

## RADIUS-VITAL EDGES IN A GRAPH

**Medha Itagi Huilgol**  
Department of Mathematics, Bangalore University,  
Central College Campus, Bangalore-560001  
[medha@bub.ernet.in](mailto:medha@bub.ernet.in)

### Abstract:

The graph resulting from contracting edge "e" is denoted as  $G/e$  and the graph resulting from deleting edge "e" is denoted as  $G-e$ . An edge "e" is radius-essential if  $\text{rad}(G/e) < \text{rad}(G)$ , radius-increasing if  $\text{rad}(G-e) > \text{rad}(G)$ , and radius-vital if it is both radius-essential and radius-increasing. We partition the edges that are not radius-vital into three categories. In this paper, we study realizability questions relating to the number of edges that are not radius-vital in the three defined categories. A graph is radius-vital if all its edges are radius-vital. We give a structural characterization of radius-vital graphs.

**Keywords:** Radius-vital edges, radius-increasing edges, radius-essential edges

**AMS Classification 2010:** 05C12

### 1. Introduction:

The terminology used throughout this paper is based on Buckley and Harary [1], Harary[3].

Let  $G$  be a connected graph with vertex set  $V(G)$  and edge set  $E(G)$  with  $p$  and  $q$  representing order and size of  $G$ . The distance  $d(u,v)$  between vertices  $u$  and  $v$  is the length of a shortest path joining  $u$  and  $v$ . The eccentricity  $e(v)$  of  $v$  is the distance to a farthest vertex from  $v$ . The radius  $\text{rad}(G)$  and diameter  $\text{diam}(G)$  are minimum and maximum eccentricities, respectively. The center  $C(G)$  and  $P(G)$ , periphery of a graph  $G$  consists of the sets of vertices of minimum and maximum eccentricity, respectively. Vertices in  $C(G)$  are called central vertices and those in  $P(G)$  are called the peripheral vertices. An elementary contraction of an edge  $e=uv$  in  $G$  is obtained by removing  $u$  and  $v$ , inserting a new vertex  $w$  and inserting an edge between  $w$  and any vertex to which either  $u$  or  $v$  (or both) were adjacent and  $G/e$  denotes the resulting graph. The graph resulting from deleting edge  $e$  is denoted by  $G-e$ .

As in [1], the sequential join  $G_1+G_2+G_3+\dots+G_k$  of graphs  $G_1, G_2, \dots, G_k$  is the graph formed by taking one copy of each of the graphs  $G_1, G_2, \dots, G_k$  and adding in additional edges from each vertex of  $G_i$  to each vertex in  $G_{i+1}$ , for  $1 \leq i \leq k-1$ .

An edge  $e$  is radius-essential if  $\text{rad}(G/e) < \text{rad}(G)$  and radius-increasing if

$\text{rad}(G-e) > \text{rad}(G)$ . We studied radius-essential edges in [9]. If every edge in a graph  $G$  is radius-increasing, then  $G$  is a radius-minimal graph. Gliviak [2] established various existence results for radius-minimal graphs.

**Definition 1.1:** An edge  $e$  is radius-vital if it is both radius-essential and radius-increasing; otherwise, it is radius-non-vital.

Thus, a radius-vital edge  $e$  has the property that contracting  $e$  decreases the radius and deleting  $e$  increases the radius.

An edge  $e$  is deletable if its deletion does not alter the radius, that is,  $\text{rad}(G-e) = \text{rad}(G)$ . (Gliviak [2] refers such edges superfluous). An edge  $e$  is contractible if its contraction does not alter the radius, that is,  $\text{rad}(G/e) = \text{rad}(G)$ . In view of these definitions, we can partition  $E(G)$  into four sets:

**radius-vital edges:**  $E_v(G) = \{e: \text{rad}(G/e) < \text{rad}(G) \text{ and } \text{rad}(G-e) > \text{rad}(G)\}$ ,

**contractible, radius-increasing edges:**  $E_c(G) = \{e: \text{rad}(G/e) = \text{rad}(G) \text{ and } \text{rad}(G-e) > \text{rad}(G)\}$ ,

**deletable, radius-essential edges:**  $E_d(G) = \{e: \text{rad}(G-e) = \text{rad}(G) \text{ and } \text{rad}(G/e) < \text{rad}(G)\}$ ,

and

**contractible and deletable edges:**  $E_{cd}(G) = \{e: \text{rad}(G-e) = \text{rad}(G) \text{ and } \text{rad}(G/e) = \text{rad}(G)\}$ .

