Labellings of Fuzzy Graph Structures as Fuzzy Algebra of Incidence Algebra

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Abstract

The authors established a relation between incidence algebras and the $R_i$-labellings, $R_i$-index vectors, labelling matrices and index matrices of a graph structure and a relation between fuzzy algebra of an incidence algebra and the labellings and index vectors of a fuzzy graph in previous papers. Here we extend these concepts to fuzzy graph structures.

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1. Introduction

Based on the works of Brouwer[2], Doob[9] and Stewart[16], Jeurissen[12] defined index vectors, labelings and admissible index vectors of graphs. We established some relations between graph labelings and incidence algebras in [6] and extended the concepts to graph structures in [7]. Later we established relations between a fuzzy algebra of an incidence algebra and the labellings and index vectors of a fuzzy graph in [8]. Here we extend this to fuzzy graph structures.

For concepts on Graph Theory, reference may be made to [11], for fuzzy graphs to [13] and for incidence algebras, to [15] and [10].

2. Preliminaries

We recall the concept of graph structure given by Sampathkumar[14] and fuzzy graph structure given by the authors in [3].

Definition 2.1 [14]

$G = (V,R_1,R_2,...,R_k)$ is a graph structure if $V$ is a non empty set and $R_1,R_2,...,R_k$ are relations on $V$ which are mutually disjoint such that each $R_i$, $i=1,2,...,k$, is symmetric and irreflexive. If $(u,v) \in R_i$ for some $i,1 \leq i \leq k$, $(u,v)$ is an $R_i$-edge. $R_i$-path between two vertices $u$ and $v$ consists only of $R_i$-edges. $G$ is $R_1R_2...R_k$ connected if $G$ is $R_i$-connected for each $i$.

Definition 2.2 [3]
Let $G$ be a graph structure $(V,R_1,R_2,...,R_k)$ and $\mu,\rho_1,\rho_2,...,\rho_k$ be fuzzy subsets of $V,R_1,R_2,...,R_k$ respectively such that $\rho_i(x,y) \leq \mu(x) \land \mu(y) \forall x,y \in V, i=1,2,...,k$. Then $\mathcal{G} = (\mu, \rho_1, \rho_2,..., \rho_k)$ is a fuzzy graph structure of $G$.

We now recall the concepts of $R_i$-labellings and $R_i$-index vectors of a graph structure and some results obtained in [4].

**Definition 2.3 [4]**

Let $F$ be an abelian group or a ring and $G = (V,R_1,R_2,...,R_k)$ be a graph structure with vertices $v_0,v_1,...,v_{p-1}$ and $q_i$ number of $R_i$-edges. A mapping $r_i:V \rightarrow F$ is an $R_i$-index vector with components $r_i(v_0), r_i(v_1),..., r_i(v_{p-1})$, $i=1,2,...,k$ and a mapping $r_i: R_i \rightarrow F$ is an $R_i$-labelling with components $x_i(e^i_1), x_i(e^i_2),..., x_i(e^i_{p})$, $i=1,2,...,k$.

An $R_i$-labelling $x_i$ is an $R_i$-labelling for the $R_i$-index vector $r_i$ iff $r_i(v_j) = \sum_{e \in E_j} x_i(e)$. where $E_j$ is the set of all $R_i$-edges incident with $v_j$. $R_i$-index vectors for which an $R_i$-labelling exists are called admissible $R_i$-index vectors.

Now we recall the concepts of partial order, pre-order, incidence algebra etc. from [15].

**Definition 2.4 [15]**

A set $X$ with a binary relation $\leq$ is a pre-ordered set if $\leq$ is reflexive and transitive. If $\leq$ is reflexive, transitive and antisymmetric, then $X$ is a partially ordered set (poset).

Spiegel and O'Donnell [15] gives the definition of incidence algebra as follows.

**Definition 2.5 [15]**

The incidence algebra $I(X,R)$ of the locally finite partially ordered set $X$ over the commutative ring $R$ with identity is $I(X,R) = \{ f:X \times X \rightarrow R | f(x,y) = 0 \text{ if } x \text{ is not less than or equal to } y \}$ with operations given by $(f+g)(x,y) = f(x,y) + g(x,y)$ $(f.g)(x,y) = \sum_{z \in X} f(x,z).g(z,y)$ $(r.f)(x,y) = r.f(x,y)$ for $f,g \in I(X,R)$ with $r \in R$ and $x,y,z \in X$.

