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COINCIDENCE AND COMMON FIXED POINT THEOREMS FOR OCCASIONALLY WEAKLY CO-COMPATIBLE MAPS

[1]RAHUL TIWARI AND [2]D.P SHUKLA

[1]Department of Mathematical Sciences

A.P.S. University Rewa (M.P.) 486003, India

E-mail : tiwari.rahul.rewa@gmail.com

[2]Department of Mathematics

Govt. P.G Science College Rewa (M.P) 486001, India

E-mail : shukladpmp@gmail.com

Abstract :- In this manuscript we have tried to develop some theorems based on the theorems of Jungck & Rhoades [6] by defining new weakly co-compatible maps and occasionally weakly co-compatible maps

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Mathematics Subject Classification: 47H10, 54H25

1 Introduction

Fixed point theorems are very important tools for proving the existence and uniqueness of the solutions to various mathematical models (differential, integral and partial differential equations and variational inequalities etc.) representing phenomena arising in different fields, such as steady state temperature distribution, chemical equations, neutron transport theory, economic theories, financial analysis, epidemics, biomedical research and flow of fluids. In 1976, Jungck [2] claimed a common fixed point theorem for commuting maps generalizing the fixed point theorem of Banach, which states that,if (X,d) is a complete metric space and a self map T on X satisfies $d(Tx,Ty) \leq kd(x,y)$ for each $x,y \in X$ where $0 \leq k \leq 1$, then T has unique fixed point in X. This theorem has many applications, but suffers from one drawback- the definition requires that T be continuous throughout X. There then follows a flood of papers involving contractive definition that do not require the continuity of T. Such result was further generalized and extended in various ways by many authors .

In 1982, Sessa [9] introduced the concept of weakly commuting mappings which extends the notion of commuting mappings. After four years, Jungck [3] defined compatible mappings as an extension of weakly commuting mappings. Later on, the same author with Murthy and Cho [4] gave another extension of weakly commuting mappings under the name of compatible mappings of type (A). Again, Pathak and Khan [8] extended compatible of type (A) mappings to compatible mappings of type (B). On this direction, Pathak et a1., [6] introduced the new concept i.e, compatible type of (C) as another extension a compatible type of (A) and proved a common fixed point Result in a Banach space.

In their paper [6], Jungck and Rhoades defined the notion of weakly compatible mappings as an extension of all above notions. Al- Thagafi and Shahzad [1] gave the concept of occasionally weakly compatible mappings which is more general than weakly compatible mappings and all above notions.

2. Preliminaries

Definition 2.1. Let X be a set and $f,g : X \to X$.Then a point $x \in X$ is called a coincidence point of f and g if $fx = gx = w$ and w is called as point of coincidence.

Definition 2.2. Two self maps f and g of a metric space (X, d) are said to be weakly commuting pair if,

$$d(fgx, gfx) \leq d(fx, gx). \qquad \text{for all } x \in X$$

Definition 2.3. Two self maps f and g of a metric space (X, d) are said to be compatible if,

$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$. whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \qquad \text{for some } t \in X.$$

Definition 2.4. Let f,g be self maps of a metric space (X,d) and (x_n) be a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. Then

(1) f , g are said to be compatible of type (A) if, $\lim_{n\to\infty} d(fgx_n, g^2x_n) = 0$ and $\lim_{n\to\infty} d(gfx_n, f^2x_n) = 0$.

(2) f,g are said to be compatible of type (B) if, $\lim_{n\to\infty} d(fgx_n, g^2x_n) \leq \frac{1}{2}[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, f^2x_n)]$and $\lim_{n\to\infty} d(gfx_n, f^2x_n) \leq \frac{1}{2}[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, g^2x_n)]$

(3) f,g are said to be compatible of type (C) if,

$\lim_{n\to\infty} d(fgx_n, g^2x_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, f^2x_n)+ \lim_{n\to\infty} d(ft, g^2x_n)]$ and

$\lim_{n\to\infty} d(gfx_n, f^2x_n) \leq \frac{1}{3} [\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, g^2x_n) + \lim_{n\to\infty} d(gt, f^2x_n)]$

(4) f,g are said to be compatible of type (P) if, $\lim_{n\to\infty} d(f^2x_n, g^2x_n) = 0$.

Note that compatibility, compatibility of type (A), (B),(C) and (P) are equivalent if f,g are continuous.

Definition 2.5. Two self maps f and g of a metric space (X, d) are said to be weakly compatible if they commute at their coincidence points.

Example 2.5.1. Define usual metric d on X = [0,3] and define self maps f ,g on [0,3] by fx=x if x∈ [0,1) , fx=3 if x∈ [1,3] and gx= 3-x if x∈ [0,1), gx= 3 if x∈ [1,3]. Then fgx= gfx for all x∈ [1,3], i.e f,g are weakly compatible maps on [0,3].

Definition 2.6. [6] Two self maps f and g of a metric space X are said to be occasionally weakly compatible (owc) if and only if , there is a point t in X which is a coincidence point of f and g at which f and g commute; i.e, there exists a point x in X such that fx=gx and fgx=gfx.

