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A Class Of Graceful Lobsters with even number of branches incident on the central path

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Abstract

We observe that a lobster with diameter at least five has a unique path $x_0, x_1, \dots\dots, x_m$ (called the central path) such that x_0 and x_m are adjacent to the centers of at least one $k_{1,s}$, where $s > 0$, and besides adjacencies in the central path each x_i, $1 \le i \le m-1$, is at most adjacent to the centers of some $k_{1,s}$, $s \ge 0$.We call $k_{1,s}$ an odd branch if s is odd , an even branch if s is non-zero even, and a pendant branch if $s = 0$. In this paper we give graceful labeling to some new classes of lobsters with each vertex of the central path is attached to an even number of branches. Furthermore the branches incident on the central path are either even branches or some combinations of even and pendant branches.

Keywords: graceful labeling, lobsters, component moving transformation, odd and even branches, transfer of the first type and second type.

AMS classification:05C78

1. Introduction

Definition 1.1 A *graceful labeling* of a tree T with q edges is a bijection $f: V(T) \to \{0,1,2, \dots\dots\dots q\}$ such that $\{|f(u) - f(v)|: \{u, v\} \ is\ an\ edge\ of\ T\} = \{1,2, \dots\dots\dots q\}$. A tree which has a graceful labeling is called a graceful tree.

Definition 1.2 A lobster is a tree having a path from which every vertex has a distance at most two. It is easy to check that a lobsters L of diameter at least five has a unique path $H = x_0, x_1, \dots\dots, x_m$ such that besides the adjacencies in H, each x_i, $1 \le i \le m-1$, is at most adjacent to the centers of some stars $k_{1,s}$, $s \ge 0$, whereas the vertices x_0 and x_m are adjacent to the centers of at least one star $k_{1,s}$, with $s > 1$. This path H is called the *central path* of the lobsters L.Throughout the paper we use H to denote the central path of a lobster with diameter at least five. If a vertex $x_i \in V(H)$ is adjacent to the center of $k_{1,s}$, $s \ge 0$, then we call $k_{1,s}$ an *even branch* if s is nonzero even, an *odd branch* if s is odd, and a pendant branch if $s = 0$. Therefore, the branches incident on a vertex in the central path of a lobster can be classified into three types, i.e. even, odd, and pendant defined as above. Furthermore, whenever we say x_i , for some $0 \le i \le m$, is attached to an even number of branches, we mean a " non zero" even number of branches unless otherwise stated.

Notation 1.3 In view of Definition 1.2, a combination of branches incident on any x_i , $0 \le i \le m$, can be represented by a triple (x, y, z), where $x, y, and\ z$ represent the number of odd, even, and pendant branches incident on x_i. We use the symbols e and o to represent a non-zero even number and an odd number, respectively. For example :$(e, 0, o)$ means an even number of odd branches, no even branch, and an odd number of pendant branches.

In 1979, Bermond [1] conjectured that "all lobsters are graceful" which is a special case of the famous and unsolved "the graceful tree conjecture" of Ringel and Kotzig (1964) [12], which states that "all tress are

graceful". Bermond's conjecture is also open and very few classes of lobsters are known to be graceful. Ng [10], Wang et al. [13], Chen et al. [2] , Morgan [9] (see [3]), and Mishra and Panigrahi[4,5,6,7,8,11] have given graceful labeling to some classes of lobsters. However, none of the above results gives graceful labeling to a class of lobsters in which every vertex of the central path is attached to an even number of branches.

The lobsters to which we give graceful labeling in this paper have the following characteristic feature.

i) The vertex x_0 is attached to the combination $(0, e, e)$. For an integer t_1, $1 \leq t_1 \leq m$ each x_i is attached to the combination $(0, o, o)$. If $t_1 < m$ then each x_i , $t_1 + 1 \leq i \leq t_2$, $t_2 \leq m$ is attached to the combination $(0, e, e)$. If $t_2 \leq m$, then each x_i , $t_2 + 1 \leq i \leq m$, is attached to the combination $(0, e, 0)$.

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ii) The lobsters obtained by deleting one or more combinations in the lobsters of type-I in the above

iii) The vertex x_0 is attached to the combination $(0, o, o)$ or $(0, e, 0)$. Each x_i ,$1 \leq i \leq m$, is attached to the combination $(0, e, 0)$.

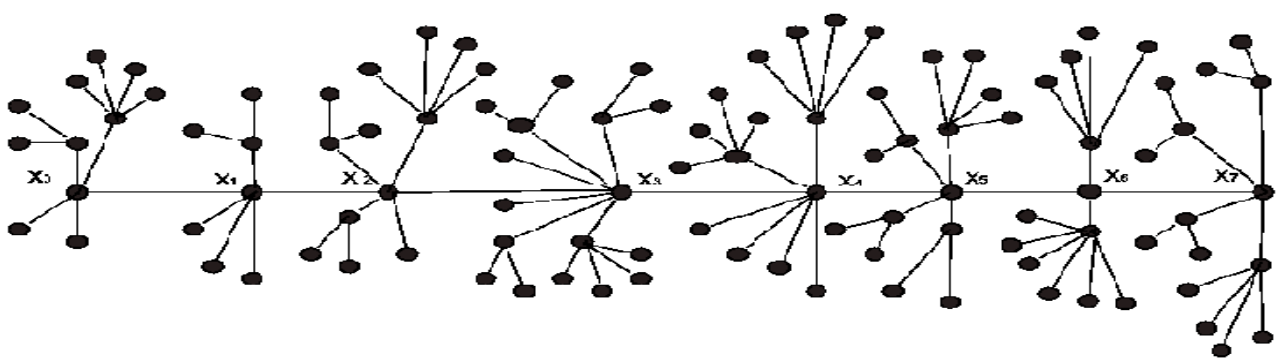

Figure 1: Example of a lobster of type (a). The vertex x_0 is attached to the combination $(e, 0, e)$. Each of x_1 and x_2 is attached to $(0, o, o)$, each of x_3 and x_4 is attached to $(0, e, e)$ and each of x_5 and x_6 and x_7 is attached to $(0, e, 0)$

2. Preliminaries

Definition 2.1 For an edge $e = \{u.v\}$ of a tree T, we define $u(T)$ as that connected component of $T - e$ which contains the vertex u. Here we say $u(T)$ is a component incident on the vertex v . If a and b are vertices of a tree T, $u(T)$ is a *component incident* on a and $b \notin u(T)$ then deleting the edge $\{a, u\}$ from T and making b and u adjacent is called the $component\ moving\ transformation$. Here we say the component $u(T)$ has been transferred or moved from a to b .This is illustrated in Figure 2. Throughout the paper we write "the component u" instead of writing "the component $u(T)$". Whenever we wish to refer u as a vertex, we write " the vertex u". By the label of the component "$u(T)$" we mean the label of the vertex u.