In [10], Foldes and Meletiou says about incidence algebra of pre-order as follows.

**Definition 2.6 [10]**

Given a field $F$, the incidence algebra $A(\rho)$, of a pre-order set $(S,\rho), S=\{1,2,...,n\}$ over $F$ is the set of maps $\alpha:S^2 \rightarrow F$ such that $\alpha(x,y)=0$ unless $x \rho y$. The addition and multiplication in $A(\rho)$ are defined as matrix sum and product.

3. $\rho_i$-labellings and $\rho_i$-index vectors of a fuzzy graph structure

Now we move on to define $\rho_i$-labellings, $\rho_i$-index vectors etc. of a fuzzy graph structure.

**Definition 3.1**

Let $\mathcal{G} = (\mu,\rho_1,\rho_2,...,\rho_k)$ be a fuzzy graph structure. Let $r_i:V \rightarrow F$ and $x_i: R_i \rightarrow F$, $i=1,2,...,k$. We have $x_i(r_i(x_i, u)) = \sup_{f \in \mu}(x_i(r_i(x_i, u))) \quad \text{and} \quad \rho_i(r_i(u)) = \sup_{r \in \mu}(r_i(r_i(u))) \mu(u)$. Then $\tilde{r}_i = (r_i, r_i(\mu))$ is a $\rho_i$-index vector of $\mathcal{G}$ if $r_i$ is an $R_i$-index vector for $\mathcal{G}^* = (\sup(\mu), \sup(\rho_1), \sup(\rho_2),..., \sup(\rho_k))$. $\tilde{x}_i = (x_i, x_i(\rho_i))$ is a $\rho_i$-labelling of $\mathcal{G}$ if $x_i$ is an $R_i$-labelling for $\mathcal{G}^* = (\sup(\mu), \sup(\rho_1), \sup(\rho_2),..., \sup(\rho_k))$. 

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Definition 3.2

For a fuzzy graph structure \( \tilde{G} = (\mu, \rho_1, \rho_2, ..., \rho_k) \),
1. \( \tilde{r}_i = (r_i, r_i(\mu)) \) is admissible if \( r_i \) is so for \( \tilde{G}^* = (\text{supp}(\mu), \text{supp}(\rho_1), \text{supp}(\rho_2), ..., \text{supp}(\rho_k)) \). Then \( \tilde{x}_i = (x_i, x_i(\rho_i)) \) is a \( \rho_i \)-labelling for \( \tilde{r}_i \).
2. \( \tilde{r}_i \) is fuzzy admissible if \( r_i(\mu)(r_i(v_i)) \geq \bigwedge_{v_j(v_i,v_j) > 0} x_i \left( \rho_j \right) x_i \left( v_j, v_j \right) \) \( \forall \ v_i \in V \).

Then \( \tilde{x}_i \) is a fuzzy \( \rho_i \)-labelling for \( \tilde{r}_i \).

In [4], we studied the operations of addition and scalar multiplication of \( R_1 \)-index vectors and \( R_1 \)-labellings of a graph structure. We introduced multiplication in [7]. We recall them now.

Let \( G = (V, R_1, R_2, ..., R_k) \) be a graph structure. \( (x_1^1 + x_2^1)(v_i, v_m) = x_1^1(v_i, v_m) + x_2^1(v_i, v_m) \)
\( (x_1^1, x_2^1)(v_i, v_m) = \sum_{(v_i, v_j), (v_j, v_m) \in R_i} x_1^1(v_i, v_j) x_2^1(v_j, v_m) \), \( \forall (v_i, v_m) \in R_i \).
\( (r^1 + r^2)(v_i) = r^1(v_i) + r^2(v_i) \)
\( (fr^1)(v_i) = f(r^1(v_i)) \)
\( (r^1, r^2)(v_i) = \sum_{(v_i, v_j), (v_j, v_m) \in R_i} r^1_i(v_j) r^2_i(v_m) \), \( \forall v_i \in V \).

Now we recall some of the results proved in [7].

