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Nearness in review

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Abstract

Nearness space is introduced by Horst Herrlich in 1974 . Its corresponding category is denoted by NEAR which has some important subcategories.
In this paper we have some new results regarding to subcategory T-Near of NEAR, whose objects are topological near spaces and subcategory CompNEAR of NEAR, whose objects are complete near spaces.

Keywords: Uniform space, Nearness space,Topological N-space, Complete N-space.

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1 Introduction

Nearness space is introduced by H. Herrlich as an axiomatization of the concept of nearness of arbitrary collection of sets. Nearness unifies various concepts of topological structures in the sense that the category **NEAR** of all nearness spaces and nearness preserving maps contains the categories $\mathbf{R_0}$**-TOP** of all symmetric topological spaces and continuous maps, **Unif** of all uniform spaces and uniformly continous maps, **EF-Prox** of all EF-proximity spaces and δ-maps and **Cont** of all contiguity spaces and contiguity maps as nicely embedded (either bireflective or bicoreflective) full subcategories.
In this paper we have some new results on objects of the category **T-Near** of all topological near spaces and nearness preserving maps which is isomorphic to $\mathbf{R_0}$**-TOP** and

also we have some new results on complete near spaces which we denote its corresponding category by **CompNEAR**.

2 Background

Definition 2.1: Let X be a set and let ξ be a subset of P^2X. Consider the following axioms:

(N1) If $\mathcal{A} << \mathcal{B}$ and $\mathcal{B} \in \xi$ then $\mathcal{A} \in \xi$, where $\mathcal{A} << \mathcal{B}$ iff $\forall A \in \mathcal{A} \exists B \in \mathcal{B}, A \supseteq B$;

(N2) If $\bigcap \mathcal{A} \neq \emptyset$ then $\mathcal{A} \in \xi$;

(N3) $\emptyset \neq \xi \neq P^2X$;

(N4) If $(\mathcal{A} \bigvee \mathcal{B}) \in \xi$ then $\mathcal{A} \in \xi$ or $\mathcal{B} \in \xi$, where $\mathcal{A} \bigvee \mathcal{B} := \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}$;

(N5) If $\{cl_\xi A | A \in \mathcal{A}\} \in \xi$ then $\mathcal{A} \in \xi$, where $cl_\xi A := \{x \in X | \{A, \{x\}\} \in \xi\}$.

ξ is called *nearness structure* on X iff ξ satisfies to above conditions and the pair (X, ξ) is called a *nearness space* or shortly *N-space* iff ξ is a nearness structure on X.

If (X, ξ) and (Y, η) are N-spaces then a function $f : X \longrightarrow Y$ is called a *nearness preserving map* (or shortly an *N-map*) from (X, ξ) to (Y, η) iff $\mathcal{A} \in \xi$ implies $f\mathcal{A} \in \eta$, where $f\mathcal{A} := \{f[A] : A \in \mathcal{A}\}$.

The corresponding category is denoted by **NEAR**.

Definition 2.2: Given a nearness space (X, ξ), then it is called *uniform* iff it satisfies the following condition :

(U) If $\mathcal{A} \notin \xi$ then there exists $\mathcal{B} \notin \xi$ such that for each $B \in \mathcal{B}$ there exists $A \in \mathcal{A}$ with $A \subset \bigcap \{C \in \mathcal{B} | C \cup B \neq X\}$.

The full subcategory of **NEAR** whose objects are uniform N-spaces is denoted by **U-Near**.

Proposition 2.3: The category **Unif** of all uniform spaces and uniformly continuous maps and the category **U-Near** of all uniform N-spaces and N-maps are isomorphic. Therefore we can identify **Unif** and **U-Near** and we can consider **Unif** as a full subcategory of **NEAR**.

Definition 2.4: Given a nearness space (X, ξ), then it is called *topological* iff it satisfies the following condition :
(T) If $\mathcal{A} \in \xi$ then $\bigcap \{cl_\xi A | A \in \mathcal{A}\} \neq \emptyset$.

The full subcategory of **NEAR** whose objects are topological N- spaces is denoted by **T-Near**.

Recall 2.5: A topological space on X is *symmetric* iff $x \in cl\,\{y\}$ implies $y \in cl\,\{x\}$.

Proposition 2.6: The category $\mathbf{R_0}$**-TOP** of all symmetric topological spaces and continuous maps and the category **T-Near** of all topological N-spaces and N-maps are isomorphic. Therefore we can identify $\mathbf{R_0}$**-TOP** and **T-Near** and we can consider $\mathbf{R_0}$**-TOP** as a full subcategory of **NEAR**.

Definition 2.7: Let (X, ξ) be a nearness space. A non-empty subset $\mathcal{A}$ of PX is called:
(1) ξ-*cluster* (or shortly *cluster*) iff $\mathcal{A}$ is a maximal element of the set ξ, ordered by inclusion.
(2) ξ-*clan* (or shortly *clan*) iff $\mathcal{A}$ is a grill and $\mathcal{A} \in \xi$. (Where $\mathcal{A}$ is *grill* iff it satisfies the following two conditions: (i) $\emptyset \notin \mathcal{A}$ (ii) $A \cup B \in \mathcal{A}$ iff $A \in \mathcal{A}$ or $B \in \mathcal{A}$.)

Proposition 2.8: In nearness space every cluster is a clan.

Definition 2.9: Let (X, ξ) be a nearness space. A cluster $\mathcal{A}$ is called a *point-ξ-cluster* iff there exists $x \in X$ with $\{x\} \in \mathcal{A}$.

