

Eulerian integral associated with product of two multivariable A-functions and a class of polynomials of several variables

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ABSTRACT

The present paper is evaluated a new Eulerian integral associated with the product of two multivariable A-functions defined by Gautam et al [1] with general arguments and general class of polynomials. Several particular cases are given .

Keywords: Eulerian integral, multivariable A-function, generalized Lauricella function of several variables, multivariable H-function, generalized hypergeometric function, class of polynomials.

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1. Introduction

In this paper, we evaluate a new Eulerian integral of most general characters associated with the products of two multivariable A-functions defined by Gautam et al [1] and general class of polynomials of several variables, but of greater order. Several particular cases are given.

The A-function is defined and represented in the following manner.

$$A(z_1, \dots, z_r) = A_{p,q;p_1,q_1;\dots;p_r,q_r}^{m,n;m_1,n_1;\dots;m_r,n_r} \left(\begin{array}{c|c} z_1 & (a_j; A_j^{(1)}, \dots, A_j^{(r)})_{1,p} : \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r & (b_j; B_j^{(1)}, \dots, B_j^{(r)})_{1,q} : \end{array} \right.$$

$$\left. \begin{array}{l} (c_j^{(1)}, C_j^{(1)})_{1,p_1}; \dots; (c_j^{(r)}, C_j^{(r)})_{1,p_r} \\ (d_j^{(1)}, D_j^{(1)})_{1,q_1}; \dots; (d_j^{(r)}, D_j^{(r)})_{1,q_r} \end{array} \right) \tag{1.1}$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \phi(s_1, \dots, s_r) \prod_{i=1}^r \theta_i(s_i) z_i^{s_i} ds_1 \dots ds_r \tag{1.2}$$

where $\phi(s_1, \dots, s_r), \theta_i(s_i), i = 1, \dots, r$ are given by :

$$\phi(s_1, \dots, s_r) = \frac{\prod_{j=1}^m \Gamma(b_j - \sum_{i=1}^r B_j^{(i)} s_i) \prod_{j=1}^n \Gamma(1 - a_j + \sum_{i=1}^r A_j^{(i)} s_j)}{\prod_{j=n+1}^p \Gamma(a_j - \sum_{i=1}^r A_j^{(i)} s_j) \prod_{j=m+1}^q \Gamma(1 - b_j + \sum_{i=1}^r B_j^{(i)} s_j)} \tag{1.3}$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{n_i} \Gamma(1 - c_j^{(i)} + C_j^{(i)} s_i) \prod_{j=1}^{m_i} \Gamma(d_j^{(i)} - D_j^{(i)} s_i)}{\prod_{j=n_i+1}^{p_i} \Gamma(c_j^{(i)} - C_j^{(i)} s_i) \prod_{j=m_i+1}^{q_i} \Gamma(1 - d_j^{(i)} - D_j^{(i)} s_i)} \tag{1.4}$$

Here $m, n, p, m_i, n_i, p_i, c_i \in \mathbb{N}^*$; $i = 1, \dots, r$; $a_j, b_j, c_j^{(i)}, d_j^{(i)}, A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{C}$

The multiple integral defining the A-function of r variables converges absolutely if :

$$|\arg(\Omega_i)z_k| < \frac{1}{2}\eta_k\pi, \xi^* = 0, \eta_i > 0 \quad (1.5)$$

$$\Omega_i = \prod_{j=1}^p \{A_j^{(i)}\}^{A_j^{(i)}} \prod_{j=1}^q \{B_j^{(i)}\}^{-B_j^{(i)}} \prod_{j=1}^{q_i} \{D_j^{(i)}\}^{D_j^{(i)}} \prod_{j=1}^{p_i} \{C_j^{(i)}\}^{-C_j^{(i)}}; i = 1, \dots, r \quad (1.6)$$

$$\xi_i^* = Im\left(\sum_{j=1}^p A_j^{(i)} - \sum_{j=1}^q B_j^{(i)} + \sum_{j=1}^{q_i} D_j^{(i)} - \sum_{j=1}^{p_i} C_j^{(i)}\right); i = 1, \dots, r \quad (1.7)$$

$$\eta_i = Re\left(\sum_{j=1}^n A_j^{(i)} - \sum_{j=n+1}^p A_j^{(i)} + \sum_{j=1}^m B_j^{(i)} - \sum_{j=m+1}^q B_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)} + \sum_{j=1}^{n_i} C_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)}\right) \quad (1.8)$$

$i = 1, \dots, r$

Consider the second multivariable A-function.