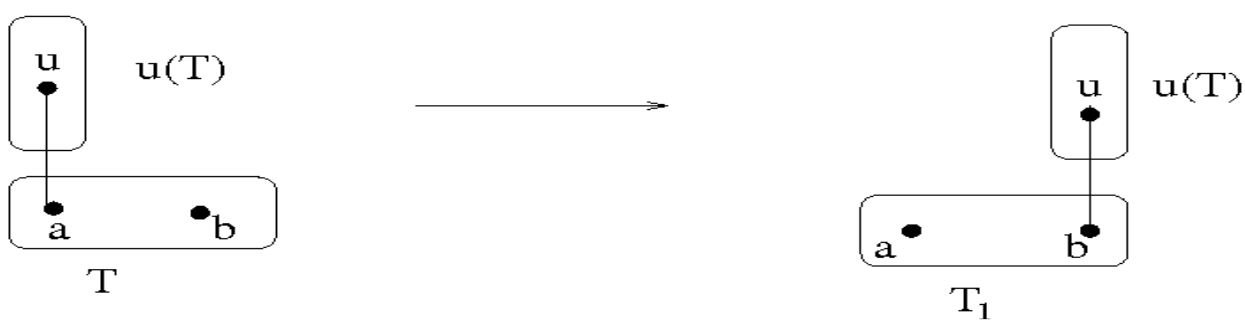

Figure 2: The tree T_1 obtained from the tree T by moving the component $u(T)$ from the vertex a to the vertex b.

Notation 2.2 For any two vertices a and b of a tree T , the notation $a \longrightarrow b$ $transfer$ means that we move some components incident on the vertex a to the vertex . If we consider successive transfers $a \longrightarrow b, b \longrightarrow c, c \longrightarrow d, \ldots\ldots\ldots$,we simply write $a \longrightarrow b \longrightarrow c \longrightarrow d \ldots\ldots\ldots$ transfer. In a transfer $a_1 \longrightarrow a_2 \longrightarrow a_3 \longrightarrow \cdots \longrightarrow a_n$, we call each vertex except a_n a vertex of the transfer.

Lemma 2.3 [8] Let f be a graceful labeling of a tree T ; Let a and b be two vertices of T; let $u(T)$and $v(T)$ be two components incident on a, where $b \notin u(T) \cup v(T)$. Then the following hold:
i)If $f(u) + f(v) = f(a) + f(b)$ then the tree T^* obtained from T by moving the components $u(T)$ and $v(T)$ from a to b is also graceful.
ii) If $2f(u) = f(a) + f(b)$ then the tree T^{**} obtained from T by moving the component $u(T)$ from a to b is also graceful.

Definition 2.4 Let T be a labelled tree and a and b be two vertices of T, and a be attached to some components. The $a \longrightarrow b$ transfer is called $a\ transfer\ of\ the\ first\ type$ if the labels of the transferred components constitute a set of consecutive integers. The $a \longrightarrow b$ transfer is called $a\ transfer\ of\ the\ second\ type$ if the labels of the transferred components can be divided into two segments, where each segment is a set of consecutive integers.

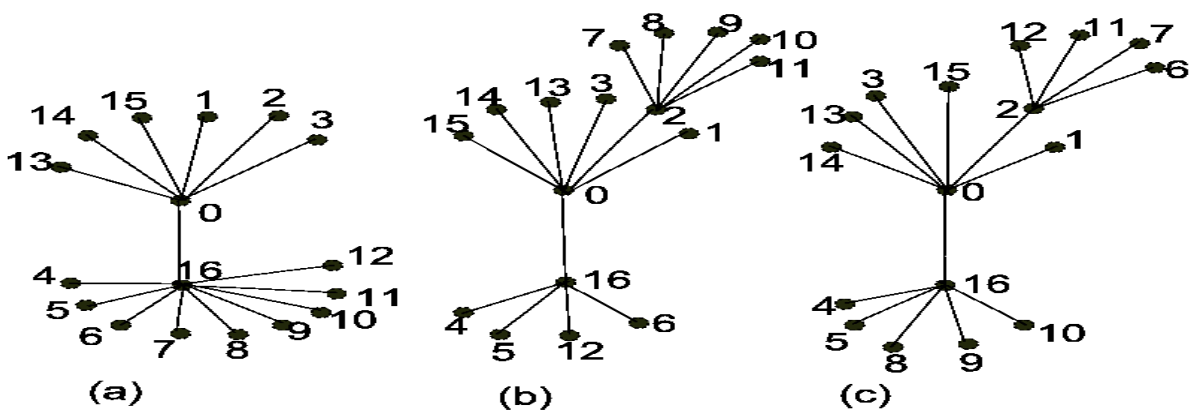

Figure 3: The tree in (a) is a tree with a graceful labeling . The trees in (b) and (c) are obtained from (a) by applying a transfer of the first type $16 \longrightarrow 2$ and a transfer of the second type $16 \longrightarrow 2$, respectively.

Though the following two results i.e. Lemma 2.5 and Lemma 2.7 already exist in the literature, still we give their here because we adopt the method described in the proofs of these lemmas to carry out a sequence of the first and second types for giving graceful labeling to the lobsters here.

Lemma 2.5 [8] In a graceful labeling f of a graceful tree T, let a and b be the labels of two vertices. Let a be attached to a set A of vertices (or components) having labels $n, n+1, n+2, \ldots\ldots\ldots, n+p(in\ f)$,which satisfy either (i) $(n+1+i) + (n+p-i) = a+b, i \geq 0$ or (ii) $(n+i) + (n+p-1-i) = x+y$, $i \geq 0$. Then the following hold.

(a) By making a transfer $a \longrightarrow b$ of the first type we can keep an odd number of components at a from the set A and move the rest to b and the resultant tree thus formed will be graceful
(b) If A has an even number of elements , then by making a transfer $a \longrightarrow b$ of the second type we can keep an even number of components at a from the set A and move the rest to b, and the resultant tree thus formed will be graceful.

