Int. J. of Mathematical Sciences and Applications,
Vol. 1, No. 3, September 2011
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Labellings of Fuzzy Graph Structures as Fuzzy Algebra of Incidence Algebra

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Abstract

The authors established a relation between incidence algebras and the R_i-labellings, R_i-index vectors, labelling matrices and index matrices of a graph structure and a relation between fuzzy algebra of an incidence algebra and the labellings and index vectors of a fuzzy graph in previous papers. Here we extend these concepts to fuzzy graph structures.

Subject classification: *Primary 05C72, Secondary 05C78,06A11.*

Keywords*: Fuzzy graph structures, R_i-labelling,R_i-index vector, Labelling matrix, Index matrix, Fuzzy algebra of incidence algebra.*

1. Introduction

Based on the works of Brouwer[2], Doob[9] and Stewart[16], Jeurissen[12] defined index vectors, labelings and admissible index vectors of graphs. We established some relations between graph labelings and incidence algebras in [6] and extended the concepts to graph structures in [7]. Later we established relations between a fuzzy algebra of an incidence algebra and the labellings and index vectors of a fuzzy graph in [8]. Here we extend this to fuzzy graph structures.

For concepts on Graph Theory, reference may be made to [11], for fuzzy graphs to [13] and for incidence algebras, to [15] and [10].

2.Preliminaries

We recall the concept of graph structure given by Sampathkumar[14] and fuzzy graph structure given by the authors in [3].

Definition 2.1 [14]

$G = (V,R_1,R_2,...,R_k)$ is a graph structure if V is a non empty set and $R_1,R_2,...,R_k$ are relations on V which are mutually disjoint such that each R_i, $i=1,2,...,k$, is symmetric and irreflexive.
If $(u,v)\in R_i$ for some $i, 1 \le i \le k$, (u,v) is an R_i-edge.
R_i-path between two vertices u and v consists only of R_i-edges.
G is $R_1R_2...R_k$ connected if G is R_i-connected for each i.

Definition 2.2 [3]

Let G be a graph structure $(V,R_1,R_2,...,R_k)$ and $\mu,\rho_1, \rho_2,..., \rho_k$ be fuzzy subsets of $V,R_1,R_2,...,R_k$ respectively such that $\rho_i (x,y) \leq \mu (x)\wedge\mu(y)\ \forall\ x,y \in V$, i=1,2,...,k. Then
$\tilde{G} = (\mu, \rho_1, \rho_2,... \rho_k)$ is a fuzzy graph structure of G.

We now recall the concepts of R_i -labellings and R_i -index vectors of a graph structure and some results obtained in [4].

Definition 2.3 [4]

Let F be an abelian group or a ring and G = $(V,R_1,R_2,...,R_k)$ be a graph structure with vertices $v_0,v_1,...,v_{p-1}$ and q_i number of R_i-edges. A mapping r_i:V→ F is an R_i-index vector with components $r_i(v_0)$, $r_i(v_1)$,..., $r_i(v_{p-1})$, i=1,2,...,k and a mapping r_i: R_i → F is an R_i -labelling with components $x_i(e_i^1)$,$x_i(e_i^2)$,..., $x_i(e_i^{q_i})$, i=1,2,...,k.

An R_i -labelling x_i is an R_i -labelling for the R_i -index vector r_i iff $r_i(v_j)=\sum_{e_r \in E_i^j} x_i(e_r)$ where E_i^j is the set of all R_i -edges incident with v_j.
R_i -index vectors for which an R_i -labelling exists are called admissible R_i -index vectors.

Now we recall the concepts of partial order, pre-order, incidence algebra etc. from [15].

Definition 2.4 [15]

A set X with a binary relation ≤ is a pre-ordered set if ≤ is reflexive and transitive. If ≤ is reflexive, transitive and antisymmetric, then X is a partially ordered set (poset).

Spiegel and O'Donnell [15] gives the definition of incidence algebra as follows.

Definition 2.5 [15]

The incidence algebra I(X,R) of the locally finite partially ordered set X over the commutative ring R with identity is I(X,R)={f:X x X→ R|f(x,y)=0 if x is not less than or equal to y} with operations given by
$(f+g)(x,y) = f(x,y)+g(x,y)$
$(f.g)(x,y) = \sum_{x\leq z\leq y} f(x,z).g(z,y)$
$(r.f)(x,y) = r.f(x,y)$
for f,g∈ I(X,R) with r∈ R and x,y,z ∈ X.