$$A(z'_1, \dots, z'_s) = A_{p', q': p'_1, q'_1; \dots; p'_r, q'_r}^{m', n': m'_1, n'_1; \dots; m'_r, n'_r} \left(\begin{array}{c|l} z_1 & (a'_j; A'_j(1), \dots, A'_j(s))_{1, p'} : \\ \cdot & \\ \cdot & \\ \cdot & (b'_j; B'_j(1), \dots, B'_j(s))_{1, q'} : \\ z_r & \end{array} \right)$$

$$\left(\begin{array}{l} (c'_j(1), C'_j(1))_{1, p'_1}; \dots; (c'_j(s), C'_j(s))_{1, p'_s} \\ (d'_j(1), D'_j(1))_{1, q'_1}; \dots; (d'_j(s), D'_j(s))_{1, q'_s} \end{array} \right) \quad (1.9)$$

$$= \frac{1}{(2\pi\omega)^s} \int_{L'_1} \dots \int_{L'_s} \phi'(t_1, \dots, t_s) \prod_{i=1}^s \theta'_i(t_i) z_i^{s_i} ds_1 \dots ds_r \quad (1.10)$$

where $\phi'(t_1, \dots, t_s), \theta'_i(t_i), i = 1, \dots, s$ are given by :

$$\phi'(t_1, \dots, t_s) = \frac{\prod_{j=1}^{m'} \Gamma(b'_j - \sum_{i=1}^s B'_j(i)t_i) \prod_{j=1}^{n'} \Gamma(1 - a'_j + \sum_{i=1}^s A'_j(i)t_j)}{\prod_{j=n'+1}^{p'} \Gamma(a'_j - \sum_{i=1}^s A'_j(i)t_j) \prod_{j=m'+1}^{q'} \Gamma(1 - b'_j + \sum_{i=1}^s B'_j(i)t_j)} \quad (1.11)$$

$$\theta'_i(t_i) = \frac{\prod_{j=1}^{n'_i} \Gamma(1 - c'_j(i) + C'_j(i)t_i) \prod_{j=1}^{m'_i} \Gamma(d'_j(i) - D'_j(i)t_i)}{\prod_{j=n'_i+1}^{p'_i} \Gamma(c'_j(i) - C'_j(i)t_i) \prod_{j=m'_i+1}^{q'_i} \Gamma(1 - d'_j(i) - D'_j(i)t_i)} \quad (1.12)$$

Here $m', n', p', m'_i, n'_i, p'_i, c'_i \in \mathbb{N}^*$; $i = 1, \dots, r$; $a'_j, b'_j, c'_j(i), d'_j(i), A'_j(i), B'_j(i), C'_j(i), D'_j(i) \in \mathbb{C}$

The multiple integral defining the A-function of r variables converges absolutely if :

$$|\arg(\Omega'_i)z'_k| < \frac{1}{2}\eta'_k\pi, \xi_i^* = 0, \eta'_i > 0 \quad (1.13)$$

$$\Omega'_i = \prod_{j=1}^{p'_i} \{A'_j{}^{(i)}\} A'_j{}^{(i)} \prod_{j=1}^{q'_i} \{B'_j{}^{(i)}\}^{-B'_j{}^{(i)}} \prod_{j=1}^{q'_i} \{D'_j{}^{(i)}\} D'_j{}^{(i)} \prod_{j=1}^{p'_i} \{C'_j{}^{(i)}\}^{-C'_j{}^{(i)}}; i = 1, \dots, s \quad (1.14)$$

$$\xi_i^* = \text{Im} \left(\sum_{j=1}^{p'_i} A'_j{}^{(i)} - \sum_{j=1}^{q'_i} B'_j{}^{(i)} + \sum_{j=1}^{q'_i} D'_j{}^{(i)} - \sum_{j=1}^{p'_i} C'_j{}^{(i)} \right); i = 1, \dots, s \quad (1.15)$$

$$\eta'_i = \text{Re} \left(\sum_{j=1}^{n'_i} A'_j{}^{(i)} - \sum_{j=n'_i+1}^{p'_i} A'_j{}^{(i)} + \sum_{j=1}^{m'_i} B'_j{}^{(i)} - \sum_{j=m'_i+1}^{q'_i} B'_j{}^{(i)} + \sum_{j=1}^{m'_i} D'_j{}^{(i)} - \sum_{j=m'_i+1}^{q'_i} D'_j{}^{(i)} + \sum_{j=1}^{n'_i} C'_j{}^{(i)} - \sum_{j=n'_i+1}^{p'_i} C'_j{}^{(i)} \right) \quad (1.16)$$