Theorem 3.1 [7]

The set \( I_{I_1(A_j)}(V, F) \) of \( R_i \)-labellings for all admissible \( R_i \)-index vectors of a graph structure \( G = (V, R_1, R_2, ..., R_k) \) is a subalgebra of \( I(V, F) \) where \( A_i \) is the collection of all admissible \( R_i \)-index vectors.

Theorem 3.2 [7]

The set \( I_{I_1(A_j)}(V, F) \) of \( R_i \)-labellings for \( \lambda_i \in F \), \( \lambda_i \) an all 1 vector, of a graph structure \( G = (V, R_1, R_2, ..., R_k) \) forms a subalgebra of the incidence algebra \( I(V, F) \).

Theorem 3.3 [7]

The set \( I_{I_1(A_j)}(V, F) \) of \( R_i \)-labellings for 0 of a graph structure \( G = (V, R_1, R_2, ..., R_k) \) forms a subalgebra of the incidence algebra \( I(V, F) \).

We now establish some relation between the fuzzy \( \rho_i \)-labellings and fuzzy \( \rho_i \)-index vectors with a fuzzy algebra of the incidence algebra related with a graph structure.

Note that by a fuzzy algebra of an incidence algebra, we mean a collection of mappings from a fuzzy subset of \( V \times V \) to a fuzzy subset of \( F \) which forms a subalgebra of \( I(V, F) \).

Theorem 3.4

The set of fuzzy \( \rho_i \)-labellings \( Fl_{I_1(A_j)}(V, F) \) for fuzzy admissible \( \rho_i \)-index vectors of a fuzzy graph structure \( \tilde{G} = (\mu, \rho_1, \rho_2, ..., \rho_k) \) is a fuzzy algebra of the incidence algebra \( I(V, F) \).

Proof

Let \( \tilde{x}_i^1, \tilde{x}_i^2 \) be fuzzy \( \rho_i \)-labellings for the fuzzy admissible \( \rho_i \)-index vectors \( \tilde{r}_i^1, \tilde{r}_i^2 \). Then by definition \( x_i^1, x_i^2 \) are \( R_i \)-labellings for \( r_i^1, r_i^2 \) in \( \tilde{G}^* = (\text{supp}(\mu), \text{supp}(\rho_1), \text{supp}(\rho_2), ..., \text{supp}(\rho_k)) \). So from theorem 3.1, \( x_i^1 + x_i^2, x_i^1 x_i^2 \) and \( fx_i^1 \) are \( R_i \)-labellings for \( r_i^1 + r_i^2, r_i^1 r_i^2 \) and \( fr_i^1 \) respectively in \( \tilde{G}^* \).

Also
\[(r_1^1 + r_2^2)(u) (r_1^1 + r_2^2)(v) = \sup_{u,(r_1^1 + r_2^2)(u) = (r_1^1 + r_2^2)(v)} \mu(u) \]
\[\geq \sup_{u,r_1^1(u) = r_1^1(v),r_2^2(u) = r_2^2(v)} \mu(u) \]

But
\[r_1^1(u) = \sum_{(u,v) \in R_{iu}} x_1^1(u,v) \]
\[r_1^1(v) = \sum_{(v,m) \in R_{iv}} x_1^1(v,m) \]
\[r_2^2(u) = \sum_{(u,v) \in R_{iu}} x_2^2(u,v) \]
\[r_2^2(v) = \sum_{(v,m) \in R_{iv}} x_2^2(v,m) \]

where \(R_{iu}\) and \(R_{iv}\) are the sets of \(\rho_i\)-edges incident with \(u\) and \(v\) respectively in \(G^*\).

Hence
\[\sup_{u,r_1^1(u) = r_1^1(v),r_2^2(u) = r_2^2(v)} \mu(u) \geq \bigwedge_{\rho_1(u,v),\rho_1(v,m) > 0} \left[ \sup_{\rho_1(u,v),\rho_1(v,m) > 0} \left\{ \rho_1(u,v) | (u,v): \sum_{(u,v) \in R_{iu}} (x_1^1 + x_2^2)(u,v) \right\} = \sum_{(v,l,v,m) \in R_{iv}} (x_1^1 + x_2^2)(v_l,v_m) \right] \]

Therefore \(x_1^1 + x_2^2\) is a fuzzy \(\rho_1\)-labelling for \(r_1^1 + r_2^2\).

\[(r_1^1, r_2^2)(u) (r_1^1, r_2^2)(v) = \sup_{u,(r_1^1, r_2^2)(u) = (r_1^1, r_2^2)(v)} \mu(u) \]
\[\geq \sup_{u,(r_1^1 + r_2^2)(u) = (r_1^1 + r_2^2)(v)} \mu(u) \]

since \((r_1^1, r_2^2)(u) = \sum_{s \in (u,s)} \sup_{\rho_2(s)} r_1^1(u)r_1^1(s)\).

