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On general Eulerian integral of certain products of special functions and a class of multivariable polynomial VI

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ABSTRACT
The object of this paper is to establish an general Eulerian integral involving the product of two multivariable I-functions defined by Nambisan et al [1], a expansion of the multivariable I-function defined by Nambisan et al [2] and a class of multivariable polynomials which provide unification and extension of numerous results. We will study the particular case concerning the multivariable H-function.

Keywords: Eulerian integral, multivariable I-function, Lauricella function of several variables, multivariable H-function, class of multivariable polynomials.

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1.Introduction

The object of this paper is to establish an general Eulerian integral involving the product of two multivariable I-functions defined by Nambisan[2], a expansion of multivariable I-function defined by Nambisan [2] and a class of polynomials of several variables defined by Srivastava et al [4].

First time, we define the multivariable $\bar{I}$-function by :

$$\bar{I}(z_1,\cdots,z_r) = \bar{I}^{0,n:m_1,n_1;\cdots;m_r,n_r}_{p,q:p_1,q_1;\cdots;p_r,q_r}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_{n+1,p}: \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_{m+1,q}: \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{c}_j^{(1)},\gamma_j^{(1)};1)_{1,n_1},(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{n_1+1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};1)_{1,n_r},(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{n_r+1,p_r} \\ \\ (\mathrm{d}_j^{(1)},\delta_j^{(1)};1)_{1,m_1},(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{m_1+1,q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};1)_{1,m_r},(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{m_r+1,q_r} \end{array}\right) \tag{1.1}$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi_1(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.2}$$

where $\phi_1(s_1,\cdots,s_r),\theta_i(s_i), i=1,\cdots,r$ are given by :

$$\phi_1(s_1,\cdots,s_r) =\frac{1}{\prod_{j=n+1}^{p}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)\prod_{j=m+1}^{q}\Gamma^{B_j}\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_j\right)} \tag{1.3}$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{n_i}\Gamma\left(1-c_j^{(i)}+\gamma_j^{(i)}s_i\right)\prod_{j=1}^{m_i}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}s_i\right)}{\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}s_i\right)\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}-\delta_j^{(i)}s_i\right)} \tag{1.4}$$

$i = 1, \cdots, r$

Serie representation

If $z_i \neq 0; i = 1, \cdots, r$

$\delta_{h_i}^{(i)}(d_j^{(i)}+k_i) \neq \delta_j^{(i)}(\delta_{h_i}^{(i)}+\eta_i) for j \neq h_i, j, h_i = 1, \cdots, m_i (i = 1, \cdots, r)$,$k_i, \eta_i = 0, 1, 2, \cdots (i = 1, \cdots, r)$, then

$$\bar{I}(z_1, \cdots, z_r) = \sum_{h_1=1}^{m_1}\cdots\sum_{h_r=1}^{m_r}\sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty}\left[\phi_1\left(\frac{dh_1^{(1)}+k_1}{\delta h_1^{(1)}}, \cdots, \frac{dh_r^{(r)}+k_r}{\delta h_r^{(r)}}\right)\right]_{j\neq h_i}\prod_{i=1}^{r}\frac{(-)^{k_i}}{\delta h_i^{(i)}k_i!}z_i^{\frac{dh_i+k_i}{\delta h_i}} \tag{1.5}$$

This result can be proved on computing the residues at the poles :

$$s_i = \frac{dh_i^{(i)}+k_i}{\delta h_i^{(i)}}, (h_i = 1, \cdots, m_i, k_i = 0, 1, 2, \cdots) for i = 1, \cdots, r \tag{1.6}$$

We may establish the the asymptotic expansion in the following convenient form :

$\bar{I}(z_1, \cdots, z_r) = 0(\,|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}\,), max(\,|z_1|, \cdots, |z_r|\,) \rightarrow 0$

$\bar{I}(z_1, \cdots, z_r) = 0(\,|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}\,), min(\,|z_1|, \cdots, |z_r|\,) \rightarrow \infty$

where $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta_k = max[Re((c_j^{(k)}-1)/\gamma_j^{(k)})], j = 1, \cdots, n_k$$

We will note $\eta_{h_i,k_i} = \dfrac{dh_i^{(i)}+k_i}{\delta h_i}$, $(h_i = 1, \cdots, m_i, k_i = 0, 1, 2, \cdots) for i = 1, \cdots, r$ (1.7)

The I-function is defined and represented in the following manner.