Proof: (a) By definition of the transfer of the first type, the labels of the components which are transferred to y must be consecutive integers. If the elements of A satisfy(i) (respectively, (ii)), then we keep any odd number of elements , say $2r+1$ elements, namely $n, n+1+i, n+p-i, i = 0,1,2, \ldots\ldots r-1$(respectively, $n+p, n+i, n+p-1-i, i = 0,1,2, \ldots\ldots r-1$) at a , and transfer the rest to b.Let A_1 be the set of elements of A which has been moved to b. For each $z \in A_1$, either we have $2z = a+b$ or there is another (unique) element w in A such that $z + w = a + b$. Therefore, the new tree thus formed is graceful by Lemma 2.3.

(b)Since A contains an even number of elements , p is odd. If the elements of A satisfy (i) (respectively,(ii)), then from A we keep an even number of elements , say $2r$ elements namely $n, n+\frac{p+1}{2}, n+1+i, n+p-i$ (respectively, $n+p$ and $n+\frac{p-1}{2}, n+i, n+p-1), 0 \le i \le r-2$ if $r \ge 2$; otherwise n and $n+\frac{p+1}{2}$ (respectively, $n+p$ and $n+\frac{p-1}{2}$) at a, and move the rest to b . The transfer that we have done here is easily seen to be a transfer of second type and the resultant tree thus formed is graceful by Lemma 2.3.

■

Observation 2.6 [8] (a) For any pair of vertex labels x and y in a graceful tree, where x is attached to a set of components A as in Lemma 2.5, whenever we make a transfer $x \longrightarrow y$ of the first type as in Lemma 2.5(a), the set A_1 of components of A that are transferred to y is of the form $A_1 = \{n+r, n+r+1, \dots\dots, n+r_1\}$ with $(n+r+i)+(n+r_1-i) = x+y$, where $0 \le i \le \left[\frac{r_1-r+1}{2}\right]$. Further , by re-pairing the elements of A_1, we get, for $0 \le i \le \left[\frac{r_1-r+1}{2}\right]$,

(1) $(n+r+i)+(n+r_1-1-i) = x+y-1$ and
(2) $(n+r+1+i)+(n+r_1-i) = x+y+1$.

Therefore, next if we make a transfer $y \longrightarrow x-1$ or $y \longrightarrow x+1$, then the set A_1 and the vertices (or labels) y and $x-1$ or y and $x+1$ satisfy the hypothesis of Lemma 2.5(a).
(b) We like to mention one important point , i.e. the vertices of the sequences of transfer we deal in this paper has the property **"P":** for any three consecutive integers u, v, and w of the sequence transfer, we have "$w = u \pm 1$". Because of this property "P" we can use Lemma 2.5(a) and part(a) of this observation repeatedly , this will be clear in Lemma 2.7.
(c) If x,y , and A are as in (a) and we make a transfer of second kind $x \longrightarrow y$ as in Lemma 2.5(b), then the set of components of A transferred to y is divided into two disjoint segments, say A_1 and A_2 where A_1 and A_2 are of the form $A_1 = \{a_1, a_1+1, \dots\dots\dots, a_1+k\}$ and $A_2 = \{a_2, a_2+1, \dots\dots\dots a_2+k\}$ with $(a_1+i)+(a_2+k-i) = x+y, 0 \le i \le k$.

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Lemma 2.7 [8] In a graceful labeling f of a tree , let $a, a-1, a-2, \dots\dots\dots a-p_1, b, b+1, b+2, \dots\dots\dots b+p_2$ (respectively,$a, a+1, a+2, \dots\dots\dots, a+r_1, b, b-1, b-2, \dots\dots\dots, b-r_2$)be some vertex labels. Let the vertex a be attached to a set A of vertices (or components) having labels $n, n+1, n+2, \dots\dots\dots, n+p$ (different from the above vertex labels) if f and satisfy either $(n+i+1)+(n+p-i) = a+b$ or $(n+i)+(n+p-1-i) = a+b, 0 \le i \le \left[\frac{p+1}{2}\right]$, then the following hold.

(a) By making a sequence of transfers of the first type $a \longrightarrow b \longrightarrow a-1 \longrightarrow b+1 \longrightarrow a-2 \longrightarrow b+2 \longrightarrow \cdots \longrightarrow x$(respectively ,$a \longrightarrow b \longrightarrow a+1 \longrightarrow b-1 \longrightarrow a+2 \longrightarrow b-2 \longrightarrow \cdots \longrightarrow x$), where $z = a-p_1$or $b+p_2$ (respectively,$z = a+r_1$or $b-r_2$), an odd number of elements from A can be kept at each vertex of the transfer and the resultant tree thus formed will be graceful.
(b) If A contains an even number of elements, then by making a sequence of transfers of second type $a \longrightarrow b \longrightarrow a-1 \longrightarrow b+1 \longrightarrow a-2 \longrightarrow b+2 \longrightarrow \cdots \longrightarrow z$ (respectively, $a \longrightarrow b \longrightarrow a+1 \longrightarrow b-1 \longrightarrow a+2 \longrightarrow b-2 \longrightarrow \cdots \longrightarrow z$), where $z = a-p_1$or $b+p_2$(respectively, $z = a+r_1$ or $b-r_2$), an even number of elements from A can be kept at each vertex of the transfer, such that the resultant tree thus formed will be graceful.
(c) Let A contain an odd number of elements. By making a transfer $a \longrightarrow b$ of the first type followed by a transfer $b \longrightarrow a-1$(respectively, $b \longrightarrow a+1$) of the second type, we can keep from A an odd number of elements at a and an even number of elements at b and move the rest to $a-1$(respectively,a+1),that the resultant tree thus formed will be graceful.

Proof: (a) The vertices of the given sequence of transfers satisfy the property **P** and therefore we can use Lemma 2.5(a) and observation 2.6(a) repeatedly and hence the proof follows.

