the domination subdivision number of G^{-+-}.

(i) The order of G^{-+-} is $n+m$.
(ii) For $x \in V(G)$, $deg_{G^{-+-}}(x) = n+m-1-2deg_G(x)$.
(iii) For $e = uv \in E(G)$, $deg_{G^{-+-}}(e) = n-4+deg_G(u)+deg_G(v)$.
(iv) $\delta(G^{-+-}) = min\{n+m-1-2\Delta(G),\ n-4+min_{uv\in E(G)}\{deg_G(u)+deg_G(v)\}\}$.

Example 1.2. *For the graph G given in Figure 1.1, $\gamma(G^{-+-}) = 3$ and $\gamma(G^{-+-}\|E') = 4$, where $E' = \{v_1v_6, v_2v_3, v_4v_5\}$. But subdivision of any two edges of G^{-+-} does not increase $\gamma(G^{-+-})$. Hence $sd_\gamma(G^{-+-}) = 3$.*

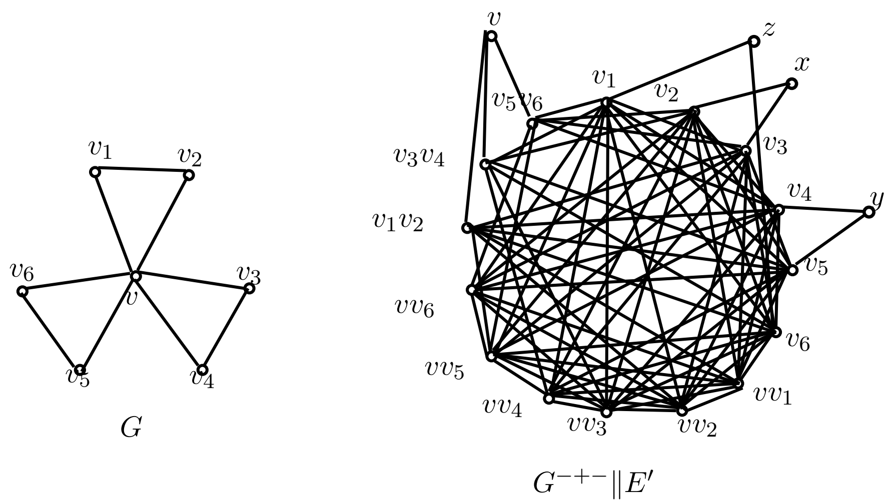


Figure 1.1. A graph G and $G^{-+-}\|E'$

The following theorems are used in the forthcoming sections.

Theorem 1.3. [1] For any graph G of order $n \geq 3$, $sd_\gamma(G) \leq \gamma(G)+1$.

Theorem 1.4. [2] For any graph G, $\gamma(G^{-+-}) \leq 3$.

Theorem 1.5. [2] For any graph G, $\gamma(G^{-+-}) = 1$ if and only if G has an isolated vertex.

Theorem 1.6. [2] Let G be an isolate-free graph of order 4 or 5. If G is not isomorphic to a star, then $\gamma(G^{-+-}) = 2$.

Theorem 1.7. [2] Let G be a connected graph with $diam(G) = 2$ and order $n \geq 4$. Then $\gamma(G^{-+-}) = 2$ if and only if $\gamma'(G) = 2$.

Theorem 1.8. [2] If $diam(G) \geq 3$, then $\gamma(G^{-+-}) = 2$.

The general description for transformation graph of some standard graphs are note worthy.

P_n^{-+-} has $2n-1$ vertices. $\Delta(P_n^{-+-}) = 2n-4$, $n \geq 4$ and

$\delta(P_n^{-+-}) = n-1,\ n \geq 5.$ P_n^{-+-} is hamiltonian.

C_n^{-+-} has $2n$ vertices and it is a bi-regular graph. $\Delta(C_n^{-+-}) = 2n-5$ and $\delta(C_n^{-+-}) = n,\ n \geq 5.$

$K_{1,r}^{-+-}$ has $2n-1$ vertices and it has one isolated vertex (maximum degree vertex of $K_{1,r}$ is isolated vertex in $K_{1,r}^{-+-}$). $\Delta(K_{1,r}^{-+-}) = 2(n-2).$ $K_{1,r}^{-+-}$ is isomorphic to K_{2r} minus a 1-factor, plus an isolated vertex. Further, it is not hamiltonian.