$i = 1, \dots, s$

Srivastava and Garg introduced and defined a general class of multivariable polynomials [3] as follows

$$S_L^{h_1, \dots, h_u} [z_1, \dots, z_u] = \sum_{R_1, \dots, R_u=0}^{h_1 R_1 + \dots + h_u R_u \leq L} (-L)_{h_1 R_1 + \dots + h_u R_u} B(E; R_1, \dots, R_u) \frac{z_1^{R_1} \dots z_u^{R_u}}{R_1! \dots R_u!} \quad (1.17)$$

2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [4, page 39 eq.30]

$$\frac{\prod_{j=1}^P \Gamma(A_j)}{\prod_{j=1}^Q \Gamma(B_j)} {}_P F_Q [(A_P); (B_Q); -(x_1 + \dots + x_r)]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \frac{\prod_{j=1}^P \Gamma(A_j + s_1 + \dots + s_r)}{\prod_{j=1}^Q \Gamma(B_j + s_1 + \dots + s_r)} \Gamma(-s_1) \dots \Gamma(-s_r) x_1^{s_1} \dots x_r^{s_r} ds_1 \dots ds_r \quad (2.1)$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j + s_1 + \dots + s_r)$ are separated from those of $\Gamma(-s_j), j = 1, \dots, r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j = 1, \dots, r$

The Lauricella function $F_D^{(k)}$ is defined as

$$F_D^{(k)} [a, b_1, \dots, b_k; c; x_1, \dots, x_k] = \frac{\Gamma(c)}{\Gamma(a) \prod_{j=1}^k \Gamma(b_j)} \frac{1}{(2\pi\omega)^k} \int_{L_1} \dots \int_{L_k} \frac{\Gamma\left(a + \sum_{j=1}^k \zeta_j\right) \Gamma(b_1 + \zeta_1), \dots, \Gamma(b_k + \zeta_k)}{\Gamma\left(c + \sum_{j=1}^k \zeta_j\right)} \prod_{j=1}^k \Gamma(-\zeta_j) (-x_j)^{\zeta_j} d\zeta_1 \dots d\zeta_k \quad (2.2)$$

where $\max[|\arg(-x_1)|, \dots, |\arg(-x_k)|] < \pi, c \neq 0, -1, -2, \dots$

In order to evaluate a number of integrals of multivariable I-function, we first establish the formula

$$\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} \prod_{j=1}^k (f_j t + g_j)^{\sigma_j} dt = (b-a)^{\alpha+\beta-1} B(\alpha, \beta) \prod_{j=1}^k (af_j + g_j)^{\sigma_j} \\ \times F_D^{(k)} \left[\alpha, -\sigma_1, \dots, -\sigma_k; \alpha + \beta; -\frac{(b-a)f_1}{af_1 + g_1}, \dots, -\frac{(b-a)f_k}{af_k + g_k} \right] \quad (2.3)$$

where $a, b \in \mathbb{R} (a < b)$, $\alpha, \beta, f_i, g_i, \sigma_i \in \mathbb{C} (i = 1, \dots, k)$; $\min(\operatorname{Re}(\alpha), \operatorname{Re}(\beta)) > 0$ and

$$\max_{1 \leq j \leq k} \left\{ \left| \frac{(b-a)f_j}{af_j + g_j} \right| \right\} < 1$$

$F_D^{(k)}$ is a Lauricella's function of k -variables, see Srivastava et al ([5], page60)

The formula (2.2) can be establish by expanding $\prod_{j=1}^k (f_j t + g_j)^{\sigma_j}$ by means of the formula :

$$(1-z)^{-\alpha} = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} z^r (|z| < 1) \quad (2.4)$$

integrating term by term with the help of the integral given by Saigo and Saxena [2, page 93, eq.(3.2)] and applying the definition of the Lauricella function $F_D^{(k)}$ [4, page 454].
, page 454].