$$I(z'_1, \cdots, z'_s) = I_{p',q':p'_1,q'_1;\cdots;p'_s,q'_s}^{0,n':m'_1,n'_1;\cdots;m'_s,n'_s}\left(\begin{array}{c|c} z'_1 & (a'_j; \alpha_j'^{(1)}, \cdots, \alpha_j'^{(s)}; A'_j)_{1,p'} : \\ \cdot & \\ \cdot & \\ \cdot & \\ z'_s & (b'_j; \beta_j'^{(1)}, \cdots, \beta_j'^{(s)}; B'_j)_{1,q'} : \end{array}\right.$$

$$\left.\begin{array}{l} (c_j'^{(1)}, \gamma_j'^{(1)}; C_j'^{(1)})_{1,p'_1}; \cdots; (c_j'^{(s)}, \gamma_j'^{(s)}; C_j'^{(r)})_{1,p'_r} \\ \\ (d_j'^{(1)}, \delta_j'^{(1)}; D_j'^{(1)})_{1,q'_1}; \cdots; (d_j'^{(s)}, \delta_j'^{(s)}; D_j'^{(s)})_{1,q'_r} \end{array}\right) \tag{1.8}$$

$$= \frac{1}{(2\pi\omega)^s}\int_{L'_1}\cdots\int_{L'_s}\phi(t_1,\cdots,t_s)\prod_{i=1}^{s}\zeta_i(t_i)z_i'^{t_i}\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.9}$$

where $\phi(t_1,\cdots,t_s), \zeta_i(t_i), i=1,\cdots,s$ are given by :

$$\phi(t_1,\cdots,t_s) = \frac{\prod_{j=1}^{n'}\Gamma^{A'_j}\left(1-a'_j+\sum_{i=1}^{s}\alpha_j'^{(i)}t_j\right)}{\prod_{j=n'+1}^{p'}\Gamma^{A'_j}\left(a'_j-\sum_{i=1}^{s}\alpha_j'^{(i)}t_j\right)\prod_{j=m'+1}^{q'}\Gamma^{B'_j}\left(1-b'_j+\sum_{i=1}^{s}\beta_j^{(i)}t_j\right)} \tag{1.10}$$

$$\zeta_i(t_i) = \frac{\prod_{j=1}^{n'_i}\Gamma^{C_j'^{(i)}}\left(1-c_j'^{(i)}+\gamma_j'^{(i)}t_i\right)\prod_{j=1}^{m'_i}\Gamma^{D_j'^{(i)}}\left(d_j'^{(i)}-\delta_j'^{(i)}t_i\right)}{\prod_{j=n'_i+1}^{p'_i}\Gamma^{C_j'^{(i)}}\left(c_j'^{(i)}-\gamma_j'^{(i)}t_i\right)\prod_{j=m'_i+1}^{q'_i}\Gamma^{D_j'^{(i)}}\left(1-d_j'^{(i)}-\delta_j'^{(i)}t_i\right)} \tag{1.11}$$

For more details, see Nambisan et al [2].

Following the result of Braaksma [1] the I-function of r variables is analytic if

$$U_i = \sum_{j=1}^{p'}A'_j\alpha_j'^{(i)} - \sum_{j=1}^{q'}B'_j\beta_j'^{(i)} + \sum_{j=1}^{p'_i}C_j'^{(i)}\gamma_j'^{(i)} - \sum_{j=1}^{q'_i}D_j'^{(i)}\delta_j'^{(i)} \leqslant 0, i=1,\cdots,s \tag{1.12}$$

The integral (2.1) converges absolutely if

where$|arg(z'_k)| < \frac{1}{2}\Delta_k\pi, k=1,\cdots,s$

$$\Delta_k = -\sum_{j=n'+1}^{p'}A'_j\alpha_j'^{(k)} - \sum_{j=1}^{q'}B'_j\beta_j^{(k)} + \sum_{j=1}^{m'_k}D_j'^{(k)}\delta_j'^{(k)} - \sum_{j=m'_k+1}^{q'_k}D_j'^{(k)}\delta_j'^{(k)} + \sum_{j=1}^{n'_k}C_j'^{(k)}\gamma_j'^{(k)} - \sum_{j=n'_k+1}^{p'_k}C_j'^{(k)}\gamma_j'^{(k)} > 0 \tag{1.13}$$

Consider the second multivariable I-function.

