A Class Of Graceful Lobsters with even number of branches incident on the central path

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Abstract

We observe that a lobster with a diameter at least five has a unique path \(x_0, x_1, \ldots, x_m\) (called the central path) such that \(x_0\) and \(x_m\) are adjacent to the centers of at least one \(k_{1,s}\), where \(s > 0\), and besides adjacencies in the central path each \(x_i, 1 \leq i \leq m - 1\), is at most adjacent to the centers of some \(k_{1,s}\), \(s \geq 0\). We call \(k_{1,s}\) an odd branch if \(s\) is odd, an even branch if \(s\) is non-zero even, and a pendant branch if \(s = 0\). In this paper we give graceful labeling to some new classes of lobsters with each vertex of the central path is attached to an even number of branches. Furthermore the branches incident on the central path are either even branches or some combinations of even and pendant branches.

Keywords: graceful labeling, lobsters, component moving transformation, odd and even branches, transfer of the first type and second type.

AMS classification: 05C78

1. Introduction

**Definition 1.1** A graceful labeling of a tree \(T\) with \(q\) edges is a bijection \(f: V(T) \to \{0, 1, 2, \ldots, q\}\) such that \(|f(u) - f(v)| = 1\} = \{1, 2, \ldots, q\}\). A tree which has a graceful labeling is called a graceful tree.

**Definition 1.2** A lobster is a tree having a path from which every vertex has a distance at most two. It is easy to check that a lobster \(L\) of diameter at least five has a unique path \(H = x_0, x_1, \ldots, x_m\) such that besides the adjacencies in \(H\), each \(x_i, 1 \leq i \leq m - 1\), is at most adjacent to the centers of some stars \(k_{1,s}\), \(s \geq 0\), whereas the vertices \(x_0\) and \(x_m\) are adjacent to the centers of at least one star \(k_{1,s}\), with \(s > 1\). This path \(H\) is called the central path of the lobsters \(L\). Throughout the paper we use \(H\) to denote the central path of a lobster with diameter at least five. If a vertex \(x_i \in V(H)\) is adjacent to the center of \(k_{1,s}\), \(s \geq 0\), then we call \(k_{1,s}\) an even branch if \(s\) is nonzero even, an odd branch if \(s\) is odd, and a pendant branch if \(s = 0\). Therefore, the branches incident on a vertex in the central path of a lobster can be classified into three types, i.e. even, odd, and pendant defined as above. Furthermore, whenever we say \(x_i\), for some \(0 \leq i \leq m\), is attached to an even number of branches, we mean a “non zero” even number of branches unless otherwise stated.

**Notation 1.3** In view of Definition 1.2, a combination of branches incident on any \(x_i, 0 \leq i \leq m\), can be represented by a triple \((x, y, z)\), where \(x, y, \text{and } z\) represent the number of odd, even, and pendant branches incident on \(x_i\). We use the symbols \(e\) and \(o\) to represent a non-zero even number and an odd number, respectively. For example \((e, 0, o)\) means an even number of odd branches, no even branch, and an odd number of pendant branches.

In 1979, Bermond [1] conjectured that “all lobsters are graceful” which is a special case of the famous and unsolved “the graceful tree conjecture” of Ringel and Kotzig (1964) [12], which states that “all tress are
Bermond’s conjecture is also open and very few classes of lobsters are known to be graceful. Ng [10], Wang et al. [13], Chen et al. [2], Morgan [9] (see [3]), and Mishra and Panigrahi[4,5,6,7,8,11] have given graceful labeling to some classes of lobsters. However, none of the above results gives graceful labeling to a class of lobsters in which every vertex of the central path is attached to an even number of branches.

The lobsters to which we give graceful labeling in this paper have the following characteristic feature.

1. The vertex $x_0$ is attached to the combination $(0, e, e)$. For an integer $t_1$, $1 \leq t_1 \leq m$ each $x_i$ is attached to the combination $(0, 0, o)$. If $t_1 < m$ then each $x_i, t_1 + 1 \leq i \leq t_2, t_2 \leq m$ is attached to the combination $(0, e, e)$. If $t_2 \leq m$, then each $x_i, t_2 + 1 \leq i \leq m$, is attached to the combination $(0, e, 0)$.