3.Eulerian integral

In this section , we note :

$$X = m_1, n_1; \dots; m_r, n_r; m'_1, n'_1; \dots; m'_1, n'_s; 1, 0; \dots; 1, 0; 1, 0; \dots; 1, 0 \quad (3.1)$$

$$Y = p_1, q_1; \dots; p_r, q_r; p'_1, q'_1; \dots; p'_s, q'_s; 0, 1; \dots; 0, 1; 0, 1; \dots; 0, 1 \quad (3.2)$$

$$A = (a_j; A_j^{(1)}, \dots, A_j^{(r)}, 0, \dots, 0, 0, \dots, 0, 0, \dots, 0)_{1,p} \quad (3.3)$$

$$B = (b_j; B_j^{(1)}, \dots, B_j^{(r)}, 0, \dots, 0, 0, \dots, 0, 0, \dots, 0)_{1,q} \quad (3.4)$$

$$A' = (a'_j; 0, \dots, 0, A_j^{(1)}, \dots, A_j^{(s)}, 0, \dots, 0, 0, \dots, 0)_{1,p'} \quad (3.5)$$

$$B' = (b'_j; 0, \dots, 0, B_j^{(1)}, \dots, B_j^{(s)}, 0, \dots, 0, 0, \dots, 0)_{1,q'} \quad (3.6)$$

$$C = (c_j^{(1)}, C_j^{(1)})_{1,p_1}; \dots; (c_j^{(r)}, C_j^{(r)})_{1,p_r}; (c_j^{(1)}, C_j^{(1)})_{1,p'_1}; \dots; (c_j^{(r)}, C_j^{(s)})_{1,p'_s}$$

$$(1, 0); \dots; (1, 0); (1, 0); \dots; (1, 0) \quad (3.7)$$

$$D = (d_j^{(1)}, D_j^{(1)})_{1,q_1}; \dots; (d_j^{(r)}, D_j^{(r)})_{1,q_r}; (d_j^{(1)}, D_j^{(1)})_{1,q'_1}; \dots; (d_j^{(s)}, D_j^{(s)})_{1,q'_s}; \\ (0, 1); \dots; (0, 1); (0, 1); \dots; (0, 1) \quad (3.8)$$

$$K_1 = (1 - \alpha - \sum_{i=1}^u R_i(\mu_i + \mu'_i); \mu_1, \dots, \mu_r, \mu'_1, \dots, \mu'_s, 1, \dots, 1, v_1, \dots, v_l) \quad (3.9)$$

$$K_2 = (1 - \beta - \sum_{i=1}^u R_i(\rho_i + \rho'_i); \rho_1, \dots, \rho_r, \rho'_1, \dots, \rho'_s, 0, \dots, 0, \tau_1, \dots, \tau_l) \quad (3.10)$$

$$K_P = [1 - A_j; 0, \dots, 0, 0, \dots, 0, 0, \dots, 0, 1, \dots, 1]_{1,P} \quad (3.11)$$

$$K_j = [1 + \sigma_j - \sum_{i=1}^u R_i(\lambda_i^{(j)} + \lambda'_i{}^{(j)}); \lambda_j^{(1)}, \dots, \lambda_j^{(r)}, \lambda_j'^{(1)}, \dots, \lambda_j'^{(s)}, 0, \dots, 1, \dots, 0, \zeta_j', \dots, \zeta_j^{(l)}]_{1,k} \quad (3.12)$$

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$$L_1 = (1 - \alpha - \beta - \sum_{i=1}^u R_i(\mu_i + \mu'_i + \rho_i + \rho'_i); \mu_1 + \rho_1, \dots, \mu_r + \rho_r, \mu'_1 + \rho'_1, \dots, \mu'_s + \rho'_s, 1, \dots, 1, v_1 + \tau_1, \dots, v_l + \tau_l) \quad (3.13)$$

$$L_Q = [1 - B_j; 0, \dots, 0, 0, \dots, 0, 0, \dots, 0, 1, \dots, 1]_{1,Q} \quad (3.14)$$

$$L_j = [1 + \sigma_j - \sum_{i=1}^u R_i(\lambda_i^{(j)} + \lambda'_i{}^{(j)}); \lambda_j^{(1)}, \dots, \lambda_j^{(r)}, \lambda_j'^{(1)}, \dots, \lambda_j'^{(s)}, 0, \dots, 0, \zeta_j', \dots, \zeta_j^{(l)}]_{1,k} \quad (3.15)$$

Let $\mathfrak{A} = A, A; \mathfrak{B} = B, B'; A, B, A'$ and B' are defined by (3.3), (3.4), (3.5) and (3.6), respectively