In [10], Foldes and Meletiou says about incidence algebra of pre-order as follows.

Definition 2.6 [10]

Given a field F, the incidence algebra A(ρ), of a pre-order set (S,ρ),S={1,2,...,n} over F is the set of maps $\alpha:S^2\to F$ such that α (x,y)=0 unless x ρ y. The addition and multiplication in A(ρ) are defined as matrix sum and product.

3. ρ_i -labellings and ρ_i -index vectors of a fuzzy graph structure

Now we move on to define ρ_i -labellings, ρ_i -index vectors etc. of a fuzzy graph structure.

Definition 3.1

Let $\tilde{G} = (\mu,\rho_1, \rho_2,... \rho_k)$ be a fuzzy graph structure. Let r_i:V→F and x_i: R_i →F, i=1,2,...,k. We have $x_i(\rho_i)(x_i(u,v))=\sup_{(e,f)\in x_i^{-1}(x_i(u,v))} r_i(e,f)$ and $\rho_i(\mu)(r_i(u))= \sup_{v\in r_i^{-1}(r_i(u))} \mu(v)$. Then $\tilde{r_i}=(r_i,r_i(\mu))$ is a ρ_i-index vector of $\tilde{G}$ if r_i is an R_i -index vector for $\tilde{G}^*$ =(supp(μ),supp(ρ_1),supp(ρ_2),...,supp(ρ_k)). $\tilde{x}_i=(x_i, x_i(\rho_i))$ is a ρ_i -labelling of $\tilde{G}$ if x_i is an R_i -index vector for $\tilde{G}^*$ =(supp(μ),supp(ρ_1),supp(ρ_2),...,supp(ρ_k)).

Definition 3.2

For a fuzzy graph structure $\tilde{G}$ = (μ,ρ_1, ρ_2,... ρ_k),
1. $\widetilde{r_i}$=(r_i,r_i (μ)) is admissible if r_i is so for $\tilde{G}^*$ =(supp(μ),supp(ρ_1),supp(ρ_2),...,supp(ρ_k)). Then $\tilde{x}_i$=(x_i, x_i (ρ_i)) is a ρ_i -labelling for $\widetilde{r_i}$.
2. $\widetilde{r_i}$ is fuzzy admissible if $r_i(\mu)(r_i(v_i)) \geq \wedge_{\rho_i(v_i,v_j)>0} x_i(\rho_i)\, x_i\left((v_i,v_j)\right)$ $\forall v_i \in V$.
Then $\tilde{x}_i$ is a fuzzy ρ_i-labelling for $\widetilde{r_i}$.

In [4], we studied the operations of addition and scalar multiplication of R_i -index vectors and R_i -labellings of a graph structure. We introduced multiplication in [7]. We recall them now.
Let G = (V,R_1,R_2,...,R_k) be a graph structure. $(x_i^1 + x_i^2)(v_l,v_m) = x_i^1(v_l,v_m) + x_i^2(v_l,v_m)$
$(fx_i^1)(v_l,v_m) = f(x_i^1(v_l,v_m))$
$(x_i^1.x_i^2)(v_l,v_m) = \sum_{(v_l,v_s),(v_s,v_m)\in E} x_i^1(v_l,v_s)\, x_i^2(v_s,v_m)$ $\forall (v_l,v_m) \in R_i$.
$(r_i^1 + r_i^2)(v_l) = r_i^1(v_l) + r_i^2(v_l)$
$(fr_i^1)(v_l) = f(r_i^1(v_l))$
$(r_i^1.r_i^2)(v_l) = \sum_{(v_l,v_s)\in E} r_i^1(v_l)\, r_i^2(v_s)$ $\forall v_l \in V$.
Now we recall some of the results proved in [7].

Theorem 3.1 [7]

The set $I_{L(A_i)}$ (V,F) of R_i -labellings for all admissible R_i -index vectors of a graph structure G = (V,R_1,R_2,...,R_k) is a subalgebra of I(V,F) where A_i is the collection of all admissible R_i -index vectors.

Theorem 3.2 [7]

The set $I_{L(\lambda_i)}$ (V,F) of R_i -labellings for $\lambda_i j_i$, $\lambda_i \in F$, j_i an all 1 vector, of a graph structure G = (V,R_1,R_2,...,R_k) forms a subalgebra of the incidence algebra I(V,F).