$$I(z''_1,\cdots,z''_u) = I^{0,n'':m''_1,n''_1;\cdots;m''_u,n''_u}_{p'',q'':p''_1,q''_1;\cdots;p''_u,q''_u}\left(\begin{array}{c|c} \mathrm{z"}_1 & (\mathrm{a"}_j;\alpha_j''^{(1)},\cdots,\alpha_j''^{(u)};A''_j)_{1,p''}: \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z"}_u & (\mathrm{b"}_j;\beta_j''^{(1)},\cdots,\beta_j''^{(u)};B''_j)_{1,q''}: \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{c"}_j^{(1)},\gamma_j''^{(1)};C_j''^{(1)})_{1,p''_1};\cdots;(c_j''^{(u)},\gamma_j''^{(u)};C_j''^{(u)})_{1,p''_u} \\ \\ (\mathrm{d"}_j^{(1)},\delta_j''^{(1)};D_j''^{(1)})_{1,q''_1};\cdots;(d_j''^{(u)},\delta_j''^{(u)};D_j''^{(u)})_{1,q''_u} \end{array}\right) \tag{1.14}$$

$$= \frac{1}{(2\pi\omega)^u}\int_{L''_1}\cdots\int_{L''_u}\psi(x_1,\cdots,x_u)\prod_{i=1}^{u}\xi_i(x_i)z_i''^{x_i}\mathrm{d}x_1\cdots\mathrm{d}x_u \tag{1.15}$$

where $\psi(x_1,\cdots,x_u), \xi_i(x_i), i=1,\cdots,u$ are given by :

$$\psi(x_1,\cdots,x_u)=\frac{\prod_{j=1}^{n''}\Gamma^{A_j''}\left(1-a_j''+\sum_{i=1}^{u}\alpha_j''^{(i)}x_j\right)}{\prod_{j=n''+1}^{p''}\Gamma^{A_j''}\left(a_j''-\sum_{i=1}^{u}\alpha_j''^{(i)}x_j\right)\prod_{j=m'+1}^{q''}\Gamma^{B_j''}\left(1-b_j''+\sum_{i=1}^{u}\beta_j''^{(i)}x_j\right)} \tag{1.16}$$

$$\xi_i(x_i)=\frac{\prod_{j=1}^{n_i''}\Gamma^{C_j''^{(i)}}\left(1-c_j''^{(i)}+\gamma_j''^{(i)}x_i\right)\prod_{j=1}^{m_i''}\Gamma^{D_j''^{(i)}}\left(d_j''^{(i)}-\delta_j''^{(i)}x_i\right)}{\prod_{j=n_i''+1}^{p_i''}\Gamma^{C_j''^{(i)}}\left(c_j''^{(i)}-\gamma_j''^{(i)}x_i\right)\prod_{j=m_i''+1}^{q_i''}\Gamma^{D_j''^{(i)}}\left(1-d_j''^{(i)}-\delta_j''^{(i)}x_i\right)} \tag{1.17}$$

For more details, see Nambisan et al [2].

Following the result of Braaksma [1] the I-function of r variables is analytic if :

$$U_i'=\sum_{j=1}^{p''}A_j''\alpha_j''^{(i)}-\sum_{j=1}^{q''}B_j'\beta_j''^{(i)}+\sum_{j=1}^{p_i''}C_j''^{(i)}\gamma_j''^{(i)}-\sum_{j=1}^{q_i''}D_j''^{(i)}\delta_j''^{(i)}\leqslant 0, i=1,\cdots,u \tag{1.18}$$

The integral (2.1) converges absolutely if

where$|arg(z_k'')|<\frac{1}{2}\Delta_k''\pi, k=1,\cdots,u$

$$\Delta_k'=-\sum_{j=n''+1}^{p''}A_j''\alpha_j''^{(k)}-\sum_{j=1}^{q''}B_j''\beta_j''^{(k)}+\sum_{j=1}^{m_k''}D_j''^{(k)}\delta_j''^{(k)}-\sum_{j=m_k''+1}^{q_k''}D_j''^{(k)}\delta_j''^{(k)}+\sum_{j=1}^{n_k''}C_j''^{(k)}\gamma_j''^{(k)}-\sum_{j=n_k''+1}^{p_k''}C_j''^{(k)}\gamma_j''^{(k)}>0 \tag{1.19}$$

Srivastava [4] introduced and defined a general class of multivariable polynomials as follows

$$S_{N_1,\cdots,N_v}^{M_1,\cdots,M_v}[y_1,\cdots,y_v]=\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_v=0}^{[N_v/M_v]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_v)_{M_vK_v}}{K_v!}$$

$$A[N_1,K_1;\cdots;N_v,K_v]y_1^{K_1}\cdots y_v^{K_v} \tag{1.20}$$

where $M_1,\cdots,M_v$ are arbitrary positive integers and the coefficients are $A[N_1,K_1;\cdots;N_v,K_v]$ arbitrary constants, real or complex.