An edge  $e$  is radius-non-vital(non-vital), if  $e \in E_c(G) \cup E_d(G) \cup E_{cd}(G)$ .

In this paper, we shall study the vital and non-vital edges of graphs. After characterizing graphs for which every edge is vital, we examine realizability questions relating to the sizes of the sets  $E_c(G)$ ,  $E_d(G)$  and  $E_{cd}(G)$  and study which triples  $(x, y, z)$  of integers are realizable for  $(|E_c(G)|, |E_d(G)|, |E_{cd}(G)|)$ .

We mention that a similar study was done for 3-connectedness in graphs. Reid and Wu[6] studied edges "e" in 3-connected graphs for which either deletion of "e" or the contraction of "e", but not both, alters the 3-connectedness of the graph.

**Definition 1.2:** A graph  $G$  is radius-vital if all its edges are radius-vital.

We recall some results from Walikar, Buckley and Itagi[9]. Let  $\sigma_r(G)$  be the number of essential edges in  $G$ . That is,

$$\sigma_r(G) = |\{e \in E(G): \text{rad}(G/e) < \text{rad}(G)\}|.$$

Since an essential edge is not contractible,  $\sigma_r(G) = |E_v(G)| + |E_d(G)|$ .

Let  $p$  and  $q$  denote the number of vertices and edges, respectively, in  $G$ . We shall need the following.

**Proposition 1.3[2]:** A non-trivial graph is radius minimal if and only if  $G$  is a tree.

**Proposition 1.4[9]:** For a tree  $T$ ,  $\sigma_r(G) = q$ , if and only if  $T$  is a path on even number of vertices.

## 2. Results:

The following result characterizes radius-vital graphs.

**Proposition 2.1:** Let  $G$  be a graph with  $\text{rad}(G) = r$ . Then  $G$  is radius-vital if and only if  $G$  is a path on even number of vertices.

**Proof:** A non-trivial graph  $G$  is radius-minimal if and only if  $G$  is a tree, by Proposition 1.1[2]. By Proposition 1.2[9],  $\sigma_r(T) = q$  if and only if  $T$  is a path on even number of vertices. Combining the two results the proof follows. ■

We now focus on the non-vital edges of a graph. We begin with a definition and several preliminary observations.

**Definition 2.2:** For any three non-negative integers  $x, y, z$ , a graph  $G$  is said to be an  $(x, y, z)$ -graph, if  $|E_c(G)| = x$ ,  $|E_d(G)| = y$  and  $|E_{cd}(G)| = z$ , and the triple  $(x, y, z)$  is realizable if there exists an  $(x, y, z)$ -graph  $G$ .

By Proposition 2.1, it is clear that only  $(0, 0, 0)$  graphs are paths on even number of vertices.

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**Remark 2.3:** If  $\sigma_r(G)=0$ , then all edges are contractible no matter whether they are deletable or not. Hence  $G$  contains no vital edges. Therefore, if  $G$  is an  $(x, y, z)$ -graph with  $\sigma_r(G)=0$ , we have  $y = 0$  and  $x + z = q$ .

A graph  $G$  is a radius-edge-invariant graph (r.e.i. graph) if for each  $e \in E(G)$ ,  $\text{rad}(G-e) = \text{rad}(G)$ , that is, every edge of  $G$  is deletable. Refer Walikar, Buckley and Itagi [8] for detailed study of these graphs.

**Remark 2.4:** If  $\sigma_r(G)=q$ , then no edge of  $G$  is contractible. Hence  $x = z = 0$  and  $y \leq q$ . If  $y = q$ , then  $G$  is radius-edge-invariant, otherwise, there exists at least one vital edge in  $G$ .

**Remark 2.5:** If  $G$  is radius-edge-invariant graph then every edge is deletable, so there are no vital edges in  $G$ . Thus, for a r.e.i. graph,  $x = 0$  and  $y + z = q$ .

**Remark 2.6:** If  $G$  is radius-minimal then no edge is deletable. Hence  $y = z = 0$  and  $x \leq q$ . Thus for a diameter minimal graph  $G$ , if  $\sigma_r(G) = 0$ , then  $x = q$  and if  $\sigma_r(G) > 0$ , there exists at least one vital edge.

Next we consider realizability of triple of integers.

**Lemma 2.7:** The triple  $(0, 1, 0)$  is not realizable.

**Proof:** On the contrary assume that  $(0, 1, 0)$  is realizable. Then there exists a graph  $G$ , containing only one edge, say  $e \in E_d(G)$ .