K_n^{-+-} has $\frac{n^2+n}{2}$ vertices. $deg_{K_n^{-+-}}(v) = \frac{(n-1)(n-2)}{2}$ for $v \in V(K_n)$ and $deg_{K_n^{-+-}}(e) = 3(n-2)$ for $e \in E(K_n)$. So K_n^{-+-} is bi-regular graph. Also it is hamiltonian.

Theorem 1.9. [2](i) $\gamma(P_n^{-+-}) = 2, \quad n > 3.$

(ii) $\gamma(C_n^{-+-}) = 2, \quad n > 3.$

(ii) $\gamma(K_{1,r}^{-+-}) = 3,$ for any positive integer r.

(iii) For $n > 5, \quad \gamma(K_n^{-+-}) = 3.$

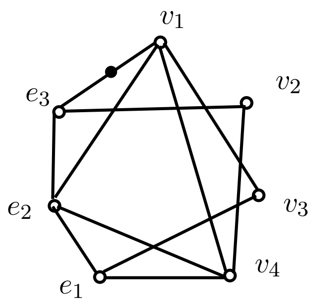
Theorem 1.10. [2] If G is disconnected graph, then $\gamma(G^{-+-}) \leq 2.$

2 Exact Value for Standard Graphs

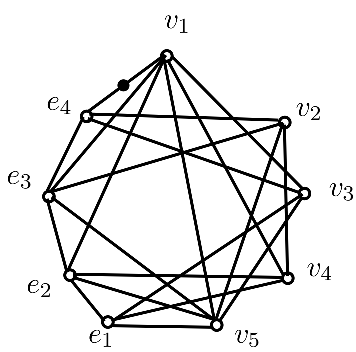
In this section, we obtain exact values of $sd_\gamma(G^{-+-})$ for some standard graphs G.

Theorem 2.1. $sd_\gamma(P_n^{-+-}) = 1$ *for* $n \geq 4.$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$, where $e_i = v_iv_{i+1}$, be the edges of P_n. By Theorem 1.9 (i), $\gamma(P_n^{-+-}) = 2$. If $n = 4$ or 5, then $\gamma(P_n^{-+-}\|\{v_1e_{n-1}\}) = 3$ as in Figure 2.1.



$P_4^{-+-}\|\{v_1e_3\}$



$P_5^{-+-}\|\{v_1e_4\}$

Figure 2.1. Examples of $P_n^{-+-}\|e'$

Now assume that $n \geq 6$.

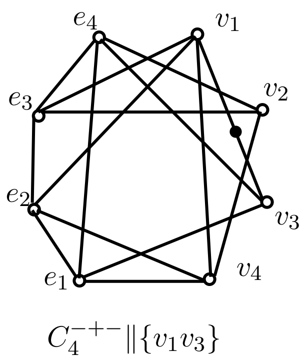
Claim: $\gamma(P_n^{-+-}\|e') \geq 3$ where e' is the edge joining e_1 and e_2 in P_n^{-+-}.

Let v be the subdivision vertex for the edge e' and D' be a dominating set of $P_n^{-+-}\|e'$. Then at least one element of $\{e_1, e_2, v\}$ is in D'. Since at least one element of $N(v_i)$ in P_n is not dominated by $\{v, v_i\}$ and $\{v, e_i\}$ in $P_n^{-+-}\|e'$, $D' \neq \{v, v_i\}$ or $\{v, e_i\}$.

Suppose that $e_1 \in D'$. Let $S = \{e_1, v_i\}$. If e_1 is incident with v_i in P_n, then the other end vertex of e_1 is adjacent to neither e_1 nor v_i in $P_n^{-+-}\|e'$. If e_1 is not incident with v_i in P_n, then at least one incident edge of v_i is adjacent to neither v_i nor e_1 in $P_n^{-+-}\|e'$. Therefore S is not a dominating set of $P_n^{-+-}\|e'$ Let $S' = \{e_1, e_i\}$. If e_1 and e_i are adjacent in P_n, then the common incident vertex of e_1 and e_i in P_n is adjacent to neither e_1 nor e_i in P_n^{-+-}. If e_1 and e_i are non-adjacent in P_n , then since $n \geq 6$ there exists an edge which is not adjacent to either e_i or e_1 in P_n^{-+-}. Hence S' is not a dominating set of $P_n^{-+-}\|e'$. Similarly, $\{e_2, v_i\}$ and $\{e_2, e_i\}$ are not dominating sets of $P_n^{-+-}\|e'$. Hence $\gamma(P_n^{-+-}\|e') \geq 3 > \gamma(P_n^{-+-})$. Thus $sd_\gamma(P_n^{-+-}) = 1$. □

Theorem 2.2. $sd_\gamma(C_n^{-+-}) = 1$ *for* $n \geq 4$.