2. Integral representation of Lauricella function of several variables

The Lauricella function $F_D^{(k)}$ is defined as

$$F_D^{(k)}[a,b_1,\cdots,b_k;c;x_1,\cdots,x_k]=\frac{\Gamma(c)}{\Gamma(a)\prod_{j=1}^{k}\Gamma(b_j)}\frac{1}{(2\pi\omega)^k}\int_{L_1}\cdots\int_{L_k}\frac{\Gamma\left(a+\sum_{j=1}^{k}\zeta_j\right)\Gamma(b_1+\zeta_1),\cdots,\Gamma(b_k+\zeta_k)}{\Gamma\left(c+\sum_{j=1}^{k}\zeta_j\right)}$$

$$\prod_{j=1}^{k}\Gamma(-\zeta_j)(-x_j)^{\zeta_i}\,\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_k \tag{2.1}$$

where $max\left[|arg(-x_1)|,\cdots,|arg(-x_k)|\right]<\pi, c\neq 0,-1,-2,\cdots$.

In order to evaluate a number of integrals of multivariable I-functions, we first establish the formula

$$\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}\mathrm{d}t=(b-a)^{\alpha+\beta-1}B(\alpha,\beta)\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}$$

$$\times F_D^{(k)}\left[\alpha,-\sigma_1,\cdots,-\sigma_k;\alpha+\beta;-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right] \tag{2.2}$$

where $a,b\in\mathbb{R}(a<b),\alpha,\beta,f_i,g_i,\sigma_i\in\mathbb{C},(i=1,\cdots,k)$; $min(Re(\alpha),Re(\beta))>0$ and

$$\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1$$

$F_D^{(k)}$ is a Lauricella's function of k-variables, see Srivastava et al ([5], page60)

The formula (2.2) can be establish by expanding $\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$ by means of the formula :

$$(1-z)^{-\alpha}=\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}z^r(|z|<1) \tag{2.3}$$

integrating term by term with the help of the integral given by Saigo and Saxena [3, page 93, eq.(3.2)] and applying the definition of the Lauricella function $F_D^{(k)}$ [5, page 60].

3. Eulerian integral

In this section , we note :

$$X=m_1',n_1';\cdots;m_s',n_s';m_1'',n_1'';\cdots;m_u'',n_u'';1,0;\cdots;1,0 \tag{3.1}$$

$$Y=p_1',q_1';\cdots;p_s',q_s';p_1'',q_1'';\cdots;p_u'',q_u'';0,1;\cdots;0,1 \tag{3.2}$$

$$A=(a_j';\alpha_j'^{(1)},\cdots,\alpha_j'^{(s)},0,\cdots,0,0,\cdots,0;A_j)_{1,p'} \tag{3.3}$$

$$B=(b_j';\beta_j'^{(1)},\cdots,\beta_j'^{(s)},0\cdots,0,0\cdots,0;B_j')_{1,q'} \tag{3.4}$$

$$A'=(a_j'';0,\cdots,0,\alpha_j''^{(1)},\cdots,\alpha_j''^{(u)},0,\cdots,0;A_j'')_{1,p''} \tag{3.5}$$

$$B'=(b_j'';0,\cdots,0,\beta_j''^{(1)},\cdots,\beta_j''^{(u)},0,\cdots,0;B_j'')_{1,q''} \tag{3.6}$$

$$\mathrm{C}=(\mathrm{c}'^{(1)}_j,\gamma_j'^{(1)};C_j'^{(1)})_{1,p_1'};\cdots;(c_j'^{(s)},\gamma_j'^{(s)};C_j'^{(s)})_{1,p_s'};$$

$$(c_j''^{(1)},\gamma_j''^{(1)};C_j''^{(1)})_{1,p_1''};\cdots;(c_j''^{(u)},\gamma_j^{(u)};C_j''^{(u)})_{1,p_u''};(1,0;1);\cdots;(1,0;1) \tag{3.7}$$

$$D=(\mathrm{d}'^{(1)}_j,\delta_j'^{(1)};D_j'^{(1)})_{1,q_1'};\cdots;(d_j'^{(s)},\delta_j'^{(s)};D_j'^{(s)})_{1,q_s};$$

$$(\mathrm{d}''^{(1)}_j,\delta_j''^{(1)};D_j''^{(1)})_{1,q_1''};\cdots;(d_j''^{(u)},\delta_j''^{(u)};D_j''^{(u)})_{1,q_u''};(0,1;1);\cdots;(0,1;1) \tag{3.8}$$