Then (i)  $\text{rad}(G-e) = \text{rad}(G)$  and (ii)  $\text{rad}(G/e) < \text{rad}(G)$  hold. And all other edges  $e'$  in  $G$  are vital, hence (i')  $\text{rad}(G-e') > \text{rad}(G)$  and (ii')  $\text{rad}(G/e') < \text{rad}(G)$ . From (ii) and (ii') it follows that  $\sigma_r(G)=q$ . Hence, for  $e$ , there exists a radius-preserving spanning tree which avoids  $e$ . But this edge can be contracted too without altering the radius of  $G$ , a contradiction to the fact that  $\sigma_r(G) = q$ . ■

**Lemma 2.8:** The triple  $(x, 0, 0)$  is realizable, for all  $x \geq 0$ .

**Proof:** If  $x = 0$ , then by Proposition 2.1,  $G$  is a path on even number of vertices. For  $x \geq 1$ , consider a graph  $G$ , obtained by joining  $x$  pendent edges to any one of the central vertices of path  $P_{2n}$ ,  $n \geq 3$ . Clearly, this graph has  $x$  edges belonging to  $E_c(G)$  and rest all vital. Hence the result. ■

**Lemma 2.9:** The triple  $(x, 1, 0)$  is not realizable for all values of  $x \geq 0$ .

**Proof:** On the contrary assume that the triple is realizable. Hence there exists a graph  $G$  containing one edge, say  $e$ ,  $e \in E_d(G)$  and "x" edges belonging to  $E_c(G)$ , that is,  $\sigma_r(G) = q-x$ . For  $e$ , there exists a radius-preserving spanning tree,

which does not contain  $e$ , as  $\text{rad}(G-e) = \text{rad}(G)$ . By contraction of this edge radius remains unaltered contradicting the fact that the edge

$e \in E_d(G)$ , proving the result. ■

**Lemma 2.10:** *The triple  $(0, y, 0)$  is realizable for  $y = 2m$ ,  $m \geq 2$ , or  $y = 2k + mn$ ,  $k \geq 2$ ,  $m \geq 2, n \geq 1$  or  $y = mn$ ,  $m \geq 2, n \geq 1$ .*

**Proof:** To show that  $(0, y, 0)$  is realizable, it is sufficient to show the existence of a graph for values given in the

hypothesis. For  $y = 0$ , the realizability follows from *Proposition 2.1*. It is clear that an edge  $e \in E_d(G)$  edge lies on a

block of  $G$ . Since  $x = z = 0$ , all other edges of  $G$  must be vital. For different values of "y", we have different structure of blocks containing "y" edges. If  $y$  is even i.e.  $y \geq 2m$ ,  $m \geq 2$ , consider a graph  $G_1 = K_1 + \overline{K}_m + K_1$ ,  $m \geq 2$ , as in Figure 1.

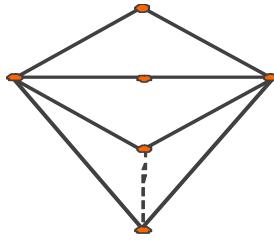


Figure 1

For the graph of Figure 2, all  $2m$  edges belong to  $E_d(G)$ . Hence  $G_1$  is a  $(0, y, 0)$  graph.

For  $y = mn$ ,  $m \geq 2$ ,  $n \geq 1$ , consider a graph  $G_2 = K_1 + \overline{K}_m \square F \square \overline{K}_m + K_1$ , where  $F$  denotes the one factor between  $\overline{K}_m$  and  $\overline{K}_m$  as in Figure 2.

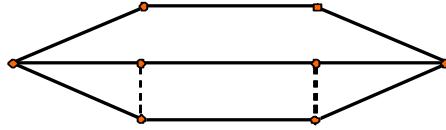


Figure 2

$G_2$  is an  $(0, y, 0)$  graph as all  $3m$  edges are deletable, radius-increasing.

Next consider a graph  $G_3 = K_1 + \overline{K}_m \square F \square \overline{K}_m \square F \square \square F \square \overline{K}_m + K_1$ , where  $F$  denotes one factor between two consecutive  $\overline{K}_m$ 's, as in Figure 3.

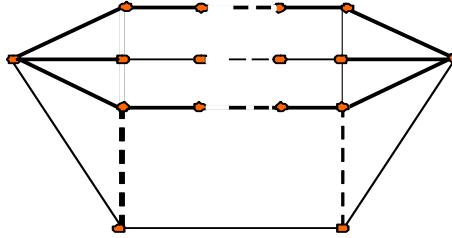


Figure 3

Clearly  $G_3$  is an  $(0, y, 0)$ -graph for  $y = mn$ ,  $m \geq 2$ ,  $n \geq 2$ , as all  $mn$  edges of  $G_3$  belong to  $E_d(G)$ . Hence any combination of the above discussed values of "y" can be realized for  $(0, y, 0)$ . So the realizing graph will be as shown in Figure 4.