2. Preliminaries

**Definition 2.1** For an edge $e = \{u, v\}$ of a tree $T$, we define $u(T)$ as that connected component of $T - e$ which contains the vertex $u$. Here we say $u(T)$ is a component incident on the vertex $v$. If $a$ and $b$ are vertices of a tree $T$, $u(T)$ is a component incident on $a$ and $b \notin u(T)$ then deleting the edge $\{a, u\}$ from $T$ and making $b$ and $u$ adjacent is called the component moving transformation. Here we say the component $u(T)$ has been transferred or moved from $a$ to $b$. This is illustrated in Figure 2. Throughout the paper we write “the component $u$” instead of writing “the component $u(T)$”. Whenever we wish to refer $u$ as a vertex, we write “the vertex $u$”. By the label of the component “$u(T)$” we mean the label of the vertex $u$.

![Figure 2](image_url)

Figure 2: The tree $T_1$ obtained from the tree $T$ by moving the component $u(T)$ from the vertex $a$ to the vertex $b$. 

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**Notation 2.2** For any two vertices $a$ and $b$ of a tree $T$, the notation $a \to b$ transfer means that we move some components incident on the vertex $a$ to the vertex $b$. If we consider successive transfers $a \to b, b \to c, c \to d, \ldots$, we simply write $a \to b \to c \to d \to \ldots$ transfer. In a transfer $a_1 \to a_2 \to a_3 \to \cdots \to a_n$, we call each vertex except $a_n$ a vertex of the transfer.

**Lemma 2.3** [8] Let $f$ be a graceful labeling of a tree $T$; let $a$ and $b$ be two vertices of $T$; let $u(T)$ and $v(T)$ be two components incident on $a$, where $b \notin u(T) \cup v(T)$. Then the following hold:

i) If $f(u) + f(v) = f(a) + f(b)$ then the tree $T''$ obtained from $T$ by moving the components $u(T)$ and $v(T)$ from $a$ to $b$ is also graceful.

ii) If $2f(u) = f(a) + f(b)$ then the tree $T''$ obtained from $T$ by moving the component $u(T)$ from $a$ to $b$ is also graceful.

**Definition 2.4** Let $T$ be a labelled tree and $a$ and $b$ be two vertices of $T$, and $a$ be attached to some components. The $a \to b$ transfer is called a transfer of the first type if the labels of the transferred components constitute a set of consecutive integers. The $a \to b$ transfer is called a transfer of the second type if the labels of the transferred components can be divided into two segments, where each segment is a set of consecutive integers.

![Diagram](image-url)

Figure 3: The tree in (a) is a tree with a graceful labeling. The trees in (b) and (c) are obtained from (a) by applying a transfer of the first type $16 \to 2$ and a transfer of the second type $16 \to 2$, respectively.

Though the following two results i.e. Lemma 2.5 and Lemma 2.7 already exist in the literature, still we give their here because we adopt the method described in the proofs of these lemmas to carry out a sequence of the first and second types for giving graceful labeling to the lobsters here.

**Lemma 2.5** [8] In a graceful labeling $f$ of a graceful tree $T$, let $a$ and $b$ be the labels of two vertices. Let $a$ be attached to a set $A$ of vertices (or components) having labels $n, n+1, n+2, \ldots, n+p (\text{in } f)$, which satisfy either (i) $(n+1+i) + (n+p-i) = a+b, i \geq 0$ or (ii) $(n+i) + (n+p-1-i) = x+y, i \geq 0$. Then the following hold.

(a) By making a transfer $a \to b$ of the first type we can keep an odd number of components at $a$ from the set $A$ and move the rest to $b$ and the resultant tree thus formed will be graceful.

(b) If $A$ has an even number of elements, then by making a transfer $a \to b$ of the second type we can keep an even number of components at $a$ from the set $A$ and move the rest to $b$, and the resultant tree thus formed will be graceful.