Proof. Let $v_1, v_2, ..., v_n$ be vertices and $e_1, e_2, ..., e_n$, where $e_i = v_i v_{i+1}$ for $1 \leq i \leq n-1$ and $e_n = v_n v_1$ be edges of C_n. By Theorem 1.9 (ii) $\gamma(C_n^{-+-}) = 2$ for $n \geq 4$. Let v be the subdivision vertex obtained by subdividing edge $e' \in E(C_n^{-+-})$ and D' be a dominating set of $C_n^{-+-}\|e'$. If $n = 4$ or 5 and e' is the edge $v_1 v_3$ of C_n^{-+-}, then $\gamma(C_n^{-+-}\|e') = 3$ as shown in Figure 2.2. Therefore $sd_\gamma(C_n^{-+-}) = 1$.



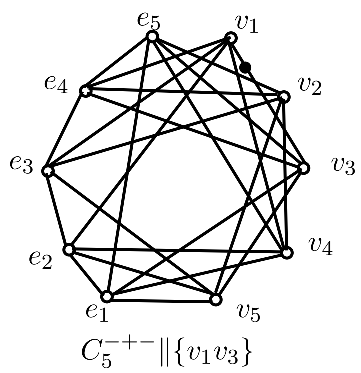


Figure 2.2. Examples of $C_n^{-+-}\|e'$

Now assume that $n \geq 6$.

Claim: $\gamma(C_n^{-+-}\|e') \geq 3$, where e' is the edge joining e_i and e_{i+1} in C_n^{-+-}. Then at least one element of $\{e_i, e_{i+1}, v\}$ is in D'.

But at least one vertex of $N(v_i)$ in C_n is not dominated by $\{v, v_i\}$ and $\{v, e_i\}$ in $C_n^{-+-}\|e'$. Therefore $D' \neq \{v, v_i\}$ or $\{v, e_i\}$. Without loss of generality, assume that $e_i \in D'$. Let $S = \{e_i, v_j\}$. If e_i is incident with v_j in C_n, then the other end of e_i is adjacent to neither e_i nor v_j in $C_n^{-+-}\|e'$. If e_i is not incident with v_j in C_n then at least one incident edge of v_j is not adjacent to either e_i or v_j in $C_n^{-+-}\|e'$. Therefore S is not a dominating set of $C_n^{-+-}\|e'$. Let $S' = \{e_i, e_j\}$. If e_i and e_j are adjacent in C_n , then the common incident vertex of e_i and e_j in C_n is not adjacent to either e_i or e_j in $C_n^{-+-}\|e'$. If e_i and e_j are not adjacent in C_n , then C_n has at least two vertices v_k and v_l

which are not incident with e_i and e_j in C_n. Then $v_kv_l \in E(G)$ is not adjacent to either e_i or e_j in $C_n^{-+-}\|e'$. Hence S' is not a dominating set of $C_n^{-+-}\|e'$. Hence $\gamma(C_n^{-+-}\|e') \geq 3$. Thus $sd_\gamma(C_n^{-+-}) = 1$. □

Theorem 2.3. $sd_\gamma(K_{1,r}^{-+-}) = 3$, *for all* $r \geq 2$.

Proof. We have $\gamma(K_{1,r}^{-+-}) = 3$, for all $r \geq 2$. Let v be the centre of $K_{1,r}$.

Claim 1: $\gamma(K_{1,r}^{-+-}\|e') = 3$ for all $e' \in E(K_{1,r}^{-+-})$.

Let e' be an edge joining u' and v' of $V(K_{1,r}^{-+-})$ in $K_{1,r}^{-+-}$. Since v is an isolated vertex in $K_{1,r}^{-+-}$, every dominating set contains v. All other vertices are dominated by $\{u', v'\}$ and hence $\{v, u', v'\}$ is a dominating set of $K_{1,r}^{-+-}\|e'$. So $\gamma(K_{1,r}^{-+-}\|e') = 3$.