Theorem 3.3 [7]

The set $I_{L(0_i)}$ (V,F) of R_i-labellings for 0 of a graph structure G = (V,R_1,R_2,...,R_k) forms a subalgebra of the incidence algebra I(V,F).

We now establish some relation between the fuzzy ρ_i -labellings and fuzzy ρ_i -index vectors with a fuzzy algebra of the incidence algebra related with a graph structure.

Note that by a fuzzy algebra of an incidence algebra, we mean a collection of mappings from a fuzzy subset of V x V to a fuzzy subset of F which forms a subalgebra of I(V,F).

Theorem 3.4

The set of fuzzy ρ_i -labellings $FI_{L(A_i)}$ (V,F) for fuzzy admissible ρ_i -index vectors of a fuzzy graph structure $\tilde{G}$ = (μ,ρ_1, ρ_2,... ρ_k) is a fuzzy algebra of the incidence algebra I(V,F).

Proof

Let $\widetilde{x_i^1}, \widetilde{x_i^2}$ be fuzzy ρ_i -labellings for the fuzzy admissible ρ_i -index vectors $\widetilde{r_i^1}$, $\widetilde{r_i^2}$. Then by definition x_i^1, x_i^2 are R_i -labellings for r_i^1, r_i^2 in $\tilde{G}^*$=(supp(μ),supp(ρ_1),supp(ρ_2),...,supp(ρ_k)). So from theorem 3.1, $x_i^1 + x_i^2$, $x_i^1.x_i^2$ and fx_i^1 are R_i -labellings for $r_i^1 + r_i^2$, $r_i^1.r_i^2$ and fr_i^1 respectively in $\tilde{G}^*$.

Also

$(r_i^1+r_i^2)(\mu)\ (r_i^1+r_i^2)(v_l) = \sup_{u:(r_i^1+r_i^2)(u)=(r_i^1+r_i^2)(v_l)} \mu(u)$
$\geq \sup_{u:r_i^1(u)=r_i^1(v_l),r_i^2(u)=r_i^2(v_l)} \mu(u)$

But
$r_i^1(u)=\sum_{(u,v)\in R_{iu}} x_i^1(u,v)$
$r_i^1(v_l)=\sum_{(u,v)\in R_{iv_l}} x_i^1(v_l,v_m)$
$r_i^2(u)=\sum_{(u,v)\in R_{iu}} x_i^2(u,v)$
$r_i^2(v_l)=\sum_{(u,v)\in R_{iv_l}} x_i^2(v_l,v_m)$

where R_{iu} and $R_{iv_l}$are the sets of ρ_i -edges incident with u and v_l respectively in $\tilde{G}^*$.
Hence
$\sup_{u:r_i^1(u)=r_i^1(v_l),r_i^2(u)=r_i^2(v_l)} \mu(u)\geq$

$$\bigwedge_{\rho_i(u,v),\rho_i(v_l,v_m)>0}\left[\sup\left\{\rho_i(u,v)|(u,v): \sum_{(u,v)\in R_{iu}}(x_i^1+x_i^2)(u,v) = \sum_{(v_l,v_m)\in R_{iv_l}}(x_i^1+x_i^2)(v_l,v_m)\right\}\right]$$

Therefore $\widetilde{x_i^1+x_i^2}$ is a fuzzy ρ_i-labelling for $\widetilde{r_i^1+r_i^2}$.

$(r_i^1.r_i^2)(\mu)\ (r_i^1.r_i^2)(v_l) = \sup_{u:(r_i^1.r_i^2)(u)=(r_i^1.r_i^2)(v_l)} \mu(u)$

$$\geq \sup_{u:r_i^1(u)=r_i^1(v_l),r_i^2(u)=r_i^2(v_l)} \mu(u)$$

since $(r_i^1.r_i^2)(u)=\sum_{s:(u,s)\in supp(\rho_i)} r_i^1(u)r_i^2(s)$.
Hence as in the previous case, $(r_i^1.r_i^2)(\mu)\ (r_i^1.r_i^2)(v_l)\geq$

$$\bigwedge_{\rho_i(u,v),\rho_i(v_l,v_m)>0}\left[\sup\left\{\rho_i(u,v)|(u,v): \sum_{(u,v)\in R_{iu}}(x_i^1.x_i^2)(u,v) = \sum_{(v_l,v_m)\in R_{iv_l}}(x_i^1.x_i^2)(v_l,v_m)\right\}\right]$$