**Proof:** (a) By definition of the transfer of the first type, the labels of the components which are transferred to $y$ must be consecutive integers. If the elements of $A$ satisfy (i) (respectively, (ii)), then we keep any odd number of elements, say $2r+1$ elements, namely $n, n+1+i, n+p-i, i = 0,1,2, \ldots, r-1$ (respectively, $n+p, n+1+i, n+p-1-i, i = 0,1,2, \ldots, r-1$) at $a$, and transfer the rest to $b$. Let $A_1$ be the set of elements of $A$ which has been moved to $b$. For each $z \in A_1$, either we have $2z = a+b$ or there is another (unique) element $w$ in $A$ such that $z+w = a+b$. Therefore, the new tree thus formed is graceful by Lemma 2.3.
(b) Since $A$ contains an even number of elements, $p$ is odd. If the elements of $A$ satisfy (i) (respectively, (ii)), then from $A$ we keep an even number of elements, say $2r$ elements namely $n, n + \frac{p+1}{2}, n + 1 + i, n + p - i$ (respectively, $n + p$ and $n + \frac{p+1}{2}, n + i, n + p - 1), 0 \leq i \leq r - 2$ if $r \geq 2$; otherwise $n$ and $n + \frac{p+1}{2}$ (respectively, $n + p$ and $n + \frac{p+1}{2}$) at $a$, and move the rest to $b$. The transfer that we have done here is easily seen to be a transfer of second type and the resultant tree thus formed is graceful by Lemma 2.3.

**Observation 2.6**[8] (a) For any pair of vertex labels $x$ and $y$ in a graceful tree, where $x$ is attached to a set of components $A$ as in Lemma 2.5, whenever we make a transfer $x \rightarrow y$ of the first type as in Lemma 2.5(a), the set $A_1$ of components of $A$ that are transferred to $y$ is of the form $A_1 = \{n + r, n + r + 1, \ldots, n + r_t\}$ with $(n + r + i) + (n + r_t - i) = x + y$, where $0 \leq i \leq \left\lfloor \frac{r-r_t+1}{2} \right\rfloor$. Further, by re-pairing the elements of $A_1$, we get,

for $0 \leq i \leq \left\lfloor \frac{r-r_t+1}{2} \right\rfloor$.

1. $(n + r + i) + (n + r_t - i) = x + y - 1$ and
2. $(n + r + 1 + i) + (n + r_t - i) = x + y + 1$.

Therefore, next if we make a transfer $y \rightarrow x - 1$ or $y \rightarrow x + 1$, then the set $A_1$ and the vertices (or labels) $y$ and $x - 1$ or $y$ and $x + 1$ satisfy the hypothesis of Lemma 2.5(a).

(b) We like to mention one important point, i.e. the vertices of the sequences of transfer we deal in this paper has the property “P”: for any three consecutive integers $u, v, w$ of the sequence transfer, we have “$w = u \pm 1$”. Because of this property “P” we can use Lemma 2.5(a) and part(a) of this observation repeatedly, this will be clear in Lemma 2.7.

(c) If $x, y$, and $A$ are as in $(a)$ and we make a transfer of second kind $x \rightarrow y$ as in Lemma 2.5(b), then the set of components of $A$ transferred to $y$ is divided into two disjoint segments, say $A_1$ and $A_2$ where $A_1$ and $A_2$ are of the form $A_1 = \{a_1, a_1 + 1, \ldots, a_1 + k\}$ and $A_2 = \{a_2, a_2 + 1, \ldots, a_2 + k\}$ with $(a_1 + i) + (a_2 + k - i) = x + y, 0 \leq i \leq k$.

**Lemma 2.7**[8] In a graceful labeling $f$ of a tree $T$, let $a, a - 1, a - 2, \ldots, a - p_1, b, b + 1, b + 2, \ldots, b + p_2$ (respectively, $a, a + 1, a + 2, \ldots, a + r_1, b - 1, b - 2, \ldots, b - r_2$) be some vertex labels. Let the vertex $a$ be attached to a set $A$ of vertices (or components) having labels $n, n + 1, n + 2, \ldots, n + p$ (different from the above vertex labels) if $f$ and satisfy either $(n + i + 1) + (n + p - i) = a + b$ or $(n + i) + (n + p - 1 - i) = a + b, 0 \leq i \leq \left\lfloor \frac{p+1}{2} \right\rfloor$, then the following hold.

(a) By making a sequence of transfers of the first type $a \rightarrow b \rightarrow a - 1 \rightarrow b + 1 \rightarrow a - 2 \rightarrow b + 2 \rightarrow \cdots \rightarrow x$ (respectively, $a \rightarrow b \rightarrow a + 1 \rightarrow b - 1 \rightarrow a + 2 \rightarrow b - 2 \rightarrow \cdots \rightarrow x$), where $z = a - p_1$ or $b + p_2$ (respectively, $z = a + r_1$ or $b - r_2$), an odd number of elements from $A$ can be kept at each vertex of the transfer and the resultant tree thus formed will be graceful.

