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A GENERALIZED CARLSON - SHAFFER OPERATOR AND NEIGHBORHOODS OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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Abstract

In this paper, we introduce a generalized Carlson - Shaffer operator denoted by $L(a,c;\eta)$ and the new subclasses $S_n^{\eta}(\beta,\gamma,a,c)$, $R_1n^{\dagger}\eta\ (\beta,\gamma,a,c;\ \mu)$, $S_n^{\eta,\alpha}(\beta,\gamma,a,c)$, $R_1n^{\dagger}(\eta,\alpha)\ (\beta,\gamma,a,c;\ \mu)$, of analytic functions and study certain $(n,\delta)-$ neighborhood properties for functions belonging to these classes.

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1. INTRODUCTION

Let $A(n)$ be the class of analytic functions f of the form

$$f(z)=z-\sum_{k=n+1}^{\infty}a_k z^k \qquad (a_k\geq 0\ ,k\in N-,1.,n\in N). \tag{1.1}$$

defined in the open unit disc $U=\{z\in C:\ |z|<1\}$

For any function $f\in A(n)$ and $\delta\geq 0$ the (n,δ) - neighborhood of f is defined as,

$$N_{n,\delta}(f)=\left\{g\in A(n):g(z)=z-\sum_{k=n+1}^{\infty}b_k z^k \text{and} \sum_{k=n+1}^{\infty}k|a_k-b_k|\leq\delta\right\}. \tag{1, 2}$$

For the identity function $e(z)=z,$ we see that,

$$N_{n,\delta}(e)=\left\{g\in A(n):g(z)=z-\sum_{k=n+1}^{\infty}b_k z^k \text{and} \sum_{k=n+1}^{\infty}k|b_k|\leq\delta\right\}. \tag{1.3}$$

The concept of neighborhoods was first introduced by Goodman [8] and then generalized by Ruscheweyh [11] . A function $f\in A(n)$ is said to be in the class $S_n^*(\gamma)$ of starlike functions of complex order γ if

$$R\left\{1+\frac{1}{\gamma}\left[\frac{zf'(z)}{f(z)}-1\right]\right\}>0 \qquad (z\in U,\gamma\in C\backslash\{0\}). \tag{1.4}$$

The subclass $Cn(\gamma)$ of $A(n)$ is the class of convex functions of complex order γ satisfying,

(1.5) $$R\{1 + 1/\gamma \ (zf''(z))/(f'(z))\} > 0 \,, \qquad (z \in U, \gamma \in C\backslash\{0\}).$$

The classes $S_n^*(\gamma)$ *and* $Cn(\gamma)$ stem essentially from the classes of starlike and convex functions of complex order, which were introduced earlier by Nasr, Aouf [10] and Wiatrow [12], respectively.
The Hadamard product of two power series

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k$$

is defined as
$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$$

In particular, we consider the convolution with the function $\emptyset(a, c)$ defined by

$$\emptyset(a, c; z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k \,, \qquad z \in U, c \neq 0, -1, -2, \ldots$$

where
$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$$

That is $(a)_0 = 1, (a)_k = a(a+1)\ldots(a+k-1), \qquad k \in N.$

Now, we define a Generalized Carlson - Shaffer operator $L(a, c; \eta)$ by

(1.6) $$L(a, c; \eta) f(z) = \emptyset(a, c; z) * D_\eta f(z)$$

for a function $f \in A(1)$, where

$$D_\eta f(z) = (1 - \eta) f(z) + \eta z f'(z), \qquad (\eta \geq 0, \quad z \in U).$$

So, we have

(1.7) $$L(a, c; \eta) f(z) = z - \sum_{k=2}^{\infty} [1 + (k-1)\eta] \frac{(a)_{k-1}}{(c)_{k-1}} a_k z^k.$$

Remark 1.1. It is easy to observe that for $\eta = 0$; we obtain the Carlson - Shaffer

linear operator [6].