$$A_1=\left[1-a-\sum_{l=1}^{r}\lambda_l\eta_{h_l,k_l}-\sum_{k=0}^{v}K_k\lambda_k';\gamma_1,\cdots,\gamma_s,\gamma_1',\cdots,\gamma_u',1,\cdots,1;1\right] \tag{3.9}$$

$$A_2 = \left[1 - b - \sum_{l=1}^{r} \mu_l \eta_{h_l,k_l} - \sum_{k=1}^{v} K_k \mu'_k ; \tau_1, \cdots, \tau_s, \tau'_1, \cdots, \tau'_u, 1, \cdots, 1; 1\right] \tag{3.10}$$

$$A_3 = \left[1 + \rho_j + \sum_{l=1}^{r} a_j^{(l)} \eta_{h_l,k_l} + \sum_{k=1}^{v} K_k b_j^{(k)} ; c_j^{(1)}, \cdots, c_j^{(s)}, c'_j{}^{(1)}, \cdots, c'_j{}^{(u)}, 0, \cdots, 1, 0, \cdots, 0; 1\right]_{1,h} \tag{3.11}$$

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$$A_4 = \left[1 + \rho_j + \sum_{l=1}^{r} a_j^{(l)} \eta_{h_l,k_l} + \sum_{k=1}^{v} K_k b_j^{(k)} ; c_j^{(1)}, \cdots, c_j^{(s)}, c'_j{}^{(1)}, \cdots, c'_j{}^{(u)}, 0, \cdots, 0; 1\right]_{1,h} \tag{3.12}$$

$$A_5 = \left[1 - a - b - \sum_{l=1}^{r} (\lambda_l + \mu_l) \eta_{g_l,k_l} - \sum_{k=1}^{v} K_k (\lambda'_k + \mu'_k) ; \gamma_1 + \tau_1, \cdots, \gamma_s + \tau_s,\right.$$

$$\left.\gamma'_u + \tau'_u, 1, \cdots, 1; 1\right] \tag{3.13}$$

$$P_1 = (n - m)^{a+b-1} \left\{ \prod_{j=1}^{h} (p_j m + q_j)^{\rho_j} \right\} \tag{3.14}$$

$$P_2 = (n - m)^{\sum_{l=1}^{r} (\lambda_l + \mu_l) \eta_{G_l,g_l} + \sum_{l=1}^{v} (\lambda'_l + \mu') K_l} \left\{ \prod_{j=1}^{h} (p_j m + q_j)^{\sum_{l=1}^{r} a_j^{(l)} \eta_{g_l,h_l} + \sum_{l=1}^{v} a_j'^{(l)} K_l} \right\} \tag{3.15}$$

$$A_v = \frac{(-N_1)_{M_1 K_1}}{K_1!} \cdots \frac{(-N_v)_{M_v K_v}}{K_v!} A[N_1, K_1; \cdots; N_v, K_v] \tag{3.16}$$

We have the following result

$$\int_m^n (t-m)^{a-1} (n-t)^{b-1} \prod_{j=1}^{h} (p_j t + q_j)^{\rho_j} S_{N_1,\cdots,N_v}^{M_1,\cdots,M_v} \begin{pmatrix} \mathrm{Z}_1 (t-m)^{\lambda'_1} (n-t)^{\mu'_1} \prod_{j=1}^{h} (p_j t + q_j)^{b'_j} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Z}_v (t-m)^{\lambda'_v} (n-t)^{\mu'_v} \prod_{j=1}^{h} (p_j t + q_j)^{b_j^{(r)}} \end{pmatrix}$$

$$\bar{I} \begin{pmatrix} \mathrm{z}_1 (t-m)^{\lambda_1} (n-t)^{\mu_1} \prod_{j=1}^{h} (p_j t + q_j)^{a'_j} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r (t-m)^{\lambda_r} (n-t)^{\mu_r} \prod_{j=1}^{h} (p_j t + q_j)^{a_j^{(r)}} \end{pmatrix} I \begin{pmatrix} \mathrm{z'}_1 (t-m)^{\gamma_1} (n-t)^{\tau_1} \prod_{j=1}^{h} (p_j t + q_j)^{-c_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z'}_s (t-m)^{\gamma_s} (n-t)^{\tau_s} \prod_{j=1}^{h} (p_j t + q_j)^{-c_j^{(s)}} \end{pmatrix}$$

$$I\begin{pmatrix} \mathrm{z"}_1(t-m)^{\gamma'_1}(n-t)^{\tau'_1}\prod_{j=1}^h(p_jt+q_j)^{-c_j'^{(1)}} \\ . \\ . \\ . \\ \mathrm{z"}_u(t-m)^{\gamma'_u}(n-t)^{\tau'_u}\prod_{j=1}^h(p_jt+q_j)^{-c_j'^{(u)}} \end{pmatrix}\mathrm{d}t$$