(b)By using Lemma 2.5(a) and Observation 2.6(a) repeatedly we can keep an odd number of vertices at each vertex of transfer $a \longrightarrow b \longrightarrow a-1 \longrightarrow b+1 \longrightarrow a \longrightarrow b \longrightarrow a-1 \longrightarrow b+1 \longrightarrow a-2$(respectively,

$a \longrightarrow b \longrightarrow a+1 \longrightarrow b-1 \longrightarrow a \longrightarrow b \longrightarrow a+1 \longrightarrow b-1 \longrightarrow a+2$),because the vertices of the sequence of transfers satisfy the property **P** .Since each of the vertices $a, b, a-1$,and $b+1$(respectively, $a, b, a+1$, and $b-1$) appear twice in the sequence of transfers and any positive even integer can be written as the sum of two odd positive integers, the result follows.
(c) The proof follows from the part (b) and Observation 2.6(a)

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3. Results

In this section we give the main result (i.e. Theorem 3.1) of this paper.

Theorem 3.1 The lobsters in Table 3.1,3.2, and 3.3 given below are Graceful.

Table 3.1

Lobsters ↓	x_0	$x_j, 1 \le j \le n_1$	$x_j, n_1+1 \le j \le n_2$	$x_j, n_2+1 \le j \le m$
(a)	$(0,e,e)$	$(0,o,o)$	$(0,e,e)$	$(0,e,0)$

Table 3.2

Lobsters ↓	x_0	$x_j, 1 \le j \le n_1$	$x_j, n_1+1 \le j \le m$
(a)	$(0,e,e)$	$(0,o,o)$	$(0,e,e)$
(b)	$(0,e,e)$	$(0,o,o)$	$(0,e,0)$
(c)	$(0,e,e)$	$(0,e,e)$	$(0,e,0)$

Table 3.3

Lobsters ↓	x_0	$x_j, 1 \le j \le m$
(a)	$(0,o,o)$ or $(0,e,0)$	$(0,e,0)$

Proof: Let L be a lobsters of type (a) in Table 3.1. Figure 1 is an example of such a lobster. We get a graceful labeling of L if we proceed as per the following steps.

1. For $0 \le i \le m$, let e_i and p_i be the number of even and pendant branches, respectively, incident on x_i . Let $|E(L)| = q$, $\sum_{i=0}^{m} o_i = s_0, \sum_{i=0}^{m} e_i = s_e$, and $\sum_{i=o}^{m} p_i = s_p$.
2. We first form the graceful tree $G(L)$ as shown in Figure 4 and Figure 5 with $|E(G(L)) = q+1|$,i.e. we attach a new pendant vertex x_{m+1} to the vertex x_m, the degree of each vertex x_i, $1 \le i \le m$, is two, and x_0 is attached to $q-m$ pendant vertices. We consider the following graceful labeling of $G(L)$.

If m is even :

$$f(v) = \begin{cases} \frac{m}{2} - i\,, \;\; v = x_{2i}\; i = 0,1,2,\dots,\frac{m}{2} \\ q - \frac{m}{2} + 1 + i\,, v = x_{2i+1},\; i = 0,1,2,\dots.,\frac{m}{2} \\ r, r = \frac{m}{2}+1, \frac{m}{2}+2, \dots\dots, q - \frac{m}{2} \\ \text{for the } q-m \text{ pendant vertices adjacent to } x_0 \end{cases} \tag{3.1}$$

If m is odd:

$$f(v) = \begin{cases} \frac{m-1}{2} - i\,, \;\; v = x_{2i+1}\; i = 0,1,2,\dots,\frac{m-1}{2} \\ q - \frac{m-1}{2} + i\,, v = x_{2i},\; i = 0,1,2,\dots.,\frac{m+1}{2} \\ r, r = \frac{m-1}{2}+1, \frac{m-1}{2}+2, \dots\dots, q - \frac{m-1}{2} - 1 \\ \text{for the } q-m \text{ pendant vertices adjacent to } x_0 \end{cases} \tag{3.2}$$

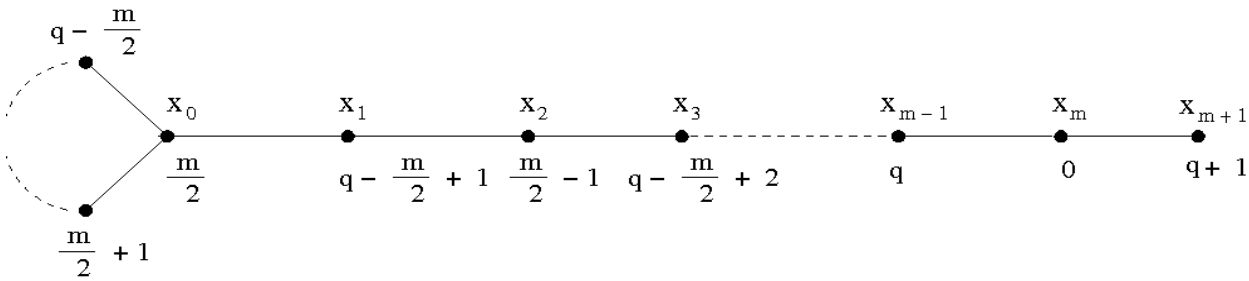

Figure 4: The tree $G(L)$ corresponding to the lobsters L for the case m is even.

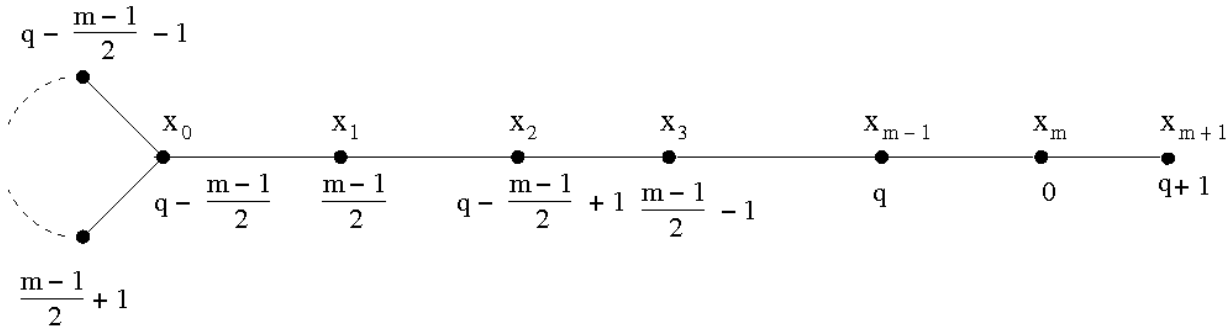

Figure 5: The tree $G(L)$ corresponding to the lobster L for the case m is odd

Let A_0 be the set of all pendant vertices adjacent to x_0 in $G(L)$. The set A_0 can be written as $A_0 = \{a_1, a_{2,} \dots, a_{q-m}\}$, where , for $1 \le k \le q-m$,

$$a_k = \begin{cases} q - \dfrac{m}{2} + 1 - k \text{ if } m \text{ is even} \\ \dfrac{m-1}{2} + k \text{ if } m \text{ is odd} \end{cases}$$

Further, the elements of A_0 satisfy $a_k + a_{q-m+1-k} = f(x_0) + f(x_1),\ 1 \le i \le \left[\frac{q-m+1}{2}\right]$.