Proposition 2.10: Let (X, ξ) be a nearness space. If $x \in X$ then $\mathcal{C}(x) := \{A \subset X | x \in cl_\xi A\}$ is a point-ξ-cluster. And every point-ξ-cluster is of this form.

Definition 2.11: A nearness space (X, ξ) is called *complete* iff every cluster contains an element $\{x\}$, for some $x \in X$. Equivalently, a nearness space (X, ξ) is called *complete* iff every cluster is a point-ξ-cluster.

Proposition 2.12 [3]: A uniform N-space is complete iff it is complete as a uniform space.

3 Topological N-space and complete N-space

Proposition 3.1: The category **CompNEAR** of all complete nearness spaces and nearness preserving maps is a subcategory of **NEAR**.

Corollary 3.2: Each complete uniform space is complete as a nearness space.

Proof: By proposition 2.12 , each complete uniform space is complete uniform N-space and since each uniform N-space is a nearness space, therefore each complete uniform space is complete as a nearness space.

Proposition 3.3: Each finite nearness space is complete.

Proof: Let (X,ξ) be a finite nearness space and $\mathcal{A}$ be an arbitrary ξ-cluster so $\mathcal{A} \neq \emptyset$ and by proposition 2.8, $\mathcal{A}$ is a ξ-clan therefore there exists $(\emptyset \neq) A \in \mathcal{A}$. Since X is finite so A is also finite. Suppose $A = \{a_1, a_2, ...a_n\}$ so $A = \{a_1\} \cup (A \setminus \{a_1\}) \in \mathcal{A}$ then, since $\mathcal{A}$ is clan, either $\{a_1\} \in \mathcal{A}$ or $A \setminus \{a_1\} \in \mathcal{A}$.
If $\{a_1\} \in \mathcal{A}$ then (X,ξ) is a complete N-space. If $\{a_1\} \notin \mathcal{A}$ then $A \setminus \{a_1\} \in \mathcal{A}$, and by induction there is $a_i \in A$ s.t. $\{a_i\} \in \mathcal{A}$, which again implies (X,ξ) is a complete N-space.

Example 3.4: Let $X = \{a,b\}$ then we have only two nearness on X.
$\xi_1 = \{\emptyset, \{\{a\}\}, \{\{b\}\}, \{\{a,b\}\}, \{\{a\}, \{a,b\}\}, \{\{b\}, \{a,b\}\}\}$.
$\xi_2 = \{\emptyset, \{\{a\}\}, \{\{b\}\}, \{\{a,b\}\}, \{\{a\}, \{a,b\}\}, \{\{b\}, \{a,b\}\} \{\{a\}, \{b\}\}, \{\{a\}, \{b\}, \{a,b\}\}\}$.
ξ_1-clusters are $\{\{a\}, \{a,b\}\}$ and $\{\{b\}, \{a,b\}\}$ which are point-ξ_1-clusters. Therefore (X,ξ_1) is a complete N- space.
ξ_2-cluster is only $\{\{a\}, \{b\}, \{a,b\}\}$ which is point-ξ_2-cluster. Therefore (X,ξ_2) is also a complete N- space.

Example 3.5: There are some infinite nearness spaces which are not complete.
By proposition 2.3 we mention that uniform spaces are essentially special nearness spaces. And by proposition 2.12 we can consider a metric or uniform space which is not complete as an example for not complete N-spaces.

Proposition 3.6: Each topological nearness space is complete.

Proof: Let (X,ξ) be a topological near space and $\mathcal{A}$ be an arbitrary ξ-cluster.

Since (X, ξ) is a topological N-space, $\bigcap\{cl_\xi(A) : A \in \mathcal{A}\} \neq \emptyset$ i.e. there exists $x \in X$ s.t. $x \in \bigcap\{cl_\xi(A) : A \in \mathcal{A}\}$ also obviously $x \in cl_\xi(\{x\})$ so $x \in \bigcap\{cl_\xi(B)|B \in (\mathcal{A} \cup \{\{x\}\})\}$ therefore by (N2) and (N5) we have, $\{\{x\}\} \cup \mathcal{A} \in \xi$ and by maximality of $\mathcal{A}$ it implies $\{x\} \in \mathcal{A}$, which implies (X, ξ) is a complete N-space.

Note 3.7: For an arbitrary complete nearness space (X, ξ), the empty set-system, which is near, must not be contained in a cluster, hence in general we do not know whether intersection of the closures of its members is empty or not.
In the case if X be empty set then the corresponding nearness contains only empty set which is not cluster so it is complete but not topological since X does not have any member so intersection of closure of members of empty set only can be empty set. Which shows converse of above theorem does not hold.

References

[1] J. Adamek, H. Herrlich, G. E. Strecker, Abstract and concrete categories, A wiley-interscience publication, pure and applied mathematics, JohnWiley, Sons, 1990.

[2] H. Herrlich, A concept of nearness, Gen. Topology Appl. **5** (1974) 191-212.

[3] H. Herrlich, Topological structures in : Topological structures I, Math. Centre Tracts **52** (1974) pp. 59-122.

[4] J. L. Kelly: General Topology, (VNR Publish).

[5] Weil, Sur les espaces a structure uniforme et sur la topologie generale, Actualites Scientifiques et Industrielles, no. **551**, Paris, Hermann, 1937.

[6] S. Willard, General Topology, Addision- Wesley Publishing Company, Inc. 1970.