We the following generalized Eulerian integral :

$$\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} \prod_{j=1}^k (f_j t + g_j)^{\sigma_j}$$

$$S_L^{h_1, \dots, h_u} \left(\begin{array}{c} x_1(t-a)^{\mu_1+\mu'_1} (b-t)^{\rho_1+\rho'_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}} \\ \vdots \\ x_u(t-a)^{\mu_u+\mu'_u} (b-t)^{\rho_u+\rho'_u} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(u)}-\lambda_j'^{(u)}} \end{array} \right)$$

$$A \left(\begin{array}{c} z_1(t-a)^{\mu_1} (b-t)^{\rho_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \vdots \\ z_r(t-a)^{\mu_r} (b-t)^{\rho_r} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(r)}} \end{array} \right)$$

(A) $a, b \in \mathbb{R}(a < b); \mu_i, \rho_i, \mu'_j, \rho'_j \lambda_v^{(i)}; \lambda_v^{(i)} \in \mathbb{R}^+, f_i, g_j, \tau_v, \sigma_j \in \mathbb{C} (i = 1, \dots, r; j = 1, \dots, s; v = 1, \dots, k)$

(B) $m, n, p, m_i, n_i, p_i, c_i \in \mathbb{N}^*; i = 1, \dots, r; a_j, b_j, c_j^{(i)}, d_j^{(i)}, A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{C}$

$m', n', p', m'_i, n'_i, p'_i, c'_i \in \mathbb{N}^*; i = 1, \dots, r; a'_j, b'_j, c'_j^{(i)}, d'_j^{(i)}, A'_j^{(i)}, B'_j^{(i)}, C'_j^{(i)}, D'_j^{(i)} \in \mathbb{C}$

$$(C) \max_{1 \leq j \leq k} \left\{ \left| \frac{(b-a)f_i}{af_i + g_i} \right| \right\} < 1$$

$$(D) \operatorname{Re} \left[\alpha + \sum_{i=1}^r \mu_i \min_{1 \leq j \leq m_i} \frac{d_j^{(i)}}{D_j^{(i)}} + \sum_{i=1}^s \mu'_i \min_{1 \leq j \leq m'_i} \frac{d'_j{}^{(i)}}{D'_j{}^{(i)}} \right] > 0$$

$$\operatorname{Re} \left[\beta + \sum_{i=1}^r \rho_i \min_{1 \leq j \leq m_i} \frac{d_j^{(i)}}{D_j^{(i)}} + \sum_{i=1}^s \rho'_i \min_{1 \leq j \leq m'_i} \frac{d'_j{}^{(i)}}{D'_j{}^{(i)}} \right] > 0$$

$$(E) |\arg(\Omega_i)z_k| < \frac{1}{2}\eta'_k\pi, \xi^* = 0, \eta_i > 0$$

$$\Omega_i = \prod_{j=1}^p \{A_j^{(i)}\}^{A_j^{(i)}} \prod_{j=1}^q \{B_j^{(i)}\}^{-B_j^{(i)}} \prod_{j=1}^{q_i} \{D_j^{(i)}\}^{D_j^{(i)}} \prod_{j=1}^{p_i} \{C_j^{(i)}\}^{-C_j^{(i)}}; i = 1, \dots, r$$

$$\xi_i^* = \operatorname{Im} \left(\sum_{j=1}^p A_j^{(i)} - \sum_{j=1}^q B_j^{(i)} + \sum_{j=1}^{q_i} D_j^{(i)} - \sum_{j=1}^{p_i} C_j^{(i)} \right); i = 1, \dots, r$$

$$\eta_i = \operatorname{Re} \left(\sum_{j=1}^n A_j^{(i)} - \sum_{j=n+1}^p A_j^{(i)} + \sum_{j=1}^m B_j^{(i)} - \sum_{j=m+1}^q B_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)} + \sum_{j=1}^{n_i} C_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)} \right)$$

$$-\mu_i - \rho_i - \sum_{l=1}^k \lambda_l^{(i)} > 0; i = 1, \dots, r$$

$$|\arg(\Omega'_i)z'_k| < \frac{1}{2}\eta'_k\pi, \xi'^* = 0, \eta'_i > 0$$

$$\Omega'_i = \prod_{j=1}^{p'} \{A'_j{}^{(i)}\}^{A'_j{}^{(i)}} \prod_{j=1}^{q'} \{B'_j{}^{(i)}\}^{-B'_j{}^{(i)}} \prod_{j=1}^{q'_i} \{D'_j{}^{(i)}\}^{D'_j{}^{(i)}} \prod_{j=1}^{p'_i} \{C'_j{}^{(i)}\}^{-C'_j{}^{(i)}}; i = 1, \dots, s$$