Claim 2: $\gamma(K_{1,r}^{-+-}\|E') = 3$, where E' is a set of edges e' and e'' in $K_{1,r}^{-+-}$.

If e' and e'' are adjacent in $K_{1,r}^{-+-}$, let v' be the common vertex incident to e' and e'' and u' be the other end of e' in $K_{1,r}^{-+-}$. Therefore, $\{v, v', u'\}$ is a dominating set of $K_{1,r}^{-+-}\|E'$. If e' is not adjacent to e'' in $K_{1,r}^{-+-}$, let v' and v'' be incident vertices of e' and e'' respectively in $K_{1,r}^{-+-}$. Then $\{v, v', v''\}$ is a dominating set of $K_{1,r}^{-+-}\|E'$. Thus $\gamma(K_{1,r}^{-+-}\|E') = 3$.

Claim 3: $\gamma(K_{1,r}^{-+-}\|E'') \geq 4$, where $E'' = \{e', e'', e'''\}$ is a set of independent edges in $K_{1,r}^{-+-}$.

Then any minimum dominating set of $K_{1,r}^{-+-}\|E''$ contains three distinct vertices of $K_{1,r}^{-+-}$ to dominate the three subdivision vertices of e', e'' and e'''. Further, it contains the isolated vertex v. Hence $\gamma(K_{1,r}^{-+-}\|E'') \geq 4$. Thus $sd_\gamma(K_{1,r}^{-+-}) = 3$. □

Remark 2.4. $sd_\gamma(K_3^{-+-}) = 4$.
$sd_\gamma(K_4^{-+-}) = 1$.
$sd_\gamma(K_5^{-+-}) = 2$.
$sd_\gamma(K_6^{-+-}) = 3$.
$sd_\gamma(K_7^{-+-}) = 3$.

Theorem 2.5. $sd_\gamma(K_n^{-+-}) = 2$ *where* $n \geq 8$.

Proof. We have $\gamma(K_n^{-+-}) = 3$, $n > 5$. Let v' be the subdivision vertex of $e' \in E(K_n^{-+-})$ and D' be a dominating set of $K_n^{-+-}\|e'$.

Claim 1: For every edge $e' \in E(K_n^{-+-})$, $\gamma(K_n^{-+-}\|e') = 3$.

If e' is an edge of K_n^{-+-} joining two edges $e_1 = uv$ and e_2 of K_n, then $\{e_1, u, v\}$ is a dominating set of $K_n^{-+-}\|e'$. If e' is an edge of K_n^{-+-} joining $x \in V(K_n)$ and $e \in E(K_n)$, let $f = xy \in E(K_n)$. Then $\{x, y, f\}$ is a dominating set of $K_n^{-+-}\|e'$. Hence the claim holds.

Since $n \geq 8$, there exist four adjacent edges e_1, e_2, e_3, e_4 with common end vertex v. Let $g' = e_1e_2$; $g'' = e_3e_4$; $E' = \{g', g''\}$.

Claim 2: $\gamma(K_n^{-+-}\|E') \geq 4$.

Then every minimum dominating set of $K_n^{-+-}\|E'$ contains one of the sets $\{e_1, e_3\}$, $\{e_1, e_4\}$, $\{e_2, e_3\}$ or $\{e_2, e_4\}$. Since $n \geq 8$, there is an edge which is adjacent to none of the above edges. Therefore $|D'| \geq 4$. Hence the claim holds. $\square$

3 Extremal Graphs for Lower Bound

In this section, we get bounds for domination subdivision number and characterize the extremal graphs for lower bounds.

Theorem 3.1. *For a connected graph G with $n \geq 3$, $1 \leq sd_\gamma(G^{-+-}) \leq 3$.*

Proof. Obviously $sd_\gamma(G^{-+-}) \geq 1$. By Theorems 1.3 and 1.4, $sd_\gamma(G^{-+-}) \leq \gamma(G^{-+-}) + 1 \leq 4$.

Claim: There is no graph G with $sd_\gamma(G^{-+-}) = 4$.

By Theorems 1.3 and 1.8, we only need to consider graphs G such that $diam(G) = 2$ and $\gamma(G^{-+-}) = 3$. Let G be such a graph and let v be a vertex of degree Δ and $e_1 = vv_1, e_2 = vv_2, \ldots, e_\Delta = vv_\Delta$ be the edges incident with v in G.

Case (i): $\Delta(G) \geq 6$.