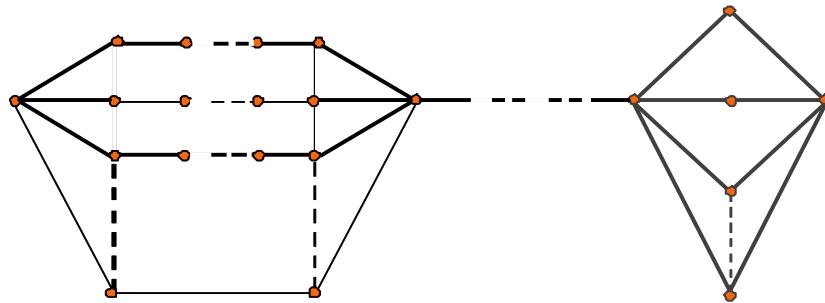


Figure 4.

■

**Lemma 2.11:** *The triple  $(x, y, 0)$  is realizable for  $x \geq 0$ ;  $y = 2m$ , or  $y = 2k + mn$ ,  $k \geq 2$ ,  $m \geq 2$ ,*

*$n \geq 1$ , or  $y = mn$ ,  $m \geq 2$ ,  $n \geq 1$ .*

**Proof:** Consider the graph  $G$  of Figure 1. Join "x" pendent edges at any one of the vertices of degree  $m$ , to get an  $G'_1=(x, y, 0)$ -graph for  $y \geq 2m$ ,  $m \geq 2$ . Clearly, contraction of these pendent "x" edges does not alter radius of  $G'_1$ . Similarly, to each of  $G_2, G_3, G_4$  of above Lemma 2.10, we can join "x" pendent edges at any vertex whose degree is not equal to two, to get graphs  $G'_2, G'_3, G'_4$  which are  $(x, y, 0)$  graphs for different values of "y". We note that  $G'_2$  is  $(x, 3m, 0)$ -graph,  $G'_3$  is  $(x, mn, 0)$ -graph and  $G'_4$  is  $(x, 2k+mn, 0)$ -graph. ■

**Lemma 2.12:** *The triple  $(0, 0, 1)$  is not realizable.*

**Proof:** Suppose,  $(0, 0, 1)$  is realizable, let  $G$  be the realizing graph. In  $G$ , let "e" be the only edge such that  $\text{rad}(G-e) = \text{rad}(G) = \text{rad}(G/e)$ .  $G$  cannot contain only one edge as  $K_2$  is neither deletable nor contractible. Hence all other edges of  $G$  must be vital. Since,  $\text{rad}(G/e) = \text{rad}(G)$ , for some central vertex, say  $u$ , there are at least two eccentric vertices say  $u_1$  and  $u_2$ , joined by disjoint paths. Hence if "e" lies on any one path, say  $u-u_1$  path, then any other edge of  $u-u_2$  path can also be contracted without altering the radius of  $G$ . This contradicts the fact that  $G$  contains only one radius-vital edge and hence the result. ■

**Lemma 2.13:** *The triple  $(0, 0, z)$  is realizable except for  $z = 1$ .*

**Proof:** From Proposition 2.1,  $(0, 0, 0)$  is realizable. From above lemma,  $(0, 0, 1)$  is not realizable. For  $z \geq 2$ , consider a graph  $G$  obtained by identification of each end vertex of a path  $P_n$  with each one central vertex of a path  $P_{2n-4}$ . The graph so obtained is as in Figure 5.

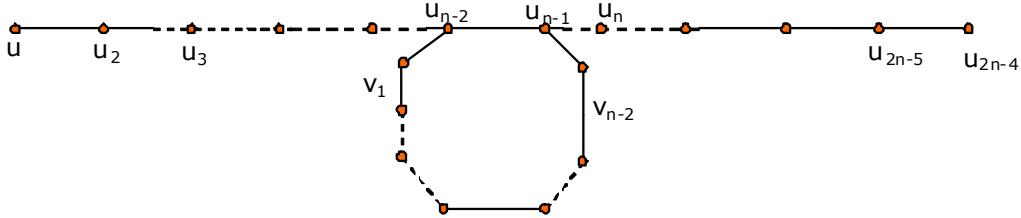


Figure 5

Label the vertices of  $G$  as in Figure 5. So  $\text{rad}(G) = n-2$ . Clearly, the edges of the form  $u_iu_{i+1}$ ,  $1 \leq i \leq 2n-4$ , are vital. Edges of the form  $v_iv_{i+1}$ ,  $v_iu_{n-2}$ ,  $v_{n-2}u_{n-1}$ , belong to  $E_{cd}(G)$  and there is no edge belonging to  $E_c(G)$  and  $E_d(G)$ . Hence by taking  $z=n-1$ , the triple  $(0, 0, z)$  is realizable, for  $z \geq 2$ . ■