$$\xi_i'^* = \operatorname{Im} \left(\sum_{j=1}^{p'} A'_j{}^{(i)} - \sum_{j=1}^{q'} B'_j{}^{(i)} + \sum_{j=1}^{q'_i} D'_j{}^{(i)} - \sum_{j=1}^{p'_i} C'_j{}^{(i)} \right); i = 1, \dots, s$$

$$\eta'_i = \operatorname{Re} \left(\sum_{j=1}^{n'} A'_j{}^{(i)} - \sum_{j=n'+1}^{p'} A'_j{}^{(i)} + \sum_{j=1}^{m'} B'_j{}^{(i)} - \sum_{j=m'+1}^{q'} B'_j{}^{(i)} + \sum_{j=1}^{m'_i} D'_j{}^{(i)} - \sum_{j=m'_i+1}^{q'_i} D'_j{}^{(i)} + \sum_{j=1}^{n'_i} C'_j{}^{(i)} - \sum_{j=n'_i+1}^{p'_i} C'_j{}^{(i)} \right)$$

$$-\mu'_i - \rho'_i - \sum_{l=1}^k \lambda'_l{}^{(i)} > 0; i = 1, \dots, s$$

$$(H) \left| \arg \left(z_i \prod_{j=1}^k (f_j t + g_j)^{\lambda_j^{(i)}} \right) \right| < \frac{1}{2} \eta_i \pi \quad (a \leq t \leq b; i = 1, \dots, r)$$

$$\left| \arg \left(z'_i \prod_{j=1}^k (f_j t + g_j)^{\lambda_j^{(i)'}} \right) \right| < \frac{1}{2} \eta'_i \pi \quad (a \leq t \leq b; i = 1, \dots, s)$$

(G) $P \leq Q + 1$. The equality holds, when, in addition,

$$\text{either } P > Q \text{ and } \left| z_i \left(\prod_{j=1}^k (f_j t + g_j)^{\lambda_j^{(i)}} \right) \right|^{\frac{1}{Q-P}} < 1, \left| z'_i \left(\prod_{j=1}^k (f_j t + g_j)^{\lambda_j^{(i)'}} \right) \right|^{\frac{1}{Q-P}} < 1 \quad (a \leq t \leq b)$$

$$\text{or } P \leq Q \text{ and } \max_{1 \leq i \leq k} \left[\left| \left(z'_i \prod_{j=1}^k (f_j t + g_j)^{\lambda_j^{(i)'}} \right) \right| \right] < 1, \max_{1 \leq i \leq k} \left[\left| \left(z_i \prod_{j=1}^k (f_j t + g_j)^{\lambda_j^{(i)}} \right) \right| \right] < 1 \quad (a \leq t \leq b)$$

Proof

First expressing the class of polynomial $S_L^{h_1, \dots, h_u}[\cdot]$ in series with the help of (1.17), expressing the A-function of r variables by the Mellin-Barnes contour integral with the help of the equation (1.2), the A-function of s variables by the Mellin-Barnes contour integral with the help of the equation (1.8), the generalized hypergeometric function ${}_pF_q(\cdot)$ in Mellin-Barnes contour integral with the help of (2.1). Now collect the power of $(f_j t + g_j)$ with $j = 1, \dots, k$ and use the equations (2.1) and (2.2) and we obtain k -Mellin-Barnes contour integral. Changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) and interpreting $(r + s + k + l)$ -Mellin-Barnes contour integral in multivariable A-function defined by Gautam et al [1], we obtain the desired result.

4. Particular cases

$$a) \text{ If } B(L; R_1, \dots, R_u) = \frac{\prod_{j=1}^A (a_j)_{R_1 \theta_j' + \dots + R_u \theta_j^{(u)}} \prod_{j=1}^{B'} (b_j)_{R_1 \phi_j'} \dots \prod_{j=1}^{B^{(u)}} (b_j^{(u)})_{R_u \phi_j^{(u)}}}{\prod_{j=1}^C (c_j)_{m_1 \psi_j' + \dots + m_u \psi_j^{(u)}} \prod_{j=1}^{D'} (d_j)_{R_1 \delta_j'} \dots \prod_{j=1}^{D^{(u)}} (d_j^{(u)})_{R_u \delta_j^{(u)}}} \quad (4.1)$$

then the general class of multivariable polynomial $S_L^{R_1, \dots, R_u}[x_1, \dots, x_u]$ reduces to generalized Lauricella function defined by Srivastava et al [3].