$$=P_1\sum_{h_1=1}^{m_1}\cdots\sum_{h_r=1}^{m_r}\sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_v=0}^{[N_v/M_v]}\prod_{i=1}^{r}\frac{(-)^{k_i}}{\delta h_i^{(i)}k_i!}z_i^{\eta_{h_i,k_i}}\prod_{k=1}^{v}Z^kP_2[\phi_1(\eta_{h_1,k_1},\cdots,\eta_{h_r,k_r})]_{j\neq h_i}A_v$$

$$I^{0,n'+n''+h+2;X}_{p'+p''+h+2,q'+q''+h+1;Y}\left(\begin{array}{c|c} \frac{z'_1(n-m)^{\gamma_1+\tau_1}}{\prod_{j=1}^h(mf_j+q_j)^{c_j^{(1)}}} & \\ \cdots & \\ \cdots & \mathrm{A}\ ,\mathrm{A'},\mathrm{A}_1,A_2,A_3:C \\ \frac{z'_s(n-m)^{\gamma_s+\tau_s}}{\prod_{j=1}^h(mf_j+q_j)^{c_j^{(s)}}} & . \\ \cdots & . \\ \frac{z''_1(n-m)^{\gamma'_1+\tau'_1}}{\prod_{j=1}^h(mf_j+q_j)^{c_j'^{(1)}}} & . \\ \cdots & . \\ \cdots & . \\ \frac{z''_u(n-m)^{\gamma'_u+\tau'_u}}{\prod_{j=1}^h(mf_j+q_j)^{c_j'^{(u)}}} & . \\ \cdots & . \\ \frac{(n-m)p_1}{p_1m+q_1} & \mathrm{B},\mathrm{B'},\ \mathrm{A}_4,A_5,D \\ \cdots & \\ \cdots & \\ \frac{(n-m)p_h}{p_hm+q_h} & \end{array}\right) \tag{3.17}$$

$\phi_1$is defined respectively by (1.3)

Provided that

(A) $m_j,n_j,p_j,q_j(j=1,\cdots,s),n',p',q'\in\mathbb{N}^*$ **;**$\delta_j^{(i)}\in\mathbb{R}_+(j=1,\cdots,q_i;i=1,\cdots,s)$

$\alpha_j'^{(i)}\in\mathbb{R}_+(j=1,\cdots,p';i=1,\cdots,s),\beta_j'^{(i)}\in\mathbb{R}_+(j=1,\cdots,q';i=1,\cdots,s),\gamma_j'^{(i)}\in\mathbb{R}_+(j=1,\cdots,p'_i;i=1,\cdots,s)$

$a'_j(j=1,\cdots,p),b'_j(j=1,\cdots,q),c_j'^{(i)}(j=1,\cdots,p'_i,i=1,\cdots,s),d_j'^{(i)}(j=1,\cdots,q'_i,i=1,\cdots,s)\in\mathbb{C}$

The exposants

$A'_j(j=1,\cdots,p'), B'_j(j=1,\cdots,q'), C'^{(i)}_j(j=1,\cdots,p'_i;i=1,\cdots,s), D'^{(i)}_j(j=1,\cdots,q'_i;i=1,\cdots,s)$

of various gamma function involved in (1.10) and (1.11) may take non integer values.

$m''_j, n''_j, p''_j, q'_j(j=1,\cdots,u), n'', p'', q'' \in \mathbb{N}^*$ **;** $\delta''^{(i)}_j \in \mathbb{R}_+(j=1,\cdots,q''_i;i=1,\cdots,u)$

$\alpha''^{(i)}_j \in \mathbb{R}_+(j=1,\cdots,p'';i=1,\cdots,u), \beta''^{(i)}_j \in \mathbb{R}_+(j=1,\cdots,q'';i=1,\cdots,u), \gamma''^{(i)}_j \in \mathbb{R}_+(j=1,\cdots,p''_i;i=1,\cdots,u)$

$a''_j(j=1,\cdots,p''), b''_j(j=1,\cdots,q''), c''^{(i)}_j(j=1,\cdots,p''_i, i=1,\cdots,u), d''^{(i)}_j(j=1,\cdots,q''_i, i=1,\cdots,u) \in \mathbb{C}$

The exposants

$A''_j(j=1,\cdots,p''), B''_j(j=1,\cdots,q''), C''^{(i)}_j(j=1,\cdots,p''_i;i=1,\cdots,u), D''^{(i)}_j(j=1,\cdots,q''_i;i=1,\cdots,u)$

of various gamma function involved in (1.15) and (1.16) may take non integer values.

(B) $m,n \in \mathbb{N}, \gamma_i, \tau_i, \gamma'_i, \tau'_i, c^{(i)}_j, c'^{(i)}_j, \lambda_l, \lambda'_k, \mu'_k, \mu_l, a^{(l)}_j, b^{(k)}_j \in \mathbb{R}^+, \rho_j \in \mathbb{R}, p_i, q_i \in \mathbb{C}$

(C) $\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(n-m)p_i}{mp_i+q_i}\right|\right\} < 1$

(D) $Re\big[a+\sum_{l=1}^{r}\lambda_l \min_{1\leqslant j\leqslant m_l}\frac{d^{(l)}_j}{\delta^{(l)}_j}+\sum_{i=1}^{s}\gamma_i\min_{1\leqslant j\leqslant m'_i}\frac{b^{(i)}_j}{\beta^{(i)}_j}+\sum_{i=1}^{u}\gamma'_i\min_{1\leqslant j\leqslant m''_i}\frac{b'^{(i)}_j}{\beta'^{(i)}_j}\big] > 0$

$$Re\big[b+\sum_{l=1}^{r}\mu_l \min_{1\leqslant j\leqslant m'_l}\frac{d^{(l)}_j}{\delta^{(l)}_j}+\sum_{i=1}^{s}\tau_i\min_{1\leqslant j\leqslant m'_i}\frac{b^{(i)}_j}{\beta^{(i)}_j}+\sum_{i=1}^{u}\tau'_i\min_{1\leqslant j\leqslant m''_i}\frac{b'^{(i)}_j}{\beta'^{(i)}_j}\big] > 0$$

(E) $U_i=\sum_{j=1}^{p'}A'_j\alpha'^{(i)}_j-\sum_{j=1}^{q'}B'_j\beta'^{(i)}_j+\sum_{j=1}^{p'_i}C'^{(i)}_j\gamma'^{(i)}_j-\sum_{j=1}^{q'_i}D'^{(i)}_j\delta'^{(i)}_j\leqslant 0, i=1,\cdots,s$

$$U'_i=\sum_{j=1}^{p''}A''_j\alpha''^{(i)}_j-\sum_{j=1}^{q''}B'_j\beta''^{(i)}_j+\sum_{j=1}^{p''_i}C''^{(i)}_j\gamma''^{(i)}_j-\sum_{j=1}^{q''_i}D''^{(i)}_j\delta''^{(i)}_j\leqslant 0, i=1,\cdots,u$$

(F) $\left|arg\left(z'_i\prod_{j=1}^{h}(p_jt+q_j)^{-c^{(i)}_j}\right)\right| < \frac{1}{2}\Delta_i\pi$ **,** Δ_i is defined by (1.13) and

$\left|arg\left(z''_i\prod_{j=1}^{h}(p_jt+q_j)^{-c'^{(i)}_j}\right)\right| < \frac{1}{2}\Delta'_i\pi$, Δ'_i is defined by (1.18)

Proof

To prove (3.17), first, we express in serie the multivariable I-function defined by Nambisan et al [2] with the help of (1.5), a class of multivariable polynomials defined by Srivastava $S^{M_1,\cdots,M_v}_{N_1,\cdots,N_v}[.]$ in serie with the help of (1.20), the I-functions of s-variables and u-variables defined by Nambisan et [2] et al in terms of Mellin-Barnes type contour integral with the help of (1.2) and (1.9) respectively. Now interchange the order of summations and integrations (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process). We collect the power of (p_jt+q_j) with $j=1,\cdots,h$ and use the equations (2.1) and (2.2) and we obtain h-Mellin-Barnes contour integral and interpreting $(s+u+h)-$Mellin-barnes contour integral in multivariable I-function, we obtain the desired result.

Remark :

If a) $\gamma_1=\cdots=\gamma_s=\gamma'_1=\cdots,\gamma'_u=0$ and $\lambda_1=\cdots=\lambda_r=\lambda'_1=\cdots=\lambda'_v=0$

b) $\tau_1=\cdots=\tau_s=\tau'_1=\cdots=\tau'_u=0$ and $\mu_1=\cdots=\mu_r=\mu'_1=\cdots=\mu'_v=0$

we obtain the similar formulas that (3.17) with the corresponding simplifications.

4.Particular case

1) $A'_j=B'_j=C'^{(i)}_j=D'^{(i)}_j=A''_j=B''_j=C''^{(i)}_j=D''^{(i)}_j=1$, The multivariable I-functions defined by Nambisan reduces to multivariable H-function defined by Srivastava et al [6]. We have.

$$\int_m^n (t-m)^{a-1}(n-t)^{b-1}\prod_{j=1}^h (p_jt+q_j)^{\rho_j} S^{M_1,\cdots,M_v}_{N_1,\cdots,N_v}\begin{pmatrix} \mathrm{Z}_1(t-m)^{\lambda'_1}(n-t)^{\mu'_1}\prod_{j=1}^h(p_jt+q_j)^{b'_j} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Z}_v(t-m)^{\lambda'_v}(n-t)^{\mu'_v}\prod_{j=1}^h(p_jt+q_j)^{b^{(r)}_j} \end{pmatrix}$$

$$I\begin{pmatrix} \mathrm{z}_1(t-m)^{\lambda_1}(n-t)^{\mu_1}\prod_{j=1}^h(p_jt+q_j)^{a'_j} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r(t-m)^{\lambda_r}(n-t)^{\mu_r}\prod_{j=1}^h(p_jt+q_j)^{a^{(r)}_j} \end{pmatrix} H\begin{pmatrix} \mathrm{z'}_1(t-m)^{\gamma_1}(n-t)^{\tau_1}\prod_{j=1}^h(p_jt+q_j)^{-c^{(1)}_j} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z'}_s(t-m)^{\gamma_s}(n-t)^{\tau_s}\prod_{j=1}^h(p_jt+q_j)^{-c^{(s)}_j} \end{pmatrix}$$

$$H\begin{pmatrix} \mathrm{z''}_1(t-m)^{\gamma'_1}(n-t)^{\tau'_1}\prod_{j=1}^h(p_jt+q_j)^{-c'^{(1)}_j} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z''}_u(t-m)^{\gamma'_u}(n-t)^{\tau'_u}\prod_{j=1}^h(p_jt+q_j)^{-c'^{(u)}_j} \end{pmatrix}\mathrm{d}t$$

$$=P_1\sum_{h_1=1}^{m_1}\cdots\sum_{h_r=1}^{m_r}\sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_v=0}^{[N_v/M_v]}\prod_{i=1}^r\frac{(-)^{k_i}}{\delta h_i^{(i)}k_i!}z_i^{\eta_{h_i,k_i}}\prod_{k=1}^v Z^kP_2[\phi_1(\eta_{h_1,k_1},\cdots,\eta_{h_r,k_r})]_{j\neq h_i}A_v$$

$$H_{p'+p''+h+2,q'+q''+h+1;Y}^{0,n'+n''+h+2;X}\left(\begin{array}{c|c} \frac{z_1'(n-m)^{\gamma_1+\tau_1}}{\prod_{j=1}^h (mf_j+q_j)^{c_j^{(1)}}} & \\ \cdot\ \cdot\ \cdot & \\ \cdot\ \cdot\ \cdot & \mathfrak{A}_1, \mathrm{A}_1, A_2, A_3 : \mathfrak{A}_2 \\ \frac{z_s'(n-m)^{\gamma_s+\tau_s}}{\prod_{j=1}^h (mf_j+q_j)^{c_j^{(s)}}} & \cdot \\ \cdot\ \cdot\ \cdot & \cdot \\ \frac{z_1''(n-m)^{\gamma_1'+\tau_1'}}{\prod_{j=1}^h (mf_j+q_j)^{c_j'^{(1)}}} & \cdot \\ \cdot\ \cdot\ \cdot & \cdot \\ \cdot\ \cdot\ \cdot & \cdot \\ \frac{z_u''(n-m)^{\gamma_u'+\tau_u'}}{\prod_{j=1}^h (mf_j+q_j)^{c_j'^{(u)}}} & \cdot \\ \cdot\ \cdot\ \cdot & \cdot \\ \frac{(n-m)p_1}{p_1m+q_1} & \mathfrak{B}_1, \mathrm{A}_4, A_5 : \mathfrak{B}_2 \\ \cdot\ \cdot\ \cdot & \\ \cdot\ \cdot\ \cdot & \\ \frac{(n-m)p_h}{p_hm+q_h} & \end{array}\right) \quad (4.1)$$

under the same conditions and notations that (3.16) with $A_j' = B_j' = C_j'^{(i)} = D_j'^{(i)} = A_j'' = B_j'' = C_j''^{(i)} = D_j''^{(i)} = 1$.

Remark:

By the following similar procedure, the results of this document can be extented to product of any finite number of multivariable I-functions defined by Nambisan et al [2] and a class of multivariable polynomials defined by Srivastava [4].

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of three multivariable I-functions defined by Nambisan et al [2] with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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