Therefore $\widetilde{x_i^1.x_i^2}$ is a fuzzy ρ_i-labelling for $\widetilde{r_i^1.r_i^2}$.
$(fr_i^1)(\mu)\ (fr_i^1)(v_l) = \sup_{u:(fr_i^1)(u)=(fr_i^1)(v_l)} \mu(u)$

$$\geq \sup_{u:fr_i^1(u)=fr_i^1(v_l)} \mu(u)$$

As in the previous case,
$\sup_{u:fr_i^1(u)=fr_i^1(v_l)} \mu(u)\geq$

$$\bigwedge_{\rho_i(u,v),\rho_i(v_l,v_m)>0}\left[sup\left\{\rho_i(u,v)|(u,v): \sum_{(u,v)\in R_{iu}}(fx_i^1)(u,v) = \sum_{(v_l,v_m)\in R_{iv_l}}(fx_i^1)(v_l,v_m)\right\}\right]$$

Therefore $\widetilde{fx_i^1}$ is a fuzzy ρ_i-labelling for $\widetilde{fr_i^1}$.

So the set, $FI_{L(A_i)}$ (V,F), of all fuzzy ρ_i-labellings for the set of all fuzzy admissible ρ_i-index vectors, is a fuzzy algebra of the incidence algebra I(V,F).

Theorem 3.5

The set of fuzzy ρ_i-labellings, F$I_{\mathrm{L}(0_i)}$ (V,F) for $\tilde{0}$ is a fuzzy algebra of the incidence algebra I(V,F).

Proof

Let $\widetilde{x_i^1}, \widetilde{x_i^2}$ be fuzzy ρ_i-labellings for the fuzzy ρ_i-index vector $\tilde{0}$. Then by definition x_i^1, x_i^2 are R_i -labellings for 0 in $\tilde{G}^*$ =(supp(μ),supp(ρ_1),supp(ρ_2),...,supp(ρ_k)). So from theorem 3.3, $x_i^1 + x_i^2$, $x_i^1 . x_i^2$ and f$x_i^1$are R_i -labellings for 0 in $\tilde{G}^*$.

$(0+0)(\mu)$ $(0+0)(v_l)$ =

$$\sup_{u:(0+0)(u)=(0+0)(v_l)} \mu(u)$$

$$= \sup_{u:0(u)=0(v_l),0(u)=0(v_l)} \mu(u)$$

But

$0(u)=\sum_{(u,v)\in \mathrm{R_{iu}}} x_i^1(u,v) = \sum_{(u,v)\in \mathrm{R_{iu}}} x_i^2(u,v)$

$0(v_l)=\sum_{(u,v)\in \mathrm{R}_{\mathrm{i}v_l}} x_i^1(v_l,v_m) = \sum_{(u,v)\in \mathrm{R}_{\mathrm{i}v_l}} x_i^2(v_l,v_m)$

where $\mathrm{R_{iu}}$ and $\mathrm{R}_{\mathrm{i}v_l}$ are sets of ρ_i-edges incident with u and v_l respectively in $\tilde{G}^*$.

Hence $\sup_{u:0(u)=0(v_i),0(u)=0(v_i)} \mu(u)$=

$$\bigwedge_{\rho_i(u,v),\rho_i(v_l,v_m)>0}\left[sup\left\{\rho_i(u,v)|(u,v): \sum_{(u,v)\in \mathrm{R_{iu}}}(x_i^1+x_i^2)(u,v) = \sum_{(v_l,v_m)\in \mathrm{R_{iv_l}}}(x_i^1+x_i^2)(v_l,v_m)\right\}\right]$$

Therefore $\widetilde{x_i^1+x_i^2}$ is a fuzzy ρ_i-labelling for $\widetilde{0+0}$.\\

$(0.0)(\mu)$ $(0.0)(v_l)$ = $\sup_{u:(0.0)(u)=(0.0)(v_l)} \mu(u)$

$$\geq \sup_{u:0(u)=0(v_l)0(u)=0(v_l)} \mu(u)$$

since (0.0)(u) $=\sum_{s:(u,s)\in supp(\rho_i)} 0(u)0(s)$.

Hence as in the previous case, (0.0)(μ) (0.0)(v_l) ≥

$$\bigwedge_{\rho_i(u,v),\rho_i(v_l,v_m)>0}\left[sup\left\{\rho_i(u,v)|(u,v): \sum_{(u,v)\in \mathrm{R_{iu}}}(x_i^1.x_i^2)(u,v) = \sum_{(v_l,v_m)\in \mathrm{R_{iv_l}}}(x_i^1.x_i^2)(v_l,v_m)\right\}\right]$$

Therefore $\widetilde{x_i^1.x_i^2}$ is a fuzzy ρ_i-labelling for $\widetilde{0.0}$.

