Int. J. of Mathematical Sciences and Applications,
Vol. 6, No. 2, December 2016

ISSN: 2230-9888
www.journalshub.com

Fractional integrals of the multivariable Aleph-function and generalized class of polynomials

$F.Y.\ AYANT$[1]

1 Teacher in High School , France
E-mail : fredericayant@gmail.com

ABSTRACT
In the present document, we obtain two key formulas involving fractional integral of multivariable Aleph-function and general class of polynomials. Whereas the integral operator is the generalization of Riemann-Liouville and Erdelyi-Kober fractional integral operators due to M. Saigo. Each of these formulas can be shown to yield interesting new results for various special function of several variables. Several particular cases and remarks are given at the end.

Keywords : fractional integration, fractional calulus, Aleph-function of several variables , Mellin-Barnes contour multiple integral

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20

1. Introduction and preliminaries.

The object of this document is to study the fractional integral formula from the multivariables aleph-function. These function generalize the multivariable I-function recently study by C.K. Sharma and Ahmad [2] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We have : $\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i(1)},q_{i(1)},\tau_{i(1)};R^{(1)};\cdots;p_{i(r)},q_{i(r)};\tau_{i(r)};R^{(r)}}\left(\begin{array}{c} z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{array}\right|$

$$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}]\ ,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:$$
$$\cdots\cdots\cdots\cdots\ ,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}]:$$

$$\left.\begin{array}{l}[(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;\ ;[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}]\\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;;[(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^r\theta_k(s_k)z_k^{s_k}\mathrm{d}s_1\cdots\mathrm{d}s_r$$
(1.1)

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^r\alpha_j^{(k)}s_k)}{\sum_{i=1}^R[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^r\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^r\beta_{ji}^{(k)}s_k)]}\tag{1.2}$$

and $$\theta_k(s_k)=\frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]}\tag{1.3}$$

where $j = 1$ to r and $k = 1$ to r

Suppose , as usual , that the parameters

$a_j, j = 1, \cdots, p; b_j, j = 1, \cdots, q;$

$c_j^{(k)}, j = 1, \cdots, n_k; c_{ji^{(k)}}^{(k)}, j = n_k + 1, \cdots, p_{i^{(k)}};$

$d_j^{(k)}, j = 1, \cdots, m_k; d_{ji^{(k)}}^{(k)}, j = m_k + 1, \cdots, q_{i^{(k)}};$

with $k = 1 \cdots, r, i = 1, \cdots, R$, $i^{(k)} = 1, \cdots, R^{(k)}$