A function $f \in A(n)$ is said to be in the class $S_n^{\eta}(\beta, \gamma, a, c)$, if

(1.8)
$$\left|\frac{1}{\gamma}\left(\frac{z\left(L(a,c;\eta)f(z)\right)'}{L(a,c;\eta)f(z)} - 1\right)\right| < \beta$$

where, $\gamma \in C\backslash\{0\}$, $\quad \eta \geq 0$, $\quad 0 < \beta \leq 1$, $\quad a > 0$ and $z \in U$.

A function $f \in A(n)$ is said to be in the class $R_1 n^{\intercal}\eta\ (\beta, \gamma, a, c;\ \mu)$, if

(1.9)

$$\left|\frac{1}{\gamma}\left((1-\mu)\frac{\left(L(a,c;\eta)f(z)\right)}{z} + \mu([L(a,c;\eta)f(z)])' - 1\right)\right| < \beta$$
,

Where, $\gamma \in C\backslash\{0\}$, $\quad \eta \geq 0$, $\quad 0 < \beta \leq 1$, $\quad 0 \leq \mu \leq 1$, $\quad a > 0$ and $z \in U$.

2. Neighborhoods for the classes $S_n^{\eta}(\beta, \gamma, a, c)$, and $R_1 n^{\intercal}\eta\ (\beta, \gamma, a, c;\ \mu)$,

In this section, we obtain inclusion relations involving $N_{n,\delta}$ for functions in the classes $S_n^{\eta}(\beta, \gamma, a, c)$, ***and*** $R_1 n^{\intercal}\eta\ (\beta, \gamma, a, c;\ \mu)$.

Lemma 2.1. A function $f \in S_n^{\eta}(\beta, \gamma, a, c)$, if **and** only if

(2.1)
$$\sum_{k=n+1}^{\infty}[1 + (k-1)\eta]\frac{(a)_{k-1}}{(c)_{k-1}}(\beta|\gamma| + k - 1)a_k \leq \beta|\gamma|.$$

Proof. Let $f \in S_n^{\eta}(\beta, \gamma, a, c)$, Then by (1.7) we can write,

(2.2)

$$R\left\{\frac{z\left(L(a,c;\eta)f(z)\right)'}{L(a,c;\eta)f(z)} - 1\right\} > -\beta|\gamma|. \qquad (z \in U).$$

Equivalently,

(2.3)

$$R\left\{\frac{-\sum_{k=n+1}^{\infty}[1+(k-1)\eta]\frac{(a)_{k-1}}{(c)_{k-1}}(k-1)a_k z^k}{z - \sum_{k=n+1}^{\infty}[1+(k-1)\eta]\frac{(a)_{k-1}}{(c)_{k-1}}a_k z^k}\right\} > -\beta|\gamma|. \qquad (z \in U).$$

Letting $z \to 1$; through the real values, the inequality (2.3) yields the desired condition (2.1).

Conversely, by applying the hypothesis (2.1) and letting $|z| = 1$, we obtain,

$$\left|\frac{z\left(L(a,c;\eta)f(z)\right)'}{L(a,c;\eta)f(z)} - 1\right| = \left|\frac{\sum_{k=n+1}^{\infty}[1+(k-1)\eta]\frac{(a)_{k-1}}{(c)_{k-1}}(k-1)a_k z^k}{z - \sum_{k=n+1}^{\infty}[1+(k-1)\eta]\frac{(a)_{k-1}}{(c)_{k-1}}a_k z^k}\right|$$

$$\leq \frac{\beta|\gamma|\left(1 - \sum_{k=n+1}^{\infty}[1+(k-1)\eta]\frac{(a)_{k-1}}{(c)_{k-1}}a_k\right)}{1 - \sum_{k=n+1}^{\infty}[1+(k-1)\eta]\frac{(a)_{k-1}}{(c)_{k-1}}a_k} \leq \beta|\gamma|$$

Hence, by the maximum modulus theorem, we have $f \in S_n^{\eta}(\beta,\gamma,a,c)$. Thus the proof is complete.
On similar lines, we prove the following lemma.