$(f0)(\mu)$ $(f0)(v_l)$ = $\sup_{u:(f0)(u)=(f0)(v_l)} \mu(u)$

$$\geq \sup_{u:f0(u)=f0(v_l)} \mu(u)$$

As in the previous case,

$\sup_{u:f0(u)=f0(v_l)} \mu(u) \geq$

$$\bigwedge_{\rho_i(u,v),\rho_i(v_l,v_m)>0}\left[sup\left\{\rho_i(u,v)|(u,v): \sum_{(u,v)\in \mathrm{R_{iu}}}(fx_i^1)(u,v) = \sum_{(v_l,v_m)\in \mathrm{R_{iv_l}}}(fx_i^1)(v_l,v_m)\right\}\right]$$

Therefore $\widetilde{fx_i^1}$ is a fuzzy ρ_i-labelling for $\widetilde{f0}$.

Hence the set of all fuzzy ρ_i-labellings for $\tilde{0}$ is a fuzzy algebra of the incidence algebra (I(V,F).

Theorem 3.6

The set, $FI_{L(\lambda_i)}$ (V,F) of fuzzy ρ_i-labellings for $\widetilde{\lambda_i J_i}$ is a fuzzy algebra of the incidence algebra I(V,F).

Proof

Let $\widetilde{x_i^1}, \widetilde{x_i^2}$ be fuzzy ρ_i--labellings for the fuzzy admissible ρ_i--index vectors $\widetilde{\lambda_i^1 J_i}$ $\widetilde{\lambda_i^2 J_i}$. Then by definition x_i^1, x_i^2 are R_i -labellings for $\lambda_i^1 j_i$, $\lambda_i^2 j_i$ in $\tilde{G}^*$ =(supp(μ),supp(ρ_1),supp(ρ_2),...,supp(ρ_k)). So from theorem 3.2, $x_i^1 + x_i^2$, $x_i^1 . x_i^2$ and fx_i^1 are R_i -labellings for $\lambda_i^1 + \lambda_i^2$, $\lambda_i^1 . \lambda_i^2$ and $f\lambda_i^1$ respectively in $\tilde{G}^*$.

$(\lambda_i^1 + \lambda_i^2)(\mu)\ (\lambda_i^1 + \lambda_i^2)(v_l) =$

$$\sup_{u:(\lambda_i^1 + \lambda_i^2)(u) = (\lambda_i^1 + \lambda_i^2)(v_l)} \mu(u)$$

$$\geq \sup_{u::\lambda_i^1(u) = \lambda_i^1(v_l), \lambda_i^2(u) = \lambda_i^2(v_l)} \mu(u)$$

But

$\lambda_i^1(u) = \sum_{(u,v)\in R_{iu}} x_i^1(u,v)$

$\lambda_i^1(v_l) = \sum_{(u,v)\in R_{iv_l}} x_i^1(v_l,v_m)$

$\lambda_i^2(u) = \sum_{(u,v)\in R_{iu}} x_i^2(u,v)$

$\lambda_i^2(v_l) = \sum_{(u,v)\in R_{iv_l}} x_i^2(v_l,v_m)$

where R_{iu} and R_{iv_l} are the sets of ρ_i -edges incident with u and v_l respectively in $\tilde{G}^*$.

Hence

$\sup_{u:\lambda_i^1(u) = \lambda_i^1(v_i), \lambda_i^2(u) = \lambda_i^2(v_i)} \mu(u) \geq$

$$\bigwedge_{\rho_i(u,v),\rho_i(v_l,v_m)>0} \left[sup \left\{ \rho_i(u,v) | (u,v): \sum_{(u,v)\in R_{iu}} (x_i^1 + x_i^2)(u,v) = \sum_{(v_l,v_m)\in R_{iv_l}} (x_i^1 + x_i^2)(v_l,v_m) \right\} \right]$$

Therefore $\widetilde{x_i^1 + x_i^2}$ is a fuzzy ρ_i-labelling for $\widetilde{\lambda_i^1 + \lambda_i^2}$.