Hence as in the previous case,
\[\bigwedge_{\rho_1(u,v),\rho_1(v,m) > 0} \left[ \sup_{\rho_1(u,v),\rho_1(v,m) > 0} \left\{ \rho_1(u,v) | (u,v): \sum_{(u,v) \in R_{iu}} (x_1^1 \cdot x_2^2)(u,v) = \sum_{(v,l,v,m) \in R_{iv}} (x_1^1 \cdot x_2^2)(v_l,v_m) \right\} \right] \]

Therefore \(x_1^1 \cdot x_2^2\) is a fuzzy \(\rho_1\)-labelling for \(r_1^1 \cdot r_2^2\).

\((fr_1^1)(u) (fr_1^1)(v) = \sup_{u,(fr_1^1)(u) = (fr_1^1)(v)} \mu(u) \]
\[\geq \sup_{u,(fr_1^1)(u) = (fr_1^1)(v)} \mu(u) \]

As in the previous case,
\[\bigwedge_{\rho_1(u,v),\rho_1(v,m) > 0} \left[ \sup_{\rho_1(u,v),\rho_1(v,m) > 0} \left\{ \rho_1(u,v) | (u,v): \sum_{(u,v) \in R_{iu}} (fx_1^1)(u,v) = \sum_{(v,l,v,m) \in R_{iv}} (fx_1^1)(v_l,v_m) \right\} \right] \]

Therefore \(\tilde{x}_1^1\) is a fuzzy \(\rho_1\)-labelling for \((fr_1^1)\).

So the set, \(F_{I(A)} (V,F)\), of all fuzzy \(\rho_1\)-labellings for the set of all fuzzy admissible \(\rho_1\)-index vectors, is a fuzzy algebra of the incidence algebra \(I(V,F)\).
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Theorem 3.5

The set of fuzzy $\rho_1$-labellings, $F_{\rho_1}(V,F)$ for $\tilde{0}$ is a fuzzy algebra of the incidence algebra $I(V,F)$.

Proof

Let $x^1_\tilde{0}, x^2_\tilde{0}$ be fuzzy $\rho_1$-labellings for the fuzzy $\rho_1$-index vector $\tilde{0}$. Then by definition $x^1_\tilde{0}, x^2_\tilde{0}$ are $R_1$-labellings for $0$ in $\tilde{G}^*=(supp(\mu), supp(\rho_1), supp(\rho_2), ..., supp(\rho_n))$. So from theorem 3.3, $x^1_\tilde{0} + x^2_\tilde{0}, x^1_\tilde{0} \cdot x^2_\tilde{0}$ and $fx^1_\tilde{0}$ are $R_1$-labellings for $0$ in $\tilde{G}^*$.

$(0 + 0)(\mu)(0 + 0)(\nu) = 
\sup_{u:0+(0+0)(u)=(0+0)(\nu)} \mu(u) = 
\sup_{u:0(0+0)(u)=0(0+0)(\nu)} \mu(u)$. 

But $0(u) = \sum_{(u,v)\in R_{iu}} x^1_\tilde{0}(u,v) = \sum_{(u,v)\in R_{iu}} x^2_\tilde{0}(u,v)$ 
$0(\nu) = \sum_{(u,v)\in R_{iv}} x^1_\tilde{0}(v_l, v_m) = \sum_{(u,v)\in R_{iv}} x^2_\tilde{0}(v_l, v_m)$

where $R_{iu}$ and $R_{iv}$ are sets of $\rho_1$-edges incident with $u$ and $v_l$ respectively in $\tilde{G}^*$.