3. Let us define an integer I by $I = \begin{cases} e_0 \text{ if } e_0 \text{ is even} \\ e_0 - 1 \text{ if } e_0 \text{ is odd} \end{cases}$
   We keep $\frac{I}{2}$ pairs $(a_k, a_{q-m+1-k}), k = 1,2, \dots \dots, \frac{I}{2}$ at x_0 and move the rest to x_1 and let the tree thus formed be G_1. The tree G_1 has the graceful labeling f by Lemma 2.3. Let A_1 denote the set of vertices in A_0 that are transferred to x_1, i.e. $A_1 = \{a_{\frac{I}{2}+1}, a_{\frac{I}{2}+2}, \dots \dots, a_{q-m-\frac{I}{2}}\}$. We define the integers r_i ,$1 \le i \le m$, as $r_i = e_i$ if $1 \le i \le n_1$, and $r_i = e_i - (2 \propto_i + 1)$ if $n_1 + 1 \le i \le m$, where $\propto_i$ are arbitrary non-negative integers with $2 \propto_i + 1 < e_i$. Then we carry out the transfer $T_1: x_1 \longrightarrow x_2 \longrightarrow \cdots \longrightarrow x_m \longrightarrow x_{m+1}$ with each transfer is a transfer of first type and keep r_i vertices from A_1 at each x_i for $1 \le i \le m$. The resultant tree say, G_2 thus formed after the transfer T_1 will be graceful by Lemma 2.7.
4. A_{m+1} is the set of vertices of A_1 that have come to the vertex x_{m+1} after step 3. We make a transfer $x_{m+1} \to x_m$, i.e. $q_{m+1} \to 0$, of the first type and bring back all the elements of A_{m+1} to x_m. Then we remove the vertex x_{m+1}.Obviously, the new tree thus formed, say G_3, is graceful.
5. Consider the transfer $T_2: x_m \longrightarrow x_{m-1} \longrightarrow x_{m-2} \longrightarrow \cdots \longrightarrow x_{n_1+1} \longrightarrow x_{n_1}$ with each transfer in T_2 is a transfer of the first type. By Observation 2.6, the labels of the vertices x_m and x_{m-1} and the set A_{m+1} satisfy the properties of the vertices a and b and the set A, respectively, of 2.7. So we carry out the transfer T_2 consisting of $m - n_1$ successive transfer of the first type keeping $2\alpha_i + 1$ vertices from A_{m+1} at each vertex x_i of the transfer T_2. By Lemma 2.7 the new tree, say G_4 thus formed is graceful. Let B_{n_1} be the set of vertices of A_{m+1} which have been transferred to x_{n_1}.
6. Next we carry out the transfer $x_{n_1} \longrightarrow x_{n_1+1}$ and transferring every vertex of B_{n_1} to x_{n_1+1} . Obviously, the new tree thus formed, say G_5, is graceful.

7. Next consider the transfer $T_3: x_{n_1+1}+1 \longrightarrow x_{n_1+1}+2 \ldots\ldots \longrightarrow x_{n_2} \longrightarrow x_{n_2+1}$. By Observation 2.6, the vertices $x_{n_1+1}+1$ and $x_{n_1+1}+2$ and the set B_{n_1} satisfy the properties of the vertices a and b and the set A, respectively, of Lemma 2.7. Now we carry out the transfer $T_3: x_{n_1+1}+1 \longrightarrow x_{n_1+1}+2 \ldots\ldots \longrightarrow x_{n_2} \longrightarrow x_{n_2+1}$ consisting of n_2-n_1 successive transfers of the first type keeping $2\beta_i+1$ vertices from B_{n_1} at each vertex x_i of the transfer T_3, where β_i, for $n_1+1 \le i \le n_2, 2\beta_i+1 < p_i$ are arbitrary integers. By Lemma 2.7, the resultant tree say G_6 thus formed after the transfer T_3 is graceful. Let C_{n_2} be the set of vertices of B_{n_1} which has been transferred to the vertex x_{n_2+1} after the transfer T_3.
8. Next carry out transfer $x_{n_2+1} \longrightarrow x_{n_2}$ bringing all the vertices of C_{n_2} to the vertex x_{n_2}. Obviously, the new tree thus formed, say G_7 is graceful.
9. Consider the transfer $T_4: x_{n_2} \longrightarrow x_{n_2-1} \ldots \longrightarrow x_0$. By Observation 2.6, the vertices x_{n_2} and x_{n_2-1} of T_4 and the set C_{n_2} satisfy the properties of the vertices a and b and the set A, respectively, of Lemma 2.7. For $1 \le i \le n_2$, we define r_i^*

$$r_i^* = \begin{cases} p_i \text{ if } 1 \le i \le n_1 \\ p_i - 2\beta_i - 1 \text{ if } n_1+1 \le i \le n_2 \end{cases}$$

Now we carry out the transfer $T_4: x_{n_2} \longrightarrow x_{n_2-1} \ldots \longrightarrow x_0$, where each transfer is a transfer of the first type and keep r_i^* vertices at each vertex x_i of the transfer. By Lemma 2.7, the resultant tree say G_8 thus formed after the transfer T_3 is graceful.
10. Let D_0 be the set of vertices of C_{n_2} that have come to the vertex x_0 after the transfer T_4. Finally, we consider the transfer $T_5: x_0 \longrightarrow a_1 \longrightarrow a_{q-m} \longrightarrow a_2 \longrightarrow a_{q-m-1} \longrightarrow \cdots \longrightarrow a_s$, where $s = \frac{s_e+1}{2}$ if s_e is odd and $s = q-m+1-\frac{s_e}{2}$ if s_e is even. By Observation 2.6, we see that the vertices x_0 and a_1 and the set D_0 satisfy the properties of the vertices a and b and the set A, respectively, of Lemma 2.7. Now we carry out the transfer T_5, where each transfer is a transfer of the second type keeping desired even number of vertices from D_0 in each vertex of T_5, so that we back the desired lobster. By Lemma 2.7(c), the lobster is graceful.