$(\lambda_i^1 . \lambda_i^2)(\mu)\ (\lambda_i^1 . \lambda_i^2)(v_l) = \sup_{u:(\lambda_i^1 . \lambda_i^2)(u) = (\lambda_i^1 . \lambda_i^2)(v_l)} \mu(u)$

$$\geq \sup_{u:\lambda_i^1(u) = \lambda_i^1(v_l), \lambda_i^2(u) = \lambda_i^2(v_l)} \mu(u)$$

since $(\lambda_i^1 . \lambda_i^2)(u) = \sum_{s:(u,s)\in supp(\rho_i)} \lambda_i^1(u) \lambda_i^2(s)$.

Hence as in the previous case, $(\lambda_i^1 . \lambda_i^2)(\mu)\ (\lambda_i^1 . \lambda_i^2)(v_l) \geq$

$$\bigwedge_{\rho_i(u,v),\rho_i(v_l,v_m)>0} \left[sup \left\{ \rho_i(u,v) | (u,v): \sum_{(u,v)\in R_{iu}} (x_i^1 . x_i^2)(u,v) = \sum_{(v_l,v_m)\in R_{iv_l}} (x_i^1 . x_i^2)(v_l,v_m) \right\} \right]$$

Therefore $\widetilde{x_i^1 . x_i^2}$ is a fuzzy ρ_i-labelling for $\widetilde{\lambda_i^1 . \lambda_i^2}$.

$(f\lambda_i^1)(\mu)\ (f\lambda_i^1)(v_l) = \sup_{u:(f\lambda_i^1)(u)=(f\lambda_i^1)(v_l)} \mu(u)$

$$\geq \sup_{u: f\lambda_i^1(u)=f\lambda_i^1(v_l)} \mu(u)$$

As in the previous case,
$\sup_{u: f\lambda_i^1(u)=f\lambda_i^1(v_l)} \mu(u) \geq$

$$\bigwedge_{\rho_i(u,v),\rho_i(v_l,v_m)>0} \left[sup \left\{ \rho_i(u,v) | (u,v): \sum_{(u,v)\in R_{iu}} (fx_i^1)(u,v) = \sum_{(v_l,v_m)\in R_{iv_l}} (fx_i^1)(v_l,v_m) \right\} \right]$$

Therefore $\widetilde{fx_i^1}$ is a fuzzy ρ_i-labelling for $\widetilde{f\lambda_i^1}$.

So $FI_{L(\lambda_i)}$ (V,F) is a fuzzy algebra of the incidence algebra I(V,F).

4. Fuzzy labellings of a fuzzy graph structure and
fuzzy algebra of an incidence algebra

We now extend the results in the previous section to the whole of the fuzzy graph structure. For that first we recall some concepts and results from [5] and [7].

Definition 4.1 [5]

Let F be an abelian group or a ring. Let r_i be an R_i -index vector and x_i be an R_i -labelling for i=1,2,...,k. Then

$x = \begin{bmatrix} x_1 & 0 & 0 \ldots & 0 \\ 0 & x_2 & 0 \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & 0 \\ 0 & 0 \ldots & 0 & x_k \end{bmatrix}$ is a labelling matrix and $r = \begin{bmatrix} r_1 & 0 & 0 \ldots & 0 \\ 0 & r_2 & 0 \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & 0 \\ 0 & 0 \ldots & 0 & r_k \end{bmatrix}$ is an index matrix for the graph structure G = (V,R_1,R_2,...,R_k).

$x : \begin{bmatrix} R_1 \\ R_2 \\ \cdot \\ \cdot \\ \cdot \\ R_k \end{bmatrix} \to F^k$ is a labelling for r: $V^k \to F^k$ if $\sum_{m\in E_s} x_i(m) = r_i(v_s)$ for s=0,1,...,p-1; i=1,2,...,k. If r_i is an admissible R_i-index vector i=1,2,...,k, then r is called an admissible index matrix for G.

Theorem 4.1 [7]

The set $I_{L(A)}(V^k, F^k)$ of labelling matrices for all admissible index matrices of a graph structure G = (V,R_1,R_2,...,R_k) is a subalgebra of $I(V^k, F^k)$.