$$F_{C:D'; \dots; D^{(s)}}^{1+A:B'; \dots; B^{(s)}} \left(\begin{matrix} x_1 \\ \dots \\ \dots \\ x_u \end{matrix} \middle| \begin{matrix} [(-L); R_1, \dots, R_u] [(a); \theta', \dots, \theta^{(u)}] : [(b'); \phi']; \dots; [(b^{(u)}), \phi^{(u)}] \\ [(c); \psi', \dots, \psi^{(u)}] : [(d'); \delta']; \dots; [(d^{(u)}), \delta^{(u)}] \end{matrix} \right) \quad (4.2)$$

and we have the formula

$$\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} \prod_{j=1}^k (f_j t + g_j)^{\sigma_j}$$

$$F_{C:D'; \dots; D^{(t)}}^{1+A:B'; \dots; B^{(t)}} \left(\begin{array}{c} x_1(t-a)^{\mu_1+\mu'_1} (b-t)^{\rho_1+\rho'_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)} - \lambda_j'^{(1)}} \\ \vdots \\ x_u(t-a)^{\mu_u+\mu'_u} (b-t)^{\rho_u+\rho'_u} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(u)} - \lambda_j'^{(u)}} \end{array} \right)$$

$$\left[(-L); R_1, \dots, R_u \right] [(a); \theta', \dots, \theta^{(u)}] : [(b'); \phi']; \dots; [(b^{(u)}), \phi^{(u)}] \\ [(c); \psi', \dots, \psi^{(u)}] : [(d'); \delta']; \dots; [(d^{(u)}), \delta^{(u)}]$$

$$A \left(\begin{array}{c} z_1(t-a)^{\mu_1} (b-t)^{\rho_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \vdots \\ z_r(t-a)^{\mu_r} (b-t)^{\rho_r} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(r)}} \end{array} \right)$$

$$A \left(\begin{array}{c} z'_1(t-a)^{\mu'_1} (b-t)^{\rho'_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j'^{(1)}} \\ \vdots \\ z'_s(t-a)^{\mu'_s} (b-t)^{\rho'_s} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j'^{(s)}} \end{array} \right)$$

$${}^P F_Q \left[(A_P); (B_Q); - \sum_{i=1}^l z_i''(t-a)^{v_i} (b-t)^{\tau_i} \prod_{j=1}^k (f_j t + g_j)^{-\zeta_j^{(i)}} \right] dt =$$

$$(b-a)^{\alpha+\beta+\sum_1^u R_i(\mu_i+\mu'_i+\rho_i+\rho'_i)} \frac{\prod_{j=1}^Q \Gamma(B_j)}{\prod_{j=1}^P \Gamma(A_j)} \prod_{j=1}^k (af_j + g_j)^{\sigma_j + \sum_{i=1}^u R_i(\lambda_j^{(i)} + \lambda_j'^{(i)})}$$

$$\sum_{R_1, \dots, R_u=0}^{h_1 R_1 + \dots + h_u R_u \leq L} (-L)_{h_1 R_1 + \dots + h_u R_u} B(E; R_1, \dots, R_u) \frac{x_1^{R_1} \dots x_u^{R_u}}{R_1! \dots R_u!}$$

$$A_{p+p'+l+k+2, q+q'+l+k+1; X}^{m+m', n+n'+l+k+2; Y} \left(\begin{array}{c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k (af_j+g_j)^{\lambda_j^{(1)}}} \\ \dots \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^k (af_j+g_j)^{\lambda_j^{(r)}}} \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^k (af_j+g_j)^{\lambda_j^{(1)'}}} \\ \dots \\ \frac{z'_s(b-a)^{\mu'_s+\rho'_s}}{\prod_{j=1}^k (af_j+g_j)^{\lambda_j^{(s)'}}} \\ \frac{(b-a)f_1}{af_1+g_1} \\ \dots \\ \frac{(b-a)f_k}{af_k+g_k} \\ \frac{z''_1(b-a)^{\tau_1+\nu_1}}{\prod_{j=1}^k (af_j+g_j)^{\zeta_j^{(1)'}}} \\ \dots \\ \frac{z''_l(b-a)^{\tau_l+\nu_l}}{\prod_{j=1}^k (af_j+g_j)^{\zeta_j^{(l)'}}} \end{array} \right) \mathfrak{A}, K_1, K_2, K_P, K_j : C \quad \mathfrak{B}, L_1, L_j, L_Q, : D \quad (4.3)$$