(b) If $A$ contains an even number of elements, then by making a sequence of transfers of second type $a \rightarrow b \rightarrow a - 1 \rightarrow b + 1 \rightarrow a - 2 \rightarrow b + 2 \rightarrow \cdots \rightarrow z$ (respectively, $a \rightarrow b \rightarrow a + 1 \rightarrow b - 1 \rightarrow a + 2 \rightarrow b - 2 \rightarrow \cdots \rightarrow z$), where $z = a - p_1$ or $b + p_2$ (respectively, $z = a + r_1$ or $b - r_2$), an even number of elements from $A$ can be kept at each vertex of the transfer, such that the resultant tree thus formed will be graceful.

(c) Let $A$ contain an odd number of elements. By making a transfer $a \rightarrow b$ of the first type followed by a transfer $b \rightarrow a - 1$ (respectively, $b \rightarrow a + 1$) of the second type, we can keep from $A$ an odd number of elements at $a$ and an even number of elements at $b$ and move the rest to $a - 1$ (respectively, $a + 1$), that the resultant tree thus formed will be graceful.

**Proof:** (a) The vertices of the given sequence of transfers satisfy the property P and therefore we can use Lemma 2.5(a) and observation 2.6(a) repeatedly and hence the proof follows.

(b) By using Lemma 2.5(a) and Observation 2.6(a) repeatedly we can keep an odd number of vertices at each vertex of transfer $a \rightarrow b \rightarrow a - 1 \rightarrow b + 1 \rightarrow a \rightarrow b \rightarrow a - 1 \rightarrow b + 1 \rightarrow a - 2$ (respectively,
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$a \rightarrow b \rightarrow a + 1 \rightarrow b - 1 \rightarrow a \rightarrow b \rightarrow a + 1 \rightarrow b - 1 \rightarrow a + 2$, because the vertices of the sequence of transfers satisfy the property $P$. Since each of the vertices $a, b, a - 1, b + 1$ (respectively, $a, b, a + 1, b - 1$) appear twice in the sequence of transfers and any positive even integer can be written as the sum of two odd positive integers, the result follows.

(c) The proof follows from the part (b) and Observation 2.6(a).

3. Results
In this section we give the main result (i.e. Theorem 3.1) of this paper.

Theorem 3.1 The lobsters in Table 3.1, 3.2, and 3.3 given below are Graceful.

Table 3.1

<table>
<thead>
<tr>
<th>Lobsters</th>
<th>$x_0$</th>
<th>$x_j, 1 \leq j \leq n_1$</th>
<th>$x_j, n_1 + 1 \leq j \leq n_2$</th>
<th>$x_j, n_2 + 1 \leq j \leq m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>$(0, e, e)$</td>
<td>$(0, o, o)$</td>
<td>$(0, e, e)$</td>
<td>$(0, e, 0)$</td>
</tr>
</tbody>
</table>

Table 3.2

<table>
<thead>
<tr>
<th>Lobsters</th>
<th>$x_0$</th>
<th>$x_j, 1 \leq j \leq n_1$</th>
<th>$x_j, n_1 + 1 \leq j \leq m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>$(0, e, e)$</td>
<td>$(0, o, o)$</td>
<td>$(0, e, e)$</td>
</tr>
<tr>
<td>(b)</td>
<td>$(0, e, e)$</td>
<td>$(0, o, o)$</td>
<td>$(0, e, 0)$</td>
</tr>
<tr>
<td>(c)</td>
<td>$(0, e, e)$</td>
<td>$(0, e, e)$</td>
<td>$(0, e, 0)$</td>
</tr>
</tbody>
</table>

Table 3.3

<table>
<thead>
<tr>
<th>Lobsters</th>
<th>$x_0$</th>
<th>$x_j, 1 \leq j \leq m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>$(0, o, o)$ or $(0, e, 0)$</td>
<td>$(0, e, 0)$</td>
</tr>
</tbody>
</table>

Proof: Let $L$ be a lobsters of type (a) in Table 3.1. Figure 1 is an example of such a lobster. We get a graceful labeling of $L$ if we proceed as per the following steps.

1. For $0 \leq i \leq m$, let $e_i$ and $p_i$ be the number of even and pendant branches, respectively, incident on $x_i$. Let $|E(L)| = q, \sum_{i=0}^{m} e_i = s_e, \sum_{i=0}^{m} e_i = s_p$, and $\sum_{i=0}^{m} p_i = s_p$.