Hence $\sup_{u:0(0+0)(u)=0(0+0)(\nu)} \mu(u) = 
\bigwedge_{\rho_1(u,v)\neq 0(u,v)} \left[ \sup \left\{ \rho_1(u,v) | (u,v): \sum_{(u,v)\in R_{iu}} (x^1_\tilde{0} + x^2_\tilde{0})(u,v) = \sum_{(v_l,v_m)\in R_{iv}} (x^1_\tilde{0} + x^2_\tilde{0})(v_l,v_m) \right\} \right]$.

Therefore $x^1_\tilde{0} + x^2_\tilde{0}$ is a fuzzy $\rho_1$-labelling for $\tilde{0} + \tilde{0}$.

$(0.0)(\mu)(0.0)(\nu) = \sup_{u:(0.0)(u)=(0.0)(\nu)} \mu(u) \geq \sup_{u:0(0)(u)=0(0)(\nu)} \mu(u)$.

since $(0.0)(u) = \sum_{(u,v)\in supp(\rho_1)} 0(u)(v)$.

Hence as in the previous case, $(0.0)(\mu)(0.0)(\nu) \geq 
\bigwedge_{\rho_1(u,v)\neq 0(u,v)} \left[ \sup \left\{ \rho_1(u,v) | (u,v): \sum_{(u,v)\in R_{iu}} (x^1_\tilde{0} \cdot x^2_\tilde{0})(u,v) = \sum_{(v_l,v_m)\in R_{iv}} (x^1_\tilde{0} \cdot x^2_\tilde{0})(v_l,v_m) \right\} \right]$.

Therefore $x^1_\tilde{0} \cdot x^2_\tilde{0}$ is a fuzzy $\rho_1$-labelling for $\tilde{0}\tilde{0}$.

$(f0)(\mu)(f0)(\nu) = \sup_{u:(f0)(u)=(f0)(\nu)} \mu(u) \geq \sup_{u:f0(0)(u)=f0(0)(\nu)} \mu(u)$.

As in the previous case, 
$\sup_{u:f0(0)(u)=f0(0)(\nu)} \mu(u) \geq 
\bigwedge_{\rho_1(u,v)\neq f0(u,v)} \left[ \sup \left\{ \rho_1(u,v) | (u,v): \sum_{(u,v)\in R_{iu}} (fx^1_\tilde{0})(u,v) = \sum_{(v_l,v_m)\in R_{iv}} (fx^1_\tilde{0})(v_l,v_m) \right\} \right]$.

Therefore $f\tilde{x}^1_\tilde{0}$ is a fuzzy $\rho_1$-labelling for $\tilde{f0}$. 

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Hence the set of all fuzzy \( \rho_l \)-labellings for \( \hat{0} \) is a fuzzy algebra of the incidence algebra \( I(V,F) \).

**Theorem 3.6**

The set, \( \text{Fl}_{I(V,F)}(V,F) \) of fuzzy \( \rho_l \)-labellings for \( \hat{\lambda}_{J_l} \), is a fuzzy algebra of the incidence algebra \( I(V,F) \).

**Proof**

Let \( x^1_l, x^2_l \) be fuzzy \( \rho_l \)-labellings for the fuzzy admissible \( \rho_l \)-index vectors \( \hat{\lambda}_{J_l} \). Then by definition \( x^1_l, x^2_l \) are \( R_l \)-labellings for \( \lambda^1_l, \lambda^2_l \) in \( \hat{G}^* = (\text{supp}(\mu), \text{supp}(\rho), \cdots, \text{supp}(\rho_k)) \). So from theorem 3.2, \( x^1_l + x^2_l \) and \( f x^1_l \) are \( R_l \)-labellings for \( \lambda^1_l + \lambda^2_l, \lambda^1_l \cdot \lambda^2_l \) and \( f \lambda^1_l \) respectively in \( \hat{G}^* \).

\[
(\lambda^1_l + \lambda^2_l)(\mu) (\lambda^1_l + \lambda^2_l)(v_l) = \sup_{u:(\lambda^1_l + \lambda^2_l)(u) = (\lambda^1_l + \lambda^2_l)(v_l)} \mu(u) \\
\geq \sup_{u:(\lambda^1_l + \lambda^2_l)(u) = \lambda^1_l(v_l), \lambda^2_l(u) = \lambda^2_l(v_l)} \mu(u)
\]

But

\[
\lambda^1_l(u) = \sum_{(u,v) \in R_{u_l}} x^1_l(u,v) \\
\lambda^2_l(u) = \sum_{(u,v) \in R_{v_l}} x^2_l(u,v)
\]