where $B(L; R_1, \dots, R_u) = \frac{\prod_{j=1}^A (a_j)_{R_1\theta'_j+\dots+R_u\theta_j^{(u)}} \prod_{j=1}^{B'} (b'_j)_{R_1\phi'_j} \dots \prod_{j=1}^{B^{(u)}} (b_j^{(u)})_{R_u\phi_j^{(u)}}}{\prod_{j=1}^C (c_j)_{m_1\psi'_j+\dots+m_u\psi_j^{(u)}} \prod_{j=1}^{D'} (d'_j)_{R_1\delta'_j} \dots \prod_{j=1}^{D^{(u)}} (d_j^{(u)})_{R_u\delta_j^{(u)}}}$ (4.4)

b) If $A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{R}, m = 0$ and $A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{R}$ and $m' = 0$, the multivariable A-functions reduces to multivariable H-functions defined by Srivastava et al [6]. We obtain the following formula.

$$\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} \prod_{j=1}^k (f_j t + g_j)^{\sigma_j} S_L^{h_1, \dots, h_u} \left(\begin{array}{c} x_1 (t-a)^{\mu_1+\mu'_1} (b-t)^{\rho_1+\rho'_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}-\lambda_j^{(1)'}} \\ \vdots \\ x_u (t-a)^{\mu_u+\mu'_u} (b-t)^{\rho_u+\rho'_u} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(u)}-\lambda_j^{(u)'}} \end{array} \right) H \left(\begin{array}{c} z_1 (t-a)^{\mu_1} (b-t)^{\rho_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \vdots \\ z_r (t-a)^{\mu_r} (b-t)^{\rho_r} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(r)}} \end{array} \right)$$

$$H \begin{pmatrix} z'_1(t-a)^{\mu'_1}(b-t)^{\rho'_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \vdots \\ z'_s(t-a)^{\mu'_s}(b-t)^{\rho'_s} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(s)}} \end{pmatrix}$$

$${}_P F_Q \left[(A_P); (B_Q); - \sum_{i=1}^l z''_i(t-a)^{v_i}(b-t)^{\tau_i} \prod_{j=1}^k (f_j t + g_j)^{-\zeta_j^{(i)}} \right] dt =$$

$$(b-a)^{\alpha+\beta+\sum_1^u R_i(\mu_i+\mu'_i+\rho_i+\rho'_i)} \frac{\prod_{j=1}^Q \Gamma(B_j)}{\prod_{j=1}^P \Gamma(A_j)} \prod_{j=1}^k (af_j + g_j)^{\sigma_j + \sum_{i=1}^u R_i(\lambda_j^{(i)} + \lambda_j'^{(i)})}$$

$$\sum_{R_1, \dots, R_u=0}^{h_1 R_1 + \dots + h_u R_u \leq L} (-L)_{h_1 R_1 + \dots + h_u R_u} B(E; R_1, \dots, R_u) \frac{x_1^{R_1} \dots x_u^{R_u}}{R_1! \dots R_u!}$$

$$H_{p+p'+l+k+2, q+q'+l+k+1; Y}^{0, n+n'+l+k+2; X} \left(\begin{array}{c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k (af_j+g_j)^{\lambda_j^{(1)}}} \\ \dots \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^k (af_j+g_j)^{\lambda_j^{(r)}}} \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^k (af_j+g_j)^{\lambda_j'^{(1)}}} \\ \dots \\ \frac{z'_s(b-a)^{\mu'_s+\rho'_s}}{\prod_{j=1}^k (af_j+g_j)^{\lambda_j'^{(s)}}} \\ \frac{(b-a)f_1}{af_1+g_1} \\ \dots \\ \frac{(b-a)f_k}{af_k+g_k} \\ \frac{z''_1(b-a)^{\tau_1+v_1}}{\prod_{j=1}^k (af_j+g_j)^{\zeta_j^{(1)}}} \\ \dots \\ \frac{z''_l(b-a)^{\tau_l+v_l}}{\prod_{j=1}^k (af_j+g_j)^{\zeta_j^{(l)}}} \end{array} \right) \mathfrak{A}, K_1, K_2, K_P, K_j : C \begin{array}{c} \cdot \\ \cdot \end{array} \mathfrak{B}, L_1, L_j, L_Q, : D \quad (4.5)$$

under the same conditions and notations that (3.14) with $A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{R}, m=0$ and $A_j'^{(i)}, B_j'^{(i)}, C_j'^{(i)}, D_j'^{(i)} \in \mathbb{R}$ and $m'=0$.

Remark

By the following similar procedure, the results of this document can be extended to product of any finite number of multivariable A-functions defined by Gautam et al [1].

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of two multivariable A-functions defined by Gautam et al [1] with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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