Let $E' = \{e_1e_2, e_3e_4, e_5e_6\} \subseteq E(G^{-+-})$. Since every edge which is incident with v in G is not adjacent to v in G^{-+-}, there is no minimum dominating set of $G^{-+-}\|E'$ of cardinality 3. Hence $sd_\gamma(G^{-+-}) \leq 3$.

Case (ii): $\Delta(G) = 5$.

Since v_1 is not incident with e_5 in G, they are adjacent in G^{-+-}. Let $E' = \{e_1e_2, e_3e_4, v_1e_5\}$ be a subset of $E(G^{-+-})$. Since v_1 and every edge which is incident with v in G are not adjacent to v in G^{-+-}, there is no dominating set of cardinality 3 in $G^{-+-}\|E'$ and hence $sd_\gamma(G^{-+-}) \leq 3$.

Case (iii): $\Delta(G) = 4$.

If $\langle N(v)\rangle \cong K_\Delta$ in G, then $G \cong K_5$ and hence by Remark 2.4, $sd_\gamma(G^{-+-}) = 2$; otherwise $N(v)$ in G has two non-adjacent vertices v_i and v_j in G. Hence $v_iv_j \in E(G^{-+-})$. Then $E' = \{e_1e_2, e_3e_4, v_iv_j\}$ be a subset $E(G^{-+-})$. Then there is no dominating set of cardinality 3 in $G^{-+-}\|E'$ and hence $sd_\gamma(G^{-+-}) \leq 3$.

Case (iv): $\Delta = 3$.

If $\langle N(v)\rangle \cong K_\Delta$ in G, then $G \cong K_4$ and hence by Remark 2.4, $sd_\gamma(G^{-+-})$ $= 1$; otherwise $N(v)$ in G has two non-adjacent vertices, say v_1 and v_2 in G. Then $v_1v_2 \in E(G^{-+-})$. Let $E' = \{v_1v_2, e_1v_3, e_2e_3\}$ be a subset of $E(G^{-+-})$. Then there is no dominating set of cardinality three in $G^{-+-}\|E'$ and hence $sd_\gamma(G^{-+-}) \leq 3$.

Case (v): $\Delta = 2$.

Since $diam(G) = 2$, $G \cong C_4$ or C_5. By Theorem 2.2, $sd_\gamma(G^{-+-}) = 1$. Hence in all the cases $sd_\gamma(G^{-+-}) \leq 3$.

$\square$

Theorem 3.2. *Let* G *be a connected graph with* $diam(G) = 2$. *If* $\gamma'(G) = 2$, *then* $sd_\gamma(G^{-+-}) = 1$.

Proof. Assume that $\gamma'(G) = 2$. Then by Theorem 1.7, $\gamma(G^{-+-}) = 2$. Since $diam(G) = 2$, there exist two non-adjacent vertices u and v in G. Now, consider $G^{-+-}\|uv$. Let x be the subdivision vertex of the edge joining u and v in G^{-+-}. Let D' be a minimum dominating set of $G^{-+-}\|uv$. Then D' contains u, v or x. If $x \in D'$, then x is adjacent to exactly two vertices of G^{-+-} but there is no single vertex of G^{-+-} which dominates all the remaining vertices of G^{-+-}. Now, assume that $u \in D'$. Let $w \in V(G)$. If w is adjacent to u in G, then the edge joining u and w is adjacent to neither u nor w in G^{-+-}. If w is not adjacent to u in G, then $d(u, w) = 2$ and so common adjacent vertex of u and w is adjacent to neither u nor w in G^{-+-}. Hence $D' \neq \{u, w\}$.

Let $e \in E(G)$. If e is incident with u in G, then the other end of e in G is adjacent to neither u nor e in G^{-+-}. Otherwise one of the following is true.

(i) at least one of the end vertices of e is adjacent to u in G

(ii) two end vertices of e and u are adjacent to a common vertex in G.

If u is adjacent to at least one of the end vertices of e in G, then that end vertex is adjacent to neither e nor u in G^{-+-}. If u and two end vertices of e are adjacent to a common vertex y in G, then the edge uy is adjacent to neither u nor e in G^{-+-}. Hence $D' \neq \{u, e\}$. Similarly, there is no D' containing v with cardinality 2. Thus, $\gamma(G^{-+-}\|uv) > 2$ and so $sd_\gamma(G^{-+-}) = 1$. □

Theorem 3.3. *If* G *is a connected graph with* $diam(G) \geq 3$, *then* $sd_\gamma(G^{-+-}) = 1$.