Therefore \( x^1_l + x^2_l \) is a fuzzy \( \rho_l \)-labellings for \( \lambda^1_l + \lambda^2_l \).

\[
(\lambda^1_l, \lambda^2_l)(\mu) (\lambda^1_l, \lambda^2_l)(v_l) = \sup_{u:(\lambda^1_l, \lambda^2_l)(u) = (\lambda^1_l, \lambda^2_l)(v_l)} \mu(u) \\
\geq \sup_{u:(\lambda^1_l, \lambda^2_l)(u) = \lambda^1_l(v_l), \lambda^2_l(u) = \lambda^2_l(v_l)} \mu(u)
\]

since \( (\lambda^1_l, \lambda^2_l)(u) = \sum_{e \in (u,v)} \text{supp}(\rho) \lambda^1_l(u) \lambda^2_l(v) \).

Hence as in the previous case, \( (\lambda^1_l, \lambda^2_l)(\mu) (\lambda^1_l, \lambda^2_l)(v_l) \geq \)

\[
\sup_{u:(\lambda^1_l, \lambda^2_l)(u) = \lambda^1_l(v_l), \lambda^2_l(u) = \lambda^2_l(v_l)} \mu(u)
\]

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Therefore \( x_1^1 \cdot x_2^2 \) is a fuzzy \( \rho_i \)-labelling for \( \lambda_i^1, \lambda_i^2 \).

\[
(f\lambda_i^1)(\mu)(f\lambda_i^1)(v) = \sup_{u:(f\lambda_i^1)(u)=(f\lambda_i^1)(v)} \mu(u) \geq \sup_{u:(f\lambda_i^1)(u)=(f\lambda_i^1)(v)} \mu(u)
\]

As in the previous case,

\[
\sup_{u:(f\lambda_i^1)(u)=(f\lambda_i^1)(v)} \mu(u) \geq \bigwedge_{\rho_j(u,v), \rho_j(v,v_m)>0} \left[ \sup \left\{ \rho_i(u,v), (u,v): \sum_{(u,v) \in R_{iuv}} (f\lambda_i^1)(u,v) = \sum_{(v_l,v_m) \in R_{vlv}} (f\lambda_i^1)(v_l,v_m) \right\} \right]
\]

Therefore \( \tilde{x}_i^1 \) is a fuzzy \( \rho_i \)-labelling for \( \tilde{\lambda}_i^1 \).

So \( F_{\lambda_i^1}(V,F) \) is a fuzzy algebra of the incidence algebra \( I(V,F) \).

4. Fuzzy labellings of a fuzzy graph structure and fuzzy algebra of an incidence algebra

We now extend the results in the previous section to the whole of the fuzzy graph structure. For that first we recall some concepts and results from [5] and [7].

**Definition 4.1 [5]**

Let \( F \) be an abelian group or a ring. Let \( r_i \) be an \( R_i \)-index vector and \( x_i \) be an \( R_i \)-labelling for \( i = 1,2,\ldots,k \). Then

\[
x = \begin{bmatrix}
x_1 & 0 & 0 & \ldots & 0 \\
0 & x_2 & 0 & \ldots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & x_k
\end{bmatrix}
\]

is a labelling matrix and

\[
r = \begin{bmatrix}
r_1 & 0 & 0 & \ldots & 0 \\
0 & r_2 & 0 & \ldots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & r_k
\end{bmatrix}
\]

is an index matrix for the graph structure

\[
G = (V,R_1,R_2,\ldots,R_k).
\]

\[
x : \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_k \end{bmatrix} \rightarrow F^k
\]

is a labelling for \( r : V^k \rightarrow F^k \) if \( \sum_{m \in E_x} x_i(m) = r_i(v_s) \) for \( s = 0,1,\ldots,p-1 ; i = 1,2,\ldots,k \). If \( r_i \) is an admissible \( R_i \)-index vector \( i = 1,2,\ldots,k \), then \( r \) is called an admissible index matrix for \( G \).

**Theorem 4.1 [7]**

The set \( I_{L(A)}(V^k,F^k) \) of labelling matrices for all admissible index matrices of a graph structure \( G = (V,R_1,R_2,\ldots,R_k) \) is a subalgebra of \( I(V^k,F^k) \).

