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A Fixed Point Theorem In Partially OrderedCone Metric Spaces

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Abstract

In this paper, we prove a theorem on fixed point in partially ordered cone metric spaces.

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1.Introduction

Cones metric spaces are a generalization of metric spaces. These spaces are one of the interesting spaces which first introduced by Huang and Zhang [1]. After, many of authors have studied these spaces.In recent years, several authors(see[2-7]) have studied the fixed points in cone metric spaces.

Seong Hoon Cho and Mi Sun Kim [7] have proved certain fixed point theorems by using

Multivalued mapping in the setting of contractive constant in metric spaces.

Partial metric spaces have been originally developed by Matthews [8] to provide mechanism generalizing metric space theories.Now we begin with some definition.

Definition 1.1. Let E be a real Banach space. A subset P of E is called to be a cone if it satisfies in following conditions:

i. $P \neq \emptyset, P\ is\ closed\ and\ P \neq \{\theta\}$
ii. $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \ \Rightarrow ax + by \in P;$
iii. $P \cap (-P) = \{\theta\}.$

Example 1.2. Suppose $E = \mathbb{R}^2$ and $P = \{(x, y) \in E: x, y \geq 0\}$. Then P is a cone.

Define the partial order $\leq$ on P as follow:

$$x \leq y \Leftrightarrow y - x \in P$$

Remark 1.3.$x < y$ means $x \leq y$ but $x \neq y$ and $x \ll y$ means $y - x \in Int\ P$, where Intp denotes interior of P.

Definition 1.4. A cone P is called to be normal when there exists a $k > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq k\|y\|$ for each $x, y \in E$. The least number k which satisfies in perevious inquality is called normal constant of P. Clearly, $k \geq 1$.

Definition 1.5. Let X be a nonempty set and E be a real Banach space. Suppose a function $d: X \times X \to E$ satisfies in following conditions:

i. $\theta \leq d(x,y)$ for each $x, y \in X$ and $d(x,y) = \theta \; iff \; x = y$;
ii. $d(x,y) = d(y,x)$, for each $x, y \in X$;
iii. $d(x,y) \leq d(x,z) + d(z,y)$, for each $x, y, z \in X$.

Then we say d is a cone metric on X and (X,d) is called a cone metric space.

Example 1.6. Let $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x, y \geq 0\}$, $X = \mathbb{R}$ and $d: X \times X \to E by d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a Constant. Then (X,d) is a cone metric space.

2. A Fixed Point Theorem

In this section we rewrite a theorem in [9] and give a new result from it. First of all, we have a theorem from [9].

Theorem 2.1. Let (X , ⊑) be a partial ordered set and d be a metric on X such that the cone metric space (X, d) be complete. Suppose $f: X \to X$ be continuous and nondecreasing map w.r.t ⊑. Also, Let the following conditions hold:

(i) There exist $\alpha, \beta, \gamma \geq 0$ with $\alpha + 2\beta + 2\gamma < 1$ such that

$$d(f_x, f_y) \leq \alpha d(x,y) + \beta\left[d(x, f_x) + d(y, f_y)\right] + \gamma\left[d(x, f(y) + d(y, f(x))\right]$$

For all x, y ∈ X with $y \sqsubseteq x$,

(ii) There exists $x_0 \in X$ such that $x_0 \sqsubseteq f(x_0)$.

Then f has a fixed point.

The following theorem has an effective role in main theorem. Indeed, we have used from it to prove the our main result.

Theorem 2.2. Let (X , ⊑) be a partial ordered set and there exists a cone metric d on X such that (X,d) be complete. Suppose that f, g: X ⟶ X be weakly increasing maps w.r.t⊑ and the following conditions hold:

(i) Here exist $\alpha, \beta, \gamma \geq 0$ with $\alpha, 2\beta + 2\gamma < 1$ such that

$$d(f(x), f(y)) \leq \alpha d(x,y) + \beta\left[d(x, f(x)) + d(y, g(y))\right] + \gamma\left[d(x, g(y)) + d(y, f(x))\right] \quad (1)$$

for all comparative $x, y \in X$,

(ii) If (x_n) is an increasing sequence and convergent to $x \in X$, then $x_n \sqsubseteq x$, for all n.

Then each f,g have a common fixed point.

Now, we prove the main theorem:

Theorem 2.3. Let the assumptions of theorem 2.2 hold. Then each fixed point of f is a fixed point of g and vice versa.

Proof . Let x_0 be an arbitrary element in X. Define the sequence $\{x_n\}_{n\in N}$ by

$$x_{2n+1} = fx_{2n}\ , x_{2n+2} = gx_{2n+1} n \geq 0 \qquad (2)$$

Since f,g are weakly increasing we obtain

$$x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \cdots$$

Which means $\{x_n\}_{n\in N}$ is nonedecreasing w.r.t$\sqsubseteq$.

Now, by assumption x_{2n}, x_{2n+1} are comparative. By using of (1) we have

$$d(x_{2n+1}, x_{2n+2}) = d(fx_{2n}, gx_{2n+1}) \leq (\alpha+\beta+\gamma)d(x_{2n}, x_{2n+1}) + (\beta+\gamma)d(x_{2n+1}, x_{2n+2})$$

Which implies

$$d(x_{2n+1}, x_{2n+2}) \leq kd(x_{2n}, x_{2n+1})$$

where $k = \dfrac{\alpha+\beta+\gamma}{1-\beta-\gamma} < 1$. In a similar manner we can show that

$$d(x_{2n+3}, x_{2n+2}) \leq kd(x_{2n+2}, x_{2n+1})$$

So

$$d(x_{n+1}, x_{n+2}) \leq kd(x_n, x_{n+1}) \leq k^2 d(x_{n-1}, x_n) \leq .... \leq k^{n+1} d(x_0, x_1)$$

For all $n \geq 1$. Let m>n. It is easy to show that

$$d(x_m, x_n) \leq \frac{k^n}{1-k} d(x_0, x_1)$$

Thus $\{x_n\}_{n\in N}$ is a Cauchy sequence. Now, since X is complete, so there exists $x \in X$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$ and by assumption.

(ii) We conclude $x_n \sqsubseteq x$ for all n. According to (2) for some elements in X, say $x_1, x_1 \sqsubseteq fx_1$. Also, for some element in X, say $x_2, x_2 \sqsubseteq gx_2$. So the conditions of theorem 2.1 hold. Therefore f has a fixed point $\hat{x}$ and g has a fixed point $\widetilde{x}$. We show that $g\hat{x} = \hat{x}$.

We have

$$d(f\hat{x}, g\hat{x}) \le \alpha d(\hat{x}, \hat{x}) + \beta[d(\hat{x}, f\hat{x}) + d(\hat{x}, g\hat{x})] + \gamma[d(\hat{x}, g\hat{x}) + d(\hat{x}, f\hat{x})]$$

So

$d(\hat{x}, g\hat{x}) \le (\beta + \gamma)d(\hat{x}, g\hat{x})$ which implies $(\beta + \gamma)d(\hat{x}, g\hat{x}) - d(\hat{x}, g\hat{x}) \in P$

i.e. $-(1-(\beta+\gamma))d(\hat{x}, g\hat{x}) \in P.$ Since $(1-(\beta+\gamma))^{-1} > 0$

Thus $-(1-(\beta+\gamma))^{-1}(1-(\beta+\gamma))d(\hat{x}, g\hat{x}) = -d(\hat{x}, g\hat{x}) \in P.$

Since P is a cone, Hence $\hat{x} = g\hat{x}.$ As before, we can prove $f\tilde{x} = \tilde{x}$ and the proof is completed. □

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