Definition 2.7. Two self maps f and g of a metric space X are said to be subcompatible iff there exists a sequence (x_n) in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, $t \in X$ and which satisfy $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$.

Clearly, two owc maps are subcompatible ,but converse is not true in general.

New Definition 2.8. Let f,g be self maps on metric space X. Then

(1) Any point x in X is said to be co-coincidence point if ffx =ggx.

(2) f,g are said to be co-coincidence maps if ffx=ggx, for all x in X.

(3) (f,g) is said to be pair of co-compatible maps if, d(ffx , ggx) $\leq$ d(fx , gx).

(4) (f , g) is said to be pair of co-compatible maps if fx=gx implies ffx=ggx.

(5) (f , g) is said to be pair of occasionally weakly co-compatible maps if there exists a coincidence point x such that gx= fx implies ffx = ggx .

Remark 2.9. Commuting maps does not implies co-coincidence maps.

For example, let fx = 2x , gx = 3x then fgx = 6x = gfx , but ffx = 4x, ggx = 9x.

In 2006,[6] Jungck and Rhoades proved the following theorem :

Theorem 2.10 :- Let X be a metric space and f,g be two self owc maps of X . If f and g have a unique point of coincidence w = fx = gx, then w is the unique common fixed point of f and g.

3. Main Results. .

Theorem 3.1 :- Let X be a metric space and f , g be two owc self maps of X .If f and g have unique coincidence point then it is unique common fixed point of f and g.

Proof :- Let x be a unique coincidence point of f and g. Since f and g are owc maps, so fx=gx

and fgx=gfx. Since $fx = gx \Rightarrow ffx = fgx$, so $ffx = fgx = gfx$. Thus fx is also a coincidence point of f and g. So x = fx. Hence x = fx = gx. Consequently x is common fixed point of f and g.

Theorem 3.2 :- Let X be a metric space and f,g are two self maps of X then f ,g have unique point of coincidence if and only if f ,g have unique coincidence point further if (f,g) is owc pair then both are the same.

Proof :- Since f ,g have unique coincidence point x, then fx = gx. Suppose w_1 and w_2 are two points of coincidence of f and g such that $w_1 \neq w_2$ then there exist points x_1 and x_2 in X such that $w_1 = fx_1 = gx_1$ and $w_2 = fx_2 = gx_2$ so $x_1 \neq x_2$ which contradicts the uniqueness of x. Hence $w_1 = w_2 = w$ is unique point of coincidence f and g .

Conversely, let f ,g have unique point of coincidence w = fx = gx. Suppose x_1 and x_2 are two coincidence points of f and g such that $x_1 \neq x_2$, then there exist points w_1 and w_2 in X such that $fx_1 = gx_1 = w_1$ and $fx_2 = gx_2 = w_2$,so $w_1 \neq w_2$, which contradicts the uniqueness of w Hence $x_1 = x_2$.

Further if (f , g) is owc pair and f,g have unique point of coincidence w = fx = gx, we know that by theorem 2.10, w is unique common fixed point of f and g.Therefore by theorem 3.1, x is unique common fixed point of f and g. Hence x=w.

Theorem 3.3 :- Let X be a metric space and f , g be two self maps of X. Then f ,g are weakly co-compatible if and only if f ,g are weakly compatible.

Proof :- Let f , g be weakly compatible maps, so $fx = gx \Rightarrow fgx = gfx$,Since $fx = gx \Rightarrow ffx = fgx$. Again $fx = gx \Rightarrow gfx = ggx$.Therefore $fx = gx \Rightarrow ffx = ggx$. Hence f , g are weakly co- compatible maps.

Conversly, let f,g be weakly co-compatible maps, so $fx = gx \Rightarrow ffx = ggx$.Since $fx = gx \Rightarrow ffx = fgx$. Again $fx = gx \Rightarrow gfx = ggx$. Consquently $fgx = ffx = ggx = gfx$.

Therefore fx=gx implies fgx = gfx. Hence f and g are weakly compatible.

Theorem 3.4:- Let X be a metric space and f, g are two self maps. f, g are occasionally weakly co-compatible maps if and only if f, g are occasionally weakly compatible maps.

Proof: The proof of this theorem is similar to theorem 3.3.

Theorem 3.5:- Let X be a metric space and f, g be occasionally weakly co-compatible self maps of X. If f and g have a unique point of coincidence, w = fx = gx then w is the unique common fixed point of f and g.

Proof :- It can be easily proved from theorem 2.10 and theorem 3.3.

Theorem 3.6:- Let X be a metric space and f , g be occasionally weakly co-compatible self maps of X. If f and g have unique coincidence point then it is unique common fixed point of f and g.

Proof :- It can be easily proved from theorem 3.1 & theorem 3.3.

Theorem 3.7:- Let f , g be two self occasionally weakly co-compatible maps then f ,g have unique point of coincidence if and only f ,g have unique coincidence point.

Proof :- It can be easily proved from theorem 3.2 and theorem 3.3. .

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