Theorem 4.2 [7]
The set $I_{L(\lambda)}(V^k, F^k)$ of labelling matrices for$\Lambda\, J$,

$\Lambda=\begin{bmatrix} \lambda_1 \; 0 \;\; 0 \ldots \; 0 \\ 0 \;\; \lambda_2 \; 0 \ldots \; 0 \\ \ldots \ldots \ldots \ldots \ldots \\ \ldots \ldots \ldots \ldots \ldots . \\ 0 \;\; \ldots \ldots \ldots .0 \\ 0 \;\; 0 \;\; \ldots \;\; 0 \;\; \lambda_k \end{bmatrix}$, J=$\begin{bmatrix} j_1 \; 0 \;\; 0 \ldots \; 0 \\ 0 \;\; j_2 \; 0 \ldots \; 0 \\ \ldots \ldots \ldots \ldots \ldots \\ \ldots \ldots \ldots \ldots \ldots . \\ 0 \;\; \ldots \ldots \ldots .0 \\ 0 \;\; 0 \;\; \ldots \;\; 0 \;\; j_k \end{bmatrix}$, j_i an all 1 vector for i=1,2,...,k

of a graph structure G = (V,R_1,R_2,...,R_k) is a subalgebra of $I(V^k, F^k)$.

Theorem 4.3 [7]

The set $I_{L(0)}(V^k, F^k)$ of labelling matrices for 0 of a graph structure G = (V,R_1,R_2,...,R_k) is a subalgebra of $I(V^k, F^k)$.

Now we define fuzzy labellings and fuzzy index matrices of a fuzzy graph structure as follows.

Definition 4.2

Let F be an abelian group or a ring. r_i be an R_i -index vector, x_i be an R_i -labelling for i=1,2,...,k. Then

$\tilde{x}=\begin{bmatrix} \widetilde{x_1} \; 0 \;\; 0 \ldots \; 0 \\ 0 \;\; \widetilde{x_2} \; 0 \ldots \; 0 \\ \ldots \ldots \ldots \ldots \ldots \\ \ldots \ldots \ldots \ldots \ldots . \\ 0 \;\; \ldots \ldots \ldots .0 \\ 0 \;\; 0 \;\; \ldots \;\; 0 \;\; \widetilde{x_k} \end{bmatrix}$ is a labelling matrix and $\tilde{r}=\begin{bmatrix} \widetilde{r_1} \; 0 \;\; 0 \ldots \; 0 \\ 0 \;\; \tilde{r} \; 0 \ldots \; 0 \\ \ldots \ldots \ldots \ldots \ldots \\ \ldots \ldots \ldots \ldots \ldots . \\ 0 \;\; \ldots \ldots \ldots .0 \\ 0 \;\; 0 \;\; \ldots \;\; 0 \;\; \widetilde{r_k} \end{bmatrix}$ is an index matrix of a fuzzy graph structure.

$\tilde{x}$ is a labelling for $\tilde{r}$ if $\widetilde{x_\iota}$ is a ρ_i-labelling for $\widetilde{r_\iota}$, i=1,2,...,k.
If $\widetilde{r_\iota}$ is an admissible ρ_i-index vector for i=1,2,...,k, $\tilde{r}$ is an admissible index matrix for $\tilde{G}$. If $\widetilde{r_\iota}$ is fuzzy admissible for i=1,2,...,k, $\tilde{r}$ is fuzzy admissible and $\tilde{x}$ is a fuzzy labelling.

Now we move on to prove certain results on fuzzy admissible index matrices and fuzzy labelling matrices.

Theorem 4.4
The set, F$I_{L(A)}$ (V^k,F^k), of fuzzy labelling matrices for fuzzy admissible index matrices of a fuzzy graph structure $\tilde{G}$ = (μ,ρ_1, ρ_2,... ρ_k) is a fuzzy algebra of the incidence algebra I (V^k,F^k).
Proof
Let $\widetilde{x_1}$, $\widetilde{x_2}$ ∊ F$I_{L(A)}$ (V^k, F^k). Then $\widetilde{x_\iota^1}, \widetilde{x_\iota^2}$ ∊ F$I_{L(A_i)}$ (V^k, F^k), the collection of all fuzzy ρ_i-labellings, for i=1,2,...,k. So there exist $\widetilde{r_\iota^1}, \widetilde{r_\iota^2} \in A_i$, the collection of all fuzzy admissible ρ_i--index vectors, for i=1,2,...,k.