Proof. By Theorem 1.8, $\gamma(G^{-+-}) = 2$. If G is a path, then by Theorem 2.1, $sd_\gamma(G^{-+-}) = 1$. Now, assume that G is not a path.

Case(i): G has a strong support.

Let w be a strong support which is incident with at least two pendant edges f_1 and f_2 in G. Let $e' = f_1 f_2$ be the edge in G^{-+-}. Consider $G^{-+-}\|e'$. Let y be the subdivision vertex of e' and D' be a minimum dominating set of $G^{-+-}\|e'$. Then D' contains f_1, f_2 or y. If $y \in D'$, then y is adjacent to exactly two vertices of G^{-+-} but there is no single vertex of G^{-+-} which dominates all the remaining vertices of G^{-+-}. Now, assume that f_1 is in D'. Let $S_1 = \{f_1, x\}$, where x is any vertex of G. Then at least one edge of G which is incident with x in G is not adjacent to either f_1 or x in G^{-+-} and hence S_1 is not a dominating set of $G^{-+-}\|e'$. Let $S_2 = \{f_1, e\}$, where e is any edge of G. If f_1 and e are adjacent edges of G, then the common incident vertex of f_1 and e is not adjacent to either f_1 or e in G^{-+-}. If f_1 and e are not adjacent in G, then f_2 is adjacent to neither f_1 nor e in G^{-+-}. Hence S_2 is not a dominating set of $G^{-+-}\|e'$. Similarly, there is no dominating set containing f_2 with cardinality 2.

Case(ii): G has no strong support.

Subcase(i): $\gamma'(G) \neq 2$.

Since $diam(G) \geq 3$, $G \not\cong K_{1,r}$, $r \geq 1$ and hence $\gamma'(G) \neq 1$. Then $\gamma'(G) \geq 3$. Let e_1 and e_2 be two adjacent edges of G. Then $e_1e_2 \in E(G)$. Now, consider $G^{-+-}\|e_1e_2$. Let y be the subdivision vertex of e_1 and e_2 and D' be a minimum dominating set of $G^{-+-}\|e_1e_2$. Then D' contains e_1, e_2 or y. If $y \in D'$, then y is adjacent to exactly two vertices of G^{-+-} but there is no single vertex of G^{-+-} which dominates all the remaining vertices of G^{-+-}. Now, assume that e_1 is in D'. Let $S_3 = \{e_1, w\}$, $w \in V(G)$. If w is an end of e_1 in G, then S_3 is not a dominating set of $G^{-+-}\|e_1e_2$. Now, assume that w is not an end of e_1 in G. If w is adjacent to at least one end of e_1 in G, then that end is adjacent to neither e_1 nor w in G^{-+-}; otherwise all the edges incident with w in G are adjacent to neither e_1 nor w in G^{-+-}. Hence S_3 is not a dominating set of $G^{-+-}\|e_1e_2$. Since $\gamma'(G) \geq 3$, no two edges from a dominating set of $G^{-+-}\|e_1e_2$. Similarly, there is no dominating set containing e_2 with cardinality 2. Thus $\gamma(G^{-+-}\|e_1e_2) > 2$.

Subcase(ii): $\gamma'(G) = 2$.

Let $\{e, e'\}$ be an edge independent dominating set of G.

Suppose e and e' are adjacent to a common edge f in G. Let $e = xy$; $e' = x'y'$; y and y' are adjacent in G. Then every vertex $z \in V(G) - \{x, y, x', y'\}$ is adjacent to at least one of the vertices of $\{x, y, x', y'\}$ in G. Now, consider $G^{-+-}\|xy'$. Let w be the subdivision vertex of the edge joining x and y' and D' be a minimum dominating set of $G^{-+-}\|xy'$. Then D' contains x, y' or w. If $w \in D'$, then w is adjacent to exactly two vertices of G^{-+-} but there is no single vertex of G^{-+-} which dominates all the remaining vertices of G^{-+-}. Now, assume that x is in D'. Then for any $z \in V(G)$, $\{x, z\}$ does not dominate at least one of the vertices in $\{y, x', y'\}$ or an edge xz and hence it is not a dominating set of $G^{-+-}\|xy'$. Similarly, $\{x, f\}$, where $f \in E(G)$ is not a dominating set of $G^{-+-}\|xy'$. Therefore there is no D' containing x with cardinality 2. Similarly, there is no dominating set containing y' with cardinality 2. Thus no 2-element set is a dominating set of $G^{-+-}\|xy'$.