**Theorem 4.2 [7]**

The set \( I_{L(A)}(V^k,F^k) \) of labelling matrices for \( A \),
Now we define fuzzy labellings and fuzzy index matrices of a fuzzy graph structure as follows.

\[
\begin{pmatrix}
\lambda_1 & 0 & 0 & \ldots & 0 \\
0 & \lambda_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & \lambda_k \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
j_1 & 0 & 0 & \ldots & 0 \\
0 & j_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
j_i & \ldots & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & j_k \\
\end{pmatrix}
\]

\(A=\begin{pmatrix}
\sum_{i=1}^{k} j_i & \ldots & \ldots & \ldots \\
0 & \sum_{i=1}^{k} \lambda_i \cdot j_i \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \ldots & \sum_{i=1}^{k} \lambda_i \cdot j_i \\
\end{pmatrix}
\]

\(J=\begin{pmatrix}
j_1 & 0 & 0 & \ldots & 0 \\
0 & j_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
j_i & \ldots & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & j_k \\
\end{pmatrix}
\)

\(J_i \) an all 1 vector for \(i=1,2,\ldots,k\)

of a graph structure \(G=(V,R_1,R_2,\ldots,R_k)\) is a subalgebra of \(I(V^k,F^k)\).

**Theorem 4.3 [7]**

The set \(I_{L(0)}(V^k,F^k)\) of labelling matrices for 0 of a graph structure \(G=(V,R_1,R_2,\ldots,R_k)\) is a subalgebra of \(I(V^k,F^k)\).

Now we define fuzzy labellings and fuzzy index matrices of a fuzzy graph structure as follows.

**Definition 4.2**

Let \(F\) be an abelian group or a ring. \(\tau_i\) be an \(R_i\)-index vector, \(x_i\) be an \(R_i\)-labelling for \(i=1,2,\ldots,k\). Then

\[
\begin{pmatrix}
x_1^i & 0 & 0 & \ldots & 0 \\
0 & x_2^i & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & x_k^i \\
\end{pmatrix}
\]

is a labelling matrix and

\[
\begin{pmatrix}
\tilde{r}_1 & 0 & 0 & \ldots & 0 \\
0 & \tilde{r}_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & \tilde{r}_k \\
\end{pmatrix}
\]

is an index matrix of a fuzzy graph structure.

\(\tilde{x}\) is a labelling for \(\tilde{r}\) if \(\tilde{x}\) is a \(\rho_i\)-labelling for \(i=1,2,\ldots,k\). If \(\tilde{r}\) is an admissible \(\rho_i\)-index vector for \(i=1,2,\ldots,k\), \(\tilde{r}\) is an admissible index matrix for \(\tilde{G}\). If \(\tilde{r}\) is fuzzy admissible for \(i=1,2,\ldots,k\), \(\tilde{r}\) is fuzzy admissible and \(\tilde{x}\) is a fuzzy labelling.

Now we move on to prove certain results on fuzzy admissible index matrices and fuzzy labelling matrices.

**Theorem 4.4**

The set, \(FL_{L(A)}(V^k,F^k)\), of fuzzy labellings matrices for fuzzy admissible index matrices of a fuzzy graph structure \(\tilde{G}=(\mu,\rho_1,\rho_2,\ldots,\rho_k)\) is a fuzzy algebra of the incidence algebra \(I(V^k,F^k)\).

**Proof**

Let \(\tilde{x}_1,\tilde{x}_2 \in FL_{L(A)}(V^k,F^k)\). Then \(x_1^i, x_2^i \in FL_{L(A)}(V^k,F^k)\), the collection of all fuzzy \(\rho_i\)-labellings, for \(i=1,2,\ldots,k\). So there exist \(\tilde{r}_1^i, \tilde{r}_2^i \in A_i\), the collection of all fuzzy admissible \(\rho_i\)-index vectors, for \(i=1,2,\ldots,k\).