Lemma 2.2. A function $f(z) \in R_1 n^{\dagger}\eta\ (\beta,\gamma,a,c;$ **μ),** if and only if
(2.4)

$$\sum_{k=n+1}^{\infty}[1+(k-1)\eta]\frac{(a)_{k-1}}{(c)_{k-1}}[\mu(k-1)+1]a_k \leq \beta|\gamma|.$$

Lemma 2.2. A function $f(z) \in R_1 n^{\dagger}\eta\ (\beta,\gamma,a,c;$ **μ),** if and only if
(2.4)

$$\sum_{k=n+1}^{\infty}[1+(k-1)\eta]\frac{(a)_{k-1}}{(c)_{k-1}}[\mu(k-1)+1]a_k \leq \beta|\gamma|.$$

Theorem 2.3. if
(2.5)

$$\delta = \frac{(n+1)\beta|\gamma|}{(\beta|\gamma|+n)(1+n\eta)\frac{(a)_n}{(c)_n}}, \qquad (|\gamma| < 1),$$

then $S_n^{\eta}(\beta,\gamma,a,c) \subset N_{n,\delta(e)}$.

Proof. Let $f \in S_n^{\eta}(\beta,\gamma,a,c)$. By Lemma 2.1, we have,

$$(\beta|\gamma|+n)(1+n\eta)\,(a)_n/(c)_n \sum_{k=n+1}^{\infty} a_k \leq \beta|\gamma|.$$

So,
(2.6)

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\beta|\gamma|}{(\beta|\gamma|+n)(1+n\eta)\frac{(a)_n}{(c)_n}}.$$

Using (2.1) and (2.6).we have,

$$\frac{(1+n\eta)\frac{(a)_n}{(c)_n}\sum_{k=n+1}^{\infty}[k a_k \leq \beta|\gamma| + (1-]\beta|\gamma|)(1+n\eta)(a)_n}{(c)_n\sum_{k=n+1}^{\infty}a_k}$$

$$\leq \beta(\gamma) + (1 - \beta|\gamma|)(1 + n\eta)\frac{(a)_n}{(c)_n}\frac{\beta(\gamma)}{(\beta|\gamma| + n)(1 + n\eta)\frac{(a)_n}{(c)_n}}$$

$$\leq \frac{(n+1)\beta|\gamma|}{(\beta|\gamma| + n)}$$

That is

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{(n+1)\beta(\gamma)}{(\beta|\gamma| + n)(1 + n\eta)\frac{(a)_n}{(c)_n}} = \delta$$

Thus, by the definition given by (1.3), $f \in N_{n,\delta(e)}$. This completes the proof.

Theorem 2.4. if

(2.7)

$$\delta = \frac{(n+1)\beta|\gamma|}{(\mu n + 1)(1 + n\eta)\frac{(a)_n}{(c)_n}}$$

then $R_n^{\eta}(\beta, \gamma, a, c; \mu) \subset N_{n,\delta(e)}$.

Proof. Let $f \in R_n^{\eta}(\beta, \gamma, a, c; \mu)$. Then, by Lemma 2.2, we have,

$$(1 + n\eta)\frac{(a)_n}{(c)_n(\mu n + 1)\sum_{k=n+1}^{\infty} a_k} \leq \beta|\gamma|,$$

which yields the following coefficient inequality:

(2.8)

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\beta|\gamma|}{(\mu n + 1)(1 + n\eta)\frac{(a)_n}{(c)_n}}.$$

Using (2.4) and (2.8), we also have,

$$\mu(1 + n\eta)\frac{(a)_n}{(c)_n}\sum_{k=n+1}^{\infty} k a_k \leq \beta|\gamma| + (\mu - 1)(1 + n\eta)\frac{(a)_n}{(c)_n}\sum_{k=n+1}^{\infty} a_k$$

$$\leq \beta|\gamma| + (\mu - 1)(1 + n\eta)\frac{(a)_n}{(c)_n}\frac{\beta|\gamma|}{(\mu n + 1)(1 + n\eta)\frac{(a)_n}{(c)_n}}$$

.That is,

$$\sum_{k=n+1}^{\infty} k a_k \leq \frac{(n+1)\beta|\gamma|}{(\mu n + 1)(1 + n\eta)\frac{(a)_n}{(c)_n}} = \delta$$

Thus, by the definition given by (1.3), $f \in N_1(n, \delta(e).)$ This completes the proof.

3. **Neighborhoods for the classes** $S_n^{\eta,\alpha}$ (β,γ,a,c),**and** $R_{\downarrow}n^{\uparrow}(\eta,\alpha)$ $(\beta,\gamma,a,c;$ μ),

In this section, we define the subclasses $S_n^{\eta\alpha}$ (β,γ,a,c), $and R_{\downarrow}n^{\uparrow}(\eta,\alpha)$ $(\beta,\gamma,a,c;$ μ), of $A(n)$ and certain neighborhood properties for functions belonging to these classes are obtained. For $0 \leq \alpha < 1$ and $z \in U$, a function f is said to be in the class $S_n^{\eta\ \alpha}(\beta,\gamma,a,c)$ if there exists a function $g \in S_n^{\eta}(\beta,\gamma,a,c)$ such that

(3.1)

$$\left|\frac{f(z)}{g(z)}-1\right|<1-\alpha$$

For $0 \leq \alpha < 1$ and $z \in U$, a function f is said to be in the class $R_n^{\eta,\alpha}(\beta,\gamma,a,c;\mu)$, if there exists a function $g \in R_n^{\eta}(\beta,\gamma,a,c;\mu)$ such that the inequality (3 .1) holds true.

Theorem 3.1. If $g \in S_n^{\eta}(\beta,\gamma,a,c)$ and

(3.2)

$$\alpha=1-\frac{(\beta|\gamma|+n)(1+n\eta)\frac{(a)_n}{(c)_n}}{(n+1)\left[(\beta|\gamma|+n)(1+n\eta)\frac{(a)_n}{(c)_n}-\beta|\gamma|\right]}$$

Then $N_{n,\delta(g)} \subset S_n^{\eta\ \alpha}(\beta,\gamma,a,c)$.

Proof. Let $f \in N_{n,\delta(g)}$ then,

(3.3) $$\Sigma_{\downarrow}(k=n+1)^{\uparrow}\infty \ [k|a] \ _{\downarrow}k - b_{\downarrow}k \mid \leq \delta$$

which yields the coefficient inequality,

(3.4)
$$\Sigma_{\downarrow}(k=n+1)^{\uparrow}\infty \ [k|a] \ _{\downarrow}k - b_{\downarrow}k \mid \leq \delta/(n+1) \qquad (n\in N).$$

Since $g \in S_n^{\eta}(\beta,\gamma,a,c)$ by (2.6), we have

(3.5)

$$\sum_{k=n+1}^{\infty} b_k \leq \frac{\beta|\gamma|}{(\beta|\gamma|+n)(1+n\eta)\frac{(a)_n}{(c)_n}}$$

So that
$$\left|\frac{f(z)}{g(z)}-1\right|<\frac{\left(\sum_{k=n+1}^{\infty}[\square|a]_k-b_k|\square\right)}{1-\sum_{k=n+1}^{\infty}b_k}$$

$$\leq \frac{\delta}{n+1}\frac{(\beta|\gamma|+n)(1+n\eta)\frac{(a)_n}{(c)_n}}{\left[(\beta|\gamma|+n)(1+n\eta)\frac{(a)_n}{(c)_n}-\beta|\gamma|\right]}$$

$=1-\alpha$

Thus, by definition, $f \in S_n^{\eta,\alpha}$ (β, γ, a, c), for α given by (3.2). Thus the proof is complete.
On similar lines, we can prove the following theorem.

Theorem 3.2. If $g \in R_n^{\eta}(\beta, \gamma, a, c; \mu)$ and

(3.6)

$$\alpha = 1 - \frac{(\mu n+1)\delta(1+n\eta)\frac{(a)_n}{(c)_n}}{(n+1)\left[(\mu n+1)(1+n\eta)\frac{(a)_n}{(c)_n}-\beta|\gamma|\right]}$$

then, $N_{n,\delta}(g) \subset R_n^{\eta,\alpha}(\beta, \gamma, a, c; \mu)$.

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