Suppose e and e' are not adjacent to a common edge in G. Let f' be adjacent to e in G. As in subcase(i) we can prove that no 2-element set is a dominating set of $G^{-+-}\|ef'$.

Thus in all cases, $\gamma(G^{-+-}\|e) > 2 = \gamma(G^{-+-})$ for some edge e of G^{-+-}. Hence $sd_\gamma(G^{-+-}) = 1$. □

Theorem 3.4. *For any connected graph G, $sd_\gamma(G^{-+-}) = 1$ if and only if any one of the following conditions is satisfied.*

(i) $diam(G) \geq 3$.

(ii) $diam(G) = 2$ and $\gamma'(G) = 2$.

(iii) $G \cong K_4$.

Proof. Assume that $sd_\gamma(G^{-+-}) = 1$. Suppose none of the three conditions is true. If $diam(G) = 1$, then $G \cong K_n$. By Theorem 2.5 and Remark 2.4, $sd_\gamma(G^{-+-}) \neq 1$ for $n \neq 4$. Now, assume that $diam(G) = 2$. If $\gamma'(G) = 1$, then $G \cong K_{1,r}$, $r \geq 2$ and hence by Theorem 2.3, $sd_\gamma(G^{-+-}) = 3$. If $\gamma'(G) \geq 3$, then by Theorems 1.5, 1.7 and 1.8, $\gamma(G^{-+-}) = 3$ and $\{u, v, e\}$, where $e = uv \in E(G)$, is a minimum dominating set of G^{-+-}. Let $f \in E(G)$ be adjacent to e in G. Then $ef \in E(G^{-+-})$. Without loss of generality, we assume that f is incident with u in G. Since G is a simple graph, v is adjacent to f in G^{-+-} and hence $\{u, v, e\}$ is a dominating set of $G^{-+-}\|ef$. Let $w \neq u,\ v$ in G. Then $ew \in E(G^{-+-})$. If w is adjacent to u and v in G, then $\{w, wv, v\}$ is a dominating set of $G^{-+-}\|ew$; otherwise $\{u, v, e\}$ is a dominating set of $G^{-+-}\|ew$. Now, let us consider the subdivision of an edge which is incident with u in G^{-+-}. Let u' be a non-neighbor of u in G. Then $\{u, v, e\}$ is a dominating set of $G^{-+-}\|uu'$. Let e' be an edge which is not incident with u in G. If e' is incident with v in G, then e is adjacent to e' in G^{-+-} and hence $\{u, v, e\}$ is a dominating of $G^{-+-}\|ue'$. If e' is not incident with v in G, then v is adjacent to e' in G^{-+-} and hence $\{u, v, e\}$ is a dominating set of $G^{-+-}\|ue'$. Hence $sd_\gamma(G^{-+-}) \geq 2$, which is a contradiction and so any one the three conditions holds.

Conversely, assume that any one of the three conditions is satisfied. Then by Theorems 3.2, 3.3 and Remark 2.4, $sd_\gamma(G^{-+-}) = 1$. □

If G is a disconnected graph with components $G_1, G_2, \ldots, G_k$, then every elements (vertex or edge) of G_i is adjacent to every vertices of G_j $(i \neq j)$ in G^{-+-}. In particular, if G has an isolated vertex u, then u is adjacent to all the vertices of G^{-+-}.

Theorem 3.5. *For any disconnected graph G, $sd_\gamma(G^{-+-}) \leq 2$ and equality holds if and only if $G \cong mK_2$, $m \geq 3$.*

Proof. If G has an isolated vertex, then by Theorem 1.5, $\gamma(G^{-+-}) = 1$. Therefore $sd_\gamma(G^{-+-}) = 1$.

Now assume that G has no isolated vertex. Then by Theorems 1.5 and 1.10 $\gamma(G^{-+-}) = 2$. Let G_1 and G_2 be two components of G. Let $e_1 = u_1v_1 \in E(G_1)$ and $e_2 = u_2v_2 \in E(G_2)$. Then $e_1u_2, u_1e_2 \in E(G^{-+-})$. Let $E' = \{e_1u_2, u_1e_2\}$

Claim: $\gamma(G^{-+-}\|E') \geq 3$.

Let w_1 and w_2 be subdivision vertices of e_1u_2 and u_1e_2 respectively and $\mathcal{S} = \{\{e_1', e_2'\} : e_1' \in \{e_1, u_2, w_1\} \text{ and } e_2' \in \{u_1, e_2, w_2\}\}$. Then every dominating set of $G^{-+-}\|\{e_1u_2, u_1e_2\}$ contains at least one set of $\mathcal{S}$. Now, at least one end vertex of e_1 is not dominated by $\{e_1, e_2'\}$ and $\{w_1, e_2'\}$ in $G^{-+-}\|E'$. Also either e_1 or e_2 is not dominated by $\{u_2, e_2'\}$ in $G^{-+-}\|E'$. Hence any dominating set of $G^{-+-}\|E'$

contains more than two vertices of $V(G^{-+-}) \cup \{w_1, w_2\}$ so $\gamma(G^{-+-} \| E') \geq 3$. Thus $sd_\gamma(G^{-+-}) \leq 2$.

Now, assume that $sd_\gamma(G^{-+-}) = 2$. By Theorem 1.4 and proof of Theorem 1.6, $\gamma(G^{-+-}) = 2$. Suppose $G \ncong mK_2,\ m \geq 3$.

Case 1: $G \cong 2K_2$.

Then $G_1 \cong G_2 \cong K_2$ and G^{-+-} is isomorphic to $K_{3,3}$ minus an edge. Let e' be an edge joining a vertex u of G_1 and the edge e of G_2. Then $\gamma(G \| e') = 3 > 2$ and hence $sd_\gamma(G^{-+-}) = 1$.

Case 2: G has at least one component $H \ncong K_2$.

Since H is connected with more than two vertices, there exist two adjacent edges, say e and f. Then $ef \in E(G^{-+-})$.

Claim: $\gamma(G^{-+-} \| ef) > 2$.

Let w be the subdivision vertex of ef and D' be a minimum dominating set of $G^{-+-} \| ef$. Then at least one of e, f and w is in D'. But w dominates e and f only. Since any vertex of G does not dominate its adjacent vertex and incident edges in G^{-+-}, $\{w, x\}$ where $x \in V(G)$ is not a dominating set of $G^{-+-} \| ef$. Since every edge e' of G is not adjacent to its incident vertices, $\{w, e'\}$ is not a dominating set of $G^{-+-} \| ef$. Also, $\{e, y\}$ where $y \in V(G)$ is not dominating set of $G^{-+-} \| ef$. For if y is incident with e in G, then the other end e is dominated by neither e nor y and if y is adjacent to z which is incident with e, then z is adjacent to neither e nor y; otherwise the edges which are adjacent to y are dominated by neither y nor e in $G^{-+-} \| ef$. Similarly, $\{f, y\}$ where $y \in V(G)$ is not a dominating set of $G^{-+-} \| \{ef\}$. Since any 2-element subset of $E(G)$ does not dominate at least one edge(vertex of G^{-+-}) of G, no 2-element subset of $V(G^{-+-})$ is a dominating set of G^{-+-}. Hence $\gamma(G^{-+-} \| ef) > 2$.

Thus $sd_\gamma(G^{-+-}) = 1$, which is a contradiction.

Conversely, assume that $G \cong mK_2$, $m \geq 3$. Then $\gamma(G^{-+-}) = 2$. Then no two edges of G are adjacent and no two vertices of the same component G_i are adjacent in G^{-+-}. Let $e' = u'v' \in E(G^{-+-})$. Then $u' \in V(G_i)$ and $v' \in E(G_j)$ or $u' \in V(G_i)$ and $v' \in V(G_j)$ for some i and j, $i \neq j$ Now, $\{u', w'\}$, where $w' \in E(G_k)$ $(k \neq i, j)$ is a dominating set of $G^{-+-} \| e'$. Hence $sd_\gamma(G^{-+-}) \geq 2$. But for a disconnected graph, $sd_\gamma(G^{-+-}) \leq 2$. Thus $sd_\gamma(G^{-+-}) = 2$. $\square$

Open Problem

- Characterize graphs for which $sd_\gamma(G^{-+-}) = 2$.

(OR)

- Characterize graphs for which $sd_\gamma(G^{-+-}) = 3$.

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