Hence \(\tilde{x}_1^i, \tilde{x}_2^i\) are fuzzy \(\rho_i\)-labellings for \(\tilde{r}_1^i, \tilde{r}_2^i\) respectively for \(i=1,2,\ldots,k\). By theorem 3.4, the set of fuzzy \(\rho_i\)-labellings for fuzzy admissible \(\rho_i\)-index vectors is a fuzzy algebra of the incidence algebra \(I(V,F)\) for \(i=1,2,\ldots,k\). So \(x_1^i + x_2^i, x_1^i \cdot x_1^i, x_2^i \cdot x_2^i\) and \(\tilde{x}_1^i\) are fuzzy \(\rho_i\)-labellings for fuzzy \(\rho_i\)-index vectors \(\tilde{r}_1^i, \tilde{r}_2^i\) respectively for \(i=1,2,\ldots,k\). So \(\tilde{r}_1^i + \tilde{r}_2^i, \tilde{r}_1^i \cdot \tilde{r}_2^i\) and \(\tilde{r}_1^i\) are fuzzy admissible index matrices and \(x_1^i + x_2^i, x_1^i \cdot x_1^i, x_2^i \cdot x_2^i\) are fuzzy labelling matrices for \(\tilde{r}_1^i + \tilde{r}_2^i, \tilde{r}_1^i \cdot \tilde{r}_2^i\) and \(\tilde{r}_1^i\) respectively. Hence \(FL_{L(A)}(V^k,F^k)\) is a fuzzy algebra of the incidence algebra \(I(V^k,F^k)\).

**Theorem 4.5**

The set of fuzzy labellings \(FL_{L(0)}(V^k,F^k)\), for fuzzy index matrix 0 of a fuzzy graph structure \(\tilde{G}=(\mu,\rho_1,\rho_2,\ldots,\rho_k)\) is a fuzzy algebra of the incidence algebra \(I(V^k,F^k)\).

**Proof**

Let \(\tilde{x}_1, \tilde{x}_2 \in FL_{L(0)}(V^k,F^k)\). Then \(x_1^i, x_1^i \in FL_{L(0)}(V^k,F^k)\), the collection of all fuzzy \(\rho_i\)-labellings for 0, for \(i=1,2,\ldots,k\).
By theorem 3.5, the set of fuzzy $\rho_i$-labellings for fuzzy $\rho_i$ index vector $\bar{0}$ is a fuzzy algebra of the incidence algebra $I(V,F)$ for $i=1,2,...,k$. So $x_1^i + x_2^i$, $x_1^i$, $x_2^i$ and $\bar{f}_1^i$ are fuzzy $\rho_i$-labellings for fuzzy $\rho_i$-index vectors $\bar{0} + \bar{0}$, $\bar{0}$, $\bar{0}$ respectively for $i=1,2,...,k$. So $x_1^i$, $x_2^i$ are fuzzy labelling matrices for $\bar{0}$. Hence $F_{I(0)}(V^k, F^k)$ is a fuzzy algebra of the incidence algebra $I(V^k, F^k)$.

**Theorem 4.6**

The set of fuzzy labellings $F_{I(\Lambda)}(V^k, F^k)$ for fuzzy index matrix $\Lambda$ of a fuzzy graph structure $\bar{G} = (\mu, \rho_1, \rho_2, ..., \rho_k)$ is a fuzzy algebra of the incidence algebra $I(V^k, F^k)$.

**Proof**

Let $\bar{x}_1^i, \bar{x}_2^i \in F_{I(\Lambda)}(V^k, F^k)$. Then $x_1^i, x_2^i \in F_{I(\Lambda)}(V^k, F^k)$, the collection of all fuzzy $\rho_i$-labelings for $\lambda_i$, for $i=1,2,...,k$.

By theorem 3.6, the set of fuzzy $\rho_i$-labellings for fuzzy $\rho_i$-index vectors $\lambda_i$, is a fuzzy algebra of the incidence algebra $I_{\rho_i}(V, F)$ for $i=1,2,...,k$. So $x_1^i + x_2^i$, $x_1^i$, $x_2^i$ and $\bar{f}_1^i$ are fuzzy $\rho_i$-labellings for fuzzy $\rho_i$-index vectors $\lambda^i_1 + \lambda^i_2$, $\lambda^i_1$, $\lambda^i_2$ and $\bar{f}_1^i$, respectively for $i=1,2,...,k$. So $x_1^i$, $x_2^i$ are fuzzy labelling matrices for $\Lambda$. Hence $F_{I(\Lambda)}(V^k, F^k)$ is a fuzzy algebra of the incidence algebra $I(V^k, F^k)$.

**References**