Hence $\widetilde{x_\iota^1}, \widetilde{x_\iota^2}$ are fuzzy ρ_i-labellings for $\widetilde{r_\iota^1}, \widetilde{r_\iota^2}$respectively for i=1,2,...,k. By theorem 3.4, the set of fuzzy ρ_i-labellings for fuzzy admissible ρ_i-index vectors is a fuzzy algebra of the incidence algebra I(V,F) for i=1,2,...,k. So $\widetilde{x_\iota^1 + x_\iota^2}$, $\widetilde{x_\iota^1 . x_\iota^2}$ and f$\widetilde{x_\iota^1}$ are fuzzy ρ_i-labellings for fuzzy ρ_i-index vectors $\widetilde{r_\iota^1 + r_\iota^2}$, $\widetilde{r_\iota^1 . r_\iota^2}$ and f$\widetilde{r_\iota^1}$ respectively for i=1,2,...,k. So $\widetilde{r_1 + r_2}$, $\widetilde{r_1 . r_2}$ and f$\widetilde{r_1}$are fuzzy admissible index matrices and $\widetilde{x_1 + x_2}$, $\widetilde{x_1 . x_2}$ and f$\widetilde{x_1}$ are fuzzy labelling matrices for $\widetilde{r_1 + r_2}$, $\widetilde{r_1 . r_2}$ and f$\widetilde{r_1}$ respectively. Hence F$I_{L(A)}$ (V^k, F^k) is a fuzzy algebra of the incidence algebra I (V^k,F^k).

Theorem 4.5

The set of fuzzy labellings F$I_{L(0)}$ (V^k,F^k), for fuzzy index matrix 0 of a fuzzy graph structure $\tilde{G}$ = (μ,ρ_1, ρ_2,... ρ_k) is a fuzzy algebra of the incidence algebra I (V^k,F^k).

Proof
Let $\widetilde{x_1}$, $\widetilde{x_2}$ ∊ F$I_{L(0)}$ (V^k, F^k). Then $\widetilde{x_\iota^1}, \widetilde{x_\iota^2}$ ∊ F$I_{L(0_i)}$ (V^k, F^k), the collection of all fuzzy ρ_i-labellings for 0, for i=1,2,...,k.

By theorem 3.5, the set of fuzzy $\rho_i$labellings for fuzzy ρ_i −index vector $\tilde{0}$ is a fuzzy algebra of the incidence algebra I(V,F) for i=1,2,...,k. So $\widetilde{x_i^1 + x_i^2}$, $\widetilde{x_i^1 . x_i^2}$ and $\widetilde{fx_i^1}$ are fuzzy ρ_i-labellings for fuzzy ρ_i-index vectors $\widetilde{0 + 0}$, $\widetilde{0.0}$ and $f\tilde{0}$ respectively for i=1,2,...,k. So $\widetilde{x_1}$, $\widetilde{x_2}$ are fuzzy labelling matrices for $\tilde{0}$. Hence $FI_{L(0)}$ (V^k, F^k) is a fuzzy algebra of the incidence algebra I (V^k,F^k).

Theorem 4.6

The set of fuzzy labellings $FI_{L(\Lambda)}$ (V^k,F^k) for fuzzy index matrix Λ J of a fuzzy graph structure $\tilde{G}$ = (μ,ρ_1, ρ_2,... ρ_k) is a fuzzy algebra of the incidence algebra I (V^k,F^k).

Proof

Let $\widetilde{x_1}$, $\widetilde{x_2}$ ∈ $FI_{L(\Lambda)}$ (V^k, F^k). Then $\widetilde{x_i^1}$, $\widetilde{x_i^2}$ ∈ $FI_{L(\lambda_i)}$ (V^k, F^k), the collection of all fuzzy ρ_i-labellings for λ_i, for i=1,2,...,k.

By theorem 3.6, the set of fuzzy ρ_i-labellings for fuzzy ρ_i-index vectors $\lambda_i j_i$ is a fuzzy algebra of the incidence algebra $I_{L(\lambda_i)}$ (V,F) for i=1,2,...,k. So $\widetilde{x_i^1 + x_i^2}$, $\widetilde{x_i^1 . x_i^2}$ and $\widetilde{fx_i^1}$ are fuzzy ρ_i-labellings for fuzzy ρ_i-index vectors $\widetilde{\lambda_i^1 + \lambda_i^2}$, $\widetilde{\lambda_i^1 . \lambda_i^2}$ and $f\widetilde{\lambda_i^1}$, respectively for i=1,2.,...,k. So $\widetilde{x_i^1}$, $\widetilde{x_i^2}$ are fuzzy labelling matrices for $\widetilde{\Lambda J}$. Hence $FI_{L(\Lambda)}$ (V^k,F^k) is a fuzzy algebra of the incidence algebra I (V^k,F^k).

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