■

Example: Figure 1 represents a lobster L of type (a) in table 3.1. Here $q = 105, m = 7, n_1 = 2, n_2 = 4$.Here $e_0 = 2, p_0 = 2, e_1 = 1, p_1 = 3, e_2 = 3, p_2 = 1, e_3 = 4, p_3 = 2, e_4 = 2, p_4 = 4, e_5 = 4, e_6 = 2, e_7 = 4$; $p_5 = p_6 = p_7 = 0$. Here $s_e = 12$ and $s_p = 12$. We take $\alpha_3 = 0, \alpha_4 = 0, \alpha_5 = 0, \alpha_6 = 0$, and $\alpha_7 = 0; \beta_3 = 0, \beta_4 = 0$. The method outlined in the steps 2 to 10 in the proof above is described in Figures 6 to 15. Figure 15 represents the lobster L given in figure 1 with a graceful labeling.

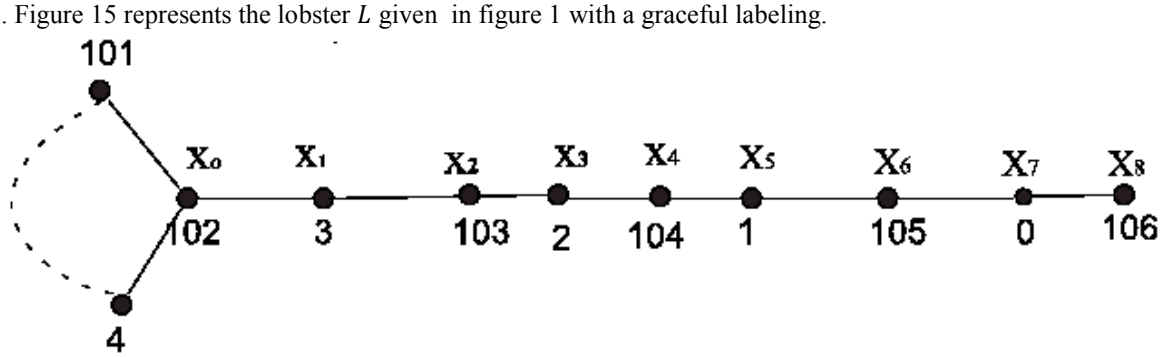

Figure 6: The tree $G(L)$ corresponding to L obtained in step 2. Here $m = 7, a_k = 3+k, 1 \le k \le 98$ and hence $A_0 = \{4,5, \ldots\ldots\ldots, 101\}$.

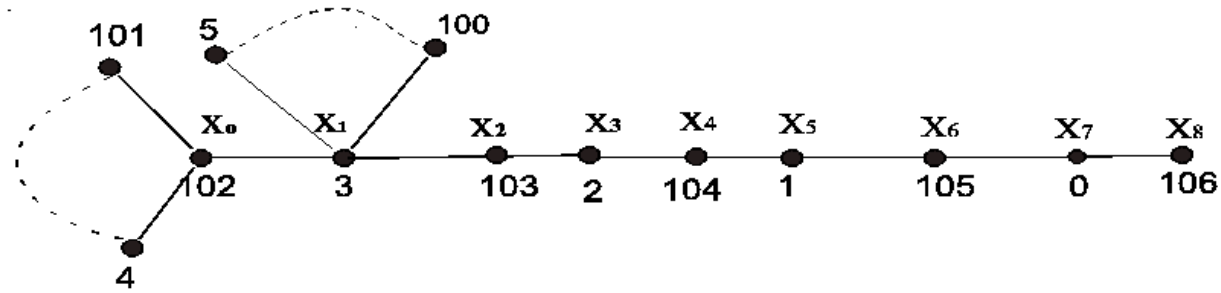

Figure 7: The tree G_1 obtained from $G(L)$ after carrying out the transfer $x_0 \longrightarrow x_1$ in step 3.Here $e_0 = 2, I = 1, A_1 = \{a_2, a_3, \dots \dots a_{97}\} = \{5,6, \dots \dots ,100\}$.

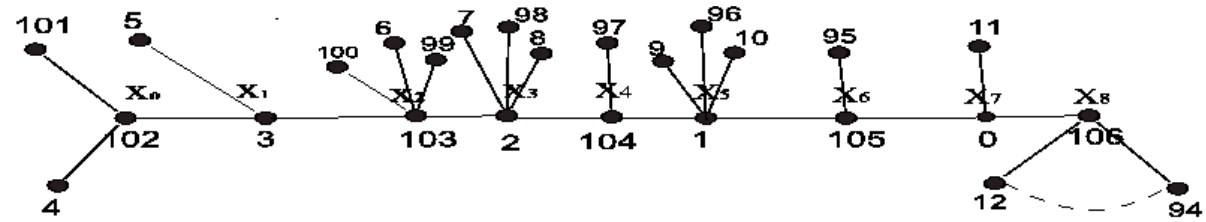

Figure 8: The tree G_2 obtained from G_1 after carrying out the transfer $T_1: x_1 \longrightarrow x_2 \longrightarrow \cdots \longrightarrow x_7 \longrightarrow x_8$ in step 3. Here $A_{m+1} = A_8 = \{12,13,14, \dots \dots ,94\}$.

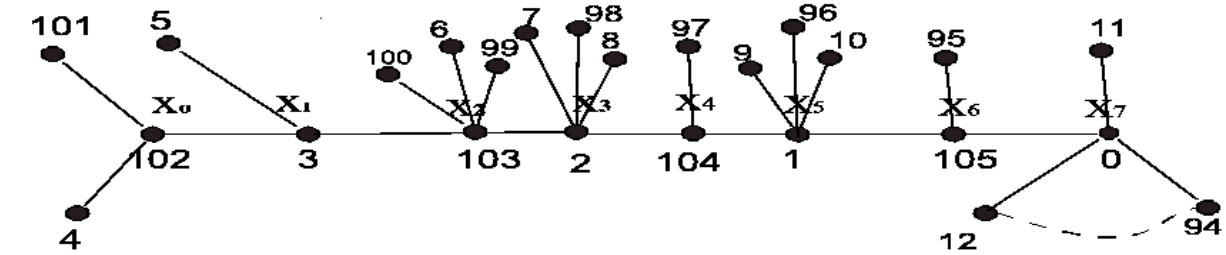

Figure 9: The tree G_3 obtained from G_2 after carrying out the transfer $x_8 \longrightarrow x_7$ in step 4.

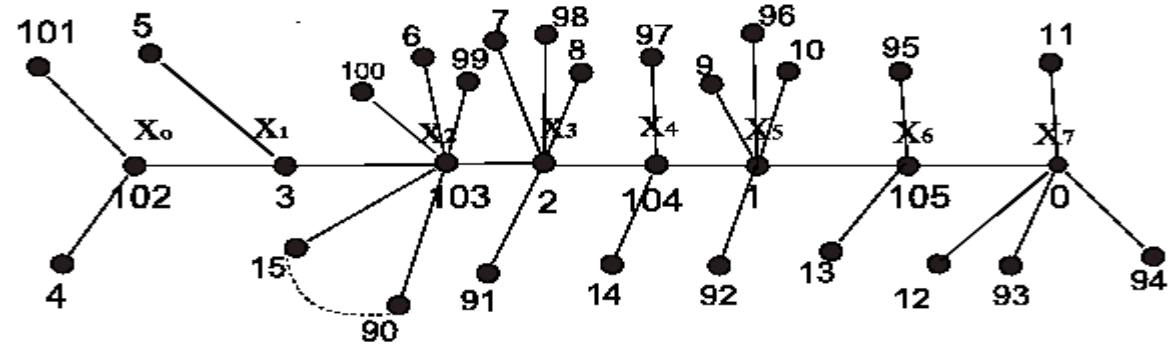

Figure 10: The tree G_4obtained from G_3 after carrying out the transfer $T_2: x_7 \longrightarrow x_6 \longrightarrow x_5 \longrightarrow x_4 \longrightarrow x_3 \longrightarrow x_2$ in step 5. Here $n_1 = 2, B_{n_1} = B_2 = \{15,16, \dots ,90\}$

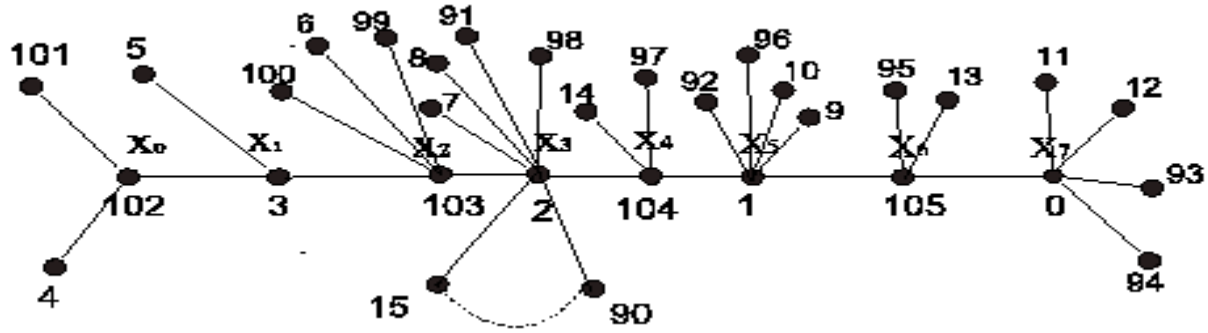

Figure 11: The tree G_5 obtained from G_4 after carrying out the transfer $x_2 \longrightarrow x_3$ in step 6.

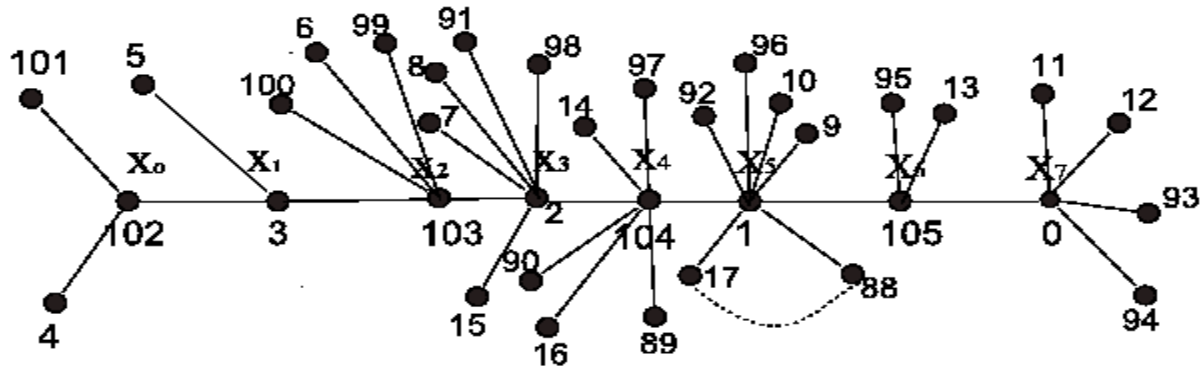

Figure 12: The tree G_6 obtained from G_5 after carrying out the transfer $T_3: x_3 \longrightarrow x_4 \longrightarrow x_5$ in step 7. Here $n_2 = 4, C_{n_2} = C_2 = \{17,18, \dots \dots ,88\}$

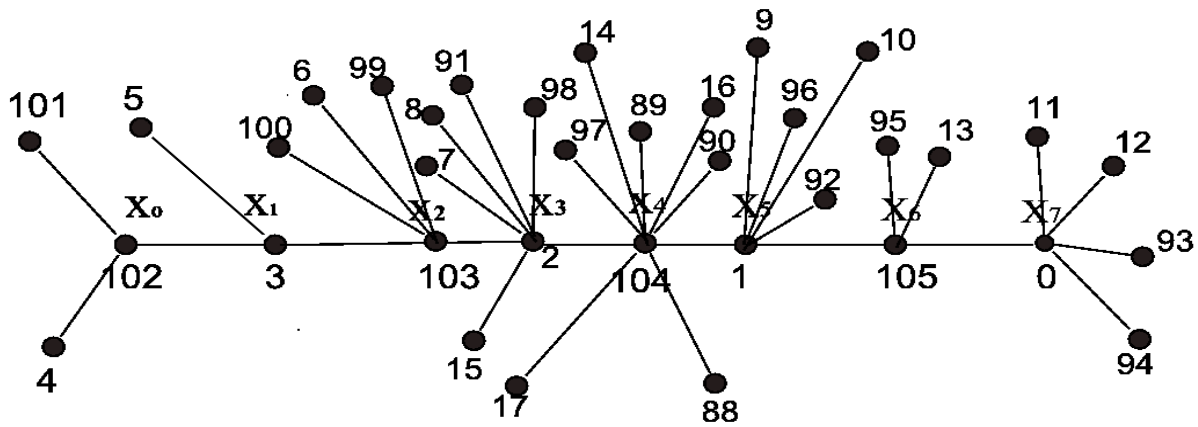

Figure 13: The tree G_7 obtained from G_6 after carrying out the transfer $x_5 \longrightarrow x_4$ in step 8.

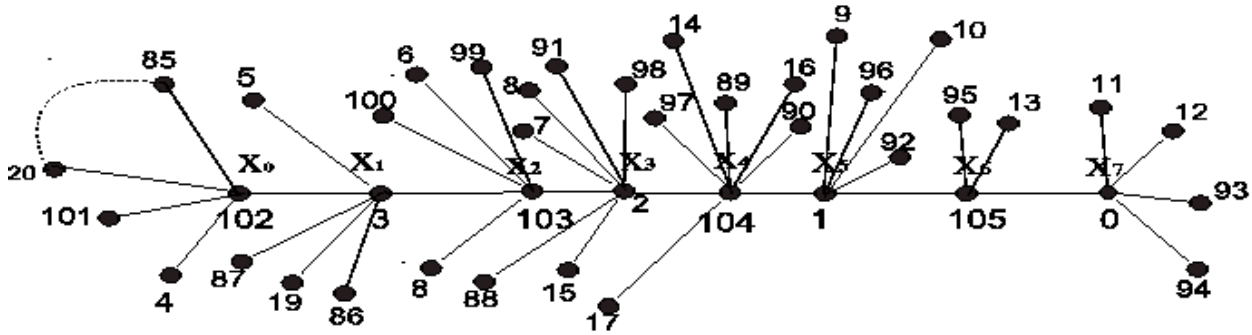

Figure 14: The tree G_8 obtained from G_7after carrying out the transfer $T_4: x_4 \longrightarrow x_3 \longrightarrow x_2 \longrightarrow x_1 \longrightarrow x_0$ in step 9. Here $D_0 = \{20,21, \dots \dots ,86\}$.

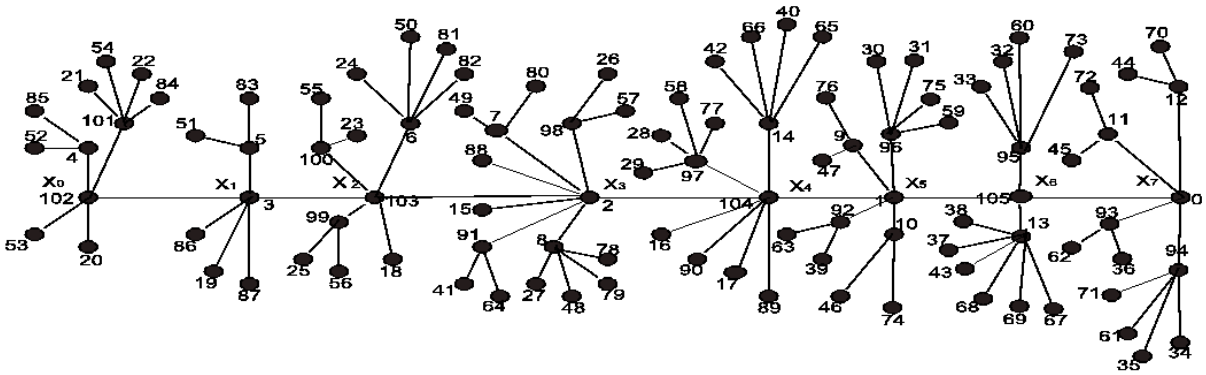

Figure 15 : The lobster L in figure 1 with a graceful labeling obtained from the graceful tree G_8 after carrying out the transfer $T_5: x_0 \longrightarrow a_1 \longrightarrow a_{q-m} \longrightarrow a_2 \longrightarrow a_{q-m-1} \longrightarrow \cdots \longrightarrow a_s, s = 93$, i.e. the transfer $x_0 \longrightarrow a_1 \longrightarrow a_{98} \longrightarrow a_2 \longrightarrow a_{97} \longrightarrow \cdots \longrightarrow a_6 \longrightarrow a_{93}$, in step 10.

For the lobsters in Table 3.2, the proof follows if we make few changes in step 1 to 10 of the proof involving the lobster of type(a) in Table 3.1. For lobsters of type (a),repeat steps 1,2 and 3, do not remove the vertex x_{m+1} in step 4, repeat steps 5 and 6, set $n_2 = m$ in steps 7,8and 9, remove the vertex x_{m+1} in step 8, and repeat step 10. For lobsters of type (b), we repeat steps 1,2,3 and 4, set $n_1 = m$ and $2\alpha_i + 1 = t_i$, where $t_i = p_i$, for $1 \leq i \leq n_1$ and $t_i = 2\alpha_i + 1$ for $n_1 + 1 \leq i \leq m$, in step 5,and set $D_0 = B_0, C_{n_2} = A_{m+1}$ and replace $T_4$with T_2 in step 10. For lobsters of type (c), repeat steps 1 and 2, set $r_i = e_i - 2\alpha_i - 1$, for $1 \leq i \leq m$, in step 3, repeat step 4,and set $n_1 = 0$ and $n_2 = n_1$ in steps 5 to 10. For the lobsters of type (a)in Table 3.3, the proof follows from the proof involving the lobsters of type (a) in Table 3.1 if we repeat steps 1 and 2, set $n_1 = 0$, i.e. $r_i = e_i - 2\alpha_i - 1$, for $1 \leq i \leq m$, in step 3, repeat step 4, set $n_1 = 0$ in step 5, and set $D_0 = B_0, C_{n_2} = A_{m+1}$ and replace T_4 with T_2 in step 10

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