2. We first form the graceful tree $G(L)$ as shown in Figure 4 and Figure 5 with $|E(G(L))| = q + 1$, i.e. we attach a new pendant vertex $x_{m+1}$ to the vertex $x_m$, the degree of each vertex $x_i, 1 \leq i \leq m$, is two, and $x_0$ is attached to $q - m$ pendant vertices. We consider the following graceful labeling of $G(L)$.

If $m$ is even:

$$f(v) = \begin{cases} 
\frac{m-i}{2}, & v = x_{2i}, i = 0, 1, 2, \ldots, \frac{m}{2} \\
q - \frac{m}{2} + 1 + i, & v = x_{2i+1}, i = 0, 1, 2, \ldots, \frac{m}{2} \\
r, r = \frac{m}{2} + 1, \frac{m}{2} + 2, \ldots, q - \frac{m}{2}
\end{cases} \quad (3.1)$$

For the $q - m$ pendant vertices adjacent to $x_0$.

If $m$ is odd:

$$f(v) = \begin{cases} 
\frac{m-1-i}{2}, & v = x_{2i+1}, i = 0, 1, 2, \ldots, \frac{m-1}{2} \\
q - \frac{m-1}{2} + 1 + i, & v = x_{2i}, i = 0, 1, 2, \ldots, \frac{m+1}{2} \\
r, r = \frac{m-1}{2} + 1, \frac{m-1}{2} + 2, \ldots, q - \frac{m-1}{2} - 1
\end{cases} \quad (3.2)$$

For the $q - m$ pendant vertices adjacent to $x_0$. 
Further, the elements of $A_0$ be the set of all pendant vertices adjacent to $x_0$ in $G(L)$. The set $A_0$ can be written as $A_0 = \{a_1, a_2, \ldots, a_{q-m}\}$, where, for $1 \leq k \leq q-m$,

$$a_k = \begin{cases} 
q - \frac{m}{2} + 1 - k & \text{if } m \text{ is even} \\
\frac{m-1}{2} + k & \text{if } m \text{ is odd}
\end{cases}$$

Further, the elements of $A_0$ satisfy $a_k + a_{q-m+1-k} = f(x_0) + f(x_1)$, $1 \leq i \leq \left\lfloor \frac{q-m+1}{2} \right\rfloor$.

3. Let us define an integer $l$ by $l = \left\lfloor \frac{e_0}{2} \right\rfloor$ if $e_0$ is even and $l = e_0 - 1$ if $e_0$ is odd.

We keep $\frac{l}{2}$ pairs $\{a_k, a_{q-m+1-k}\}$, $k = 1, 2, \ldots, \frac{l}{2}$ at $x_0$ and move the rest to $x_1$ and let the tree thus formed be $G_1$. The tree $G_1$ has the graceful labeling $f$ by Lemma 2.3. Let $A_1$ denote the set of vertices in $A_0$ that are transferred to $x_1$, i.e. $A_1 = \{a_{1+1}, a_{1+2}, \ldots, a_{q-m-1}\}$. We define the integers $r_i$, $1 \leq i \leq m$, as $r_i = e_i$ if $1 \leq i \leq n_1$, and $r_i = e_i - (2 \alpha_i + 1)$ if $n_1 + 1 \leq i \leq m$, where $\alpha_i$ are arbitrary non-negative integers with $2 \alpha_i + 1 < e_i$. Then we carry out the transfer $T_1: x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_m \rightarrow x_{m+1}$ with each transfer is a transfer of first type and keep $r_i$ vertices from $A_1$ at each $x_i$ for $1 \leq i \leq m$. The resultant tree say, $G_2$ thus formed after the transfer $T_1$ will be graceful by Lemma 2.7.

4. $A_{m+1}$ is the set of vertices of $A_1$ that have come to the vertex $x_{m+1}$ after step 3. We make a transfer $x_{m+1} \rightarrow x_m$, i.e. $q_{m+1} \rightarrow 0$, of the first type and bring back all the elements of $A_{m+1}$ to $x_m$. Then we remove the vertex $x_{m+1}$. Obviously, the new tree thus formed, say $G_3$, is graceful.