**Lemma 2.14:** *The triple  $(x, 0, z)$  is realizable for all  $x \geq 0, z \geq 0$ .*

**Proof:** For  $x = z = 0$ , the realizability of  $(x, 0, z)$ -graph is ensured by *Proposition 2.1*. For  $x = 0$ , the graph constructed in the above lemma serves the purpose for  $z \geq 2$ . For  $x \geq 1$ , consider a  $(0, 0, z)$ -graph constructed in above lemma. Join "x" pendent edges at either  $u_{n-2}$  or  $u_{n-1}$  of  $G$  of *Figure 5*. Clearly these "x" edges belong to  $E_c(G)$  and there is no edge belonging to  $E_d(G)$ . Hence,  $(x, 0, z)$  is realizable,  $x \geq 0, z \geq 2$ . For  $z=1$ , the graph of the Figure 6 is the realizer.

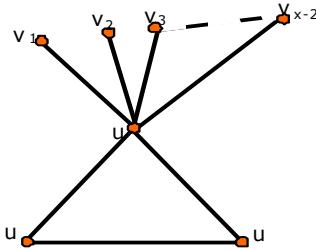


Figure 6.

Clearly, the edge  $u_1u_2$  is the only edge of  $E_{cd}(G)$  and rest all belong to  $E_c(G)$ . Hence this is  $(x, 0, 1)$ -graph. ■

**Theorem 2.15:** *The triple  $(x, y, z)$  is realizable if  $x \geq 0$ ;  $y = 2m$ ,  $m \geq 2$ , or  $y = 2k + mn$ ,  $k \geq 2$ ,  $m \geq 2$ ,  $n \geq 1$ , or  $y = mn$ ,  $m \geq 2$ ,  $n \geq 1$ ;  $z \neq 1$ .*

**Proof:** Proof follows from *Lemma 2.1 to Lemma 2.14*. The realizing graph  $G$  is as in Figure 7.

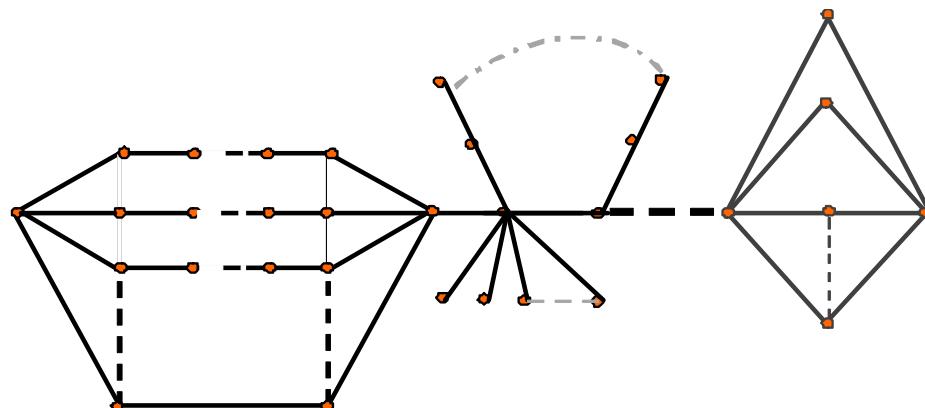


Figure 7.

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Clearly, edges of the form  $u_i u_i^l, u_i^j u_{i+1}^j, u_i^n v_i, 1 \leq i \leq m, 1 \leq j \leq n; v_{2n-4} x_i, x x_i, 1 \leq i \leq l$  belong to  $E_d(G)$ . Edges of the form  $v_{n-2} y_i, 1 \leq i \leq x$ , belong to  $E_c(G)$  and edges of the form  $v_{n-2} w_1, v_{n-1} w_{n-2}, w_i w_{i+1}; 2 \leq i \leq n-3$ , belong to  $E_{cd}(G)$ , rest all edges are vital. Hence  $G$  is an  $(x, y, z)$ -graph for the values given in the hypothesis. ■

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