5. Consider the transfer $T_2: x_m \rightarrow x_{m-1} \rightarrow x_{m-2} \rightarrow \cdots \rightarrow x_{n_1+1} \rightarrow x_{n_1}$ with each transfer in $T_2$ is a transfer of the first type. By Observation 2.6, the labels of the vertices $x_m$ and $x_{m-1}$ and the set $A_{m+1}$ satisfy the properties of the vertices $a$ and $b$ and the set $A$, respectively, of 2.7. So we carry out the transfer $T_2$ consisting of $m - n_1$ successive transfer of the first type keeping $2\alpha_i + 1$ vertices from $A_{m+1}$ at each vertex $x_i$ of the transfer $T_2$. By Lemma 2.7 the new tree, say $G_4$ thus formed is graceful. Let $B_{n_1}$ be the set of vertices of $A_{m+1}$ which have been transferred to $x_{n_1}$.

6. Next we carry out the transfer $x_{n_1} \rightarrow x_{n_1+1}$ and transferring every vertex of $B_{n_1}$ to $x_{n_1+1}$. Obviously, the new tree thus formed, say $G_5$, is graceful.
7. Next consider the transfer $T_3: x_{n_1+1} + 1 \rightarrow x_{n_1+1} + 2 \ldots \rightarrow x_{n_2} \rightarrow x_{n_2+1}$. By Observation 2.6, the vertices $x_{n_1+1} + 1$ and $x_{n_1+1} + 2$ and the set $B_{n_1}$ satisfy the properties of the vertices $a$ and $b$ and the set $A$, respectively, of Lemma 2.7. Now we carry out the transfer $T_3: x_{n_1+1} + 1 \rightarrow x_{n_1+1} + 2 \ldots \rightarrow x_{n_2} \rightarrow x_{n_2+1}$ consisting of $n_2 - n_1$ successive transfers of the first type keeping $2\beta_i + 1$ vertices from $B_{n_1}$ at each vertex $x_i$ of the transfer $T_3$, where $\beta_i$, for $n_1 + 1 \leq i \leq n_2$, $2\beta_i + 1 < p_i$ are arbitrary integers. By Lemma 2.7, the resultant tree say $G_6$ thus formed after the transfer $T_3$ is graceful. Let $C_{n_2}$ be the set of vertices of $B_{n_1}$, which has been transferred to the vertex $x_{n_2+1}$ after the transfer $T_3$.

8. Next carry out transfer $x_{n_2+1} \rightarrow x_{n_2}$ bringing all the vertices of $C_{n_2}$ to the vertex $x_{n_2}$. Obviously, the new tree thus formed, say $G_7$ is graceful.

9. Consider the transfer $T_4: x_{n_2} \rightarrow x_{n_2-1} \ldots \rightarrow x_0$. By Observation 2.6, the vertices $x_{n_2}$ and $x_{n_2-1}$ of $T_4$ and the set $C_{n_2}$ satisfy the properties of the vertices $a$ and $b$ and the set $A$, respectively, of Lemma 2.7. For $1 \leq i \leq n_2$, we define $r_i^*$

$$r_i^* = \begin{cases} p_i & \text{if } 1 \leq i \leq n_1 \\ p_i - 2\beta_i - 1 & \text{if } n_1 + 1 \leq i \leq n_2 \end{cases}$$

Now we carry out the transfer $T_4: x_{n_2} \rightarrow x_{n_2-1} \ldots \rightarrow x_0$, where each transfer is a transfer of the first type and keep $r_i^*$ vertices at each vertex $x_i$ of the transfer. By Lemma 2.7, the resultant tree say $G_8$ thus formed after the transfer $T_4$ is graceful.

10. Let $D_0$ be the set of vertices of $C_{n_2}$ that have come to the vertex $x_0$ after the transfer $T_4$. Finally, we consider the transfer $T_5: x_0 \rightarrow a_1 \rightarrow a_{q-m} \rightarrow a_2 \rightarrow a_{q-m-1} \rightarrow \ldots \rightarrow a_s$, where $s = q - m + 1 - \frac{s_e}{2}$ if $s_e$ is odd and $s = q - m + 1 - \frac{s_e}{2}$ if $s_e$ is even. By Observation 2.6, we see that the vertices $x_0$ and $a_i$ and the set $D_0$ satisfy the properties of the vertices $a$ and $b$ and the set $A$, respectively, of Lemma 2.7. Now we carry out the transfer $T_5$, where each transfer is a transfer of the second type keeping desired number of vertices from $D_0$ in each vertex of $T_5$, so that we back the desired lobster. By Lemma 2.7(c), the lobster is graceful.

**Example:** Figure 1 represents a lobster $L$ of type (a) in table 3.1. Here $q = 105, m = 7, n_1 = 2, n_2 = 4$. Here $e_0 = 2, p_0 = 2, e_1 = 1, p_1 = 3, e_2 = 3, p_2 = 1, e_3 = 4, p_3 = 2, e_4 = 2, p_4 = 4, e_5 = 4, e_6 = 2, e_7 = 4, p_5 = p_6 = p_7 = 0$. Here $s_0 = 12$ and $s_p = 12$. We take $\alpha_3 = 0, \alpha_4 = 0, \alpha_5 = 0, \alpha_6 = 0, \alpha_7 = 0, \beta_4 = 0, \beta_5 = 0$. The method outlined in the steps 2 to 10 in the proof above is described in Figures 6 to 15. Figure 15 represents the lobster $L$ given in figure 1 with a graceful labeling.

![Diagram of lobster L](image)

Figure 6: The tree $G(L)$ corresponding to $L$ obtained in step 2. Here $m = 7, \alpha_k = 3 + k, 1 \leq k \leq 98$ and hence $A_0 = \{4,5,\ldots,101\}$. 
Figure 7: The tree $G_1$ obtained from $G(L)$ after carrying out the transfer $x_0 \rightarrow x_1$ in step 3. Here $e_0 = 2$, $I = 1$, $A_1 = \{a_2, a_3, \ldots, a_{97}\} = \{5, 6, \ldots, 100\}$.

Figure 8: The tree $G_2$ obtained from $G_1$ after carrying out the transfer $T_1: x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_7 \rightarrow x_8$ in step 3. Here $A_{m+1} = A_8 = \{12, 13, 14, \ldots, 94\}$.

Figure 9: The tree $G_3$ obtained from $G_2$ after carrying out the transfer $x_8 \rightarrow x_7$ in step 4.

Figure 10: The tree $G_4$ obtained from $G_3$ after carrying out the transfer $T_2: x_7 \rightarrow x_6 \rightarrow x_5 \rightarrow x_4 \rightarrow x_3 \rightarrow x_2$ in step 5. Here $n_1 = 2$, $B_{n_1} = B_2 = \{15, 16, \ldots, 90\}$.
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Figure 11: The tree $G_5$ obtained from $G_4$ after carrying out the transfer $x_2 \rightarrow x_3$ in step 6.

Figure 12: The tree $G_6$ obtained from $G_5$ after carrying out the transfer $T_1: x_3 \rightarrow x_4 \rightarrow x_5$ in step 7. Here $n_2 = 4, C_{n_2} = C_2 = \{17,18, \ldots ,88\}$.

Figure 13: The tree $G_7$ obtained from $G_6$ after carrying out the transfer $x_5 \rightarrow x_4$ in step 8.

Figure 14: The tree $G_8$ obtained from $G_7$ after carrying out the transfer $T_4: x_4 \rightarrow x_3 \rightarrow x_2 \rightarrow x_1 \rightarrow x_0$ in step 9. Here $D_0 = \{20,21, \ldots ,86\}$. 
For the lobsters in Table 3.2, the proof follows if we make few changes in step 1 to 10 of the proof involving the lobster of type (a) in Table 3.1. For lobsters of type (a), repeat steps 1, 2, and 3, do not remove the vertex \( x_m + 1 \) in step 4, repeat steps 5 and 6, set \( n_2 = m \) in steps 7, 8, and 9, remove the vertex \( x_m + 1 \) in step 8, and repeat step 10. For lobsters of type (b), we repeat steps 1, 2, 3, and 4, set \( D_0 = B_0, C_{n_2} = A_{m+1} \) and replace \( T_4 \) with \( T_2 \) in step 10. For lobsters of type (c), repeat steps 1 and 2, set \( n_1 = 0 \), i.e., \( r_i = e_i - 2a_l - 1 \), for \( 1 \leq i \leq m \), in step 3, repeat step 4, and set \( n_1 = 0 \) and \( n_2 = n_1 \) in steps 5 to 10. For the lobsters of type (a) in Table 3.3, the proof follows from the proof involving the lobsters of type (a) in Table 3.1 if we repeat steps 1 and 2, set \( n_1 = 0 \), i.e., \( r_i = e_i - 2a_l - 1 \), for \( 1 \leq i \leq m \), in step 3, repeat step 4, set \( n_1 = 0 \) in step 5, and set \( D_0 = B_0, C_{n_2} = A_{m+1} \) and replace \( T_4 \) with \( T_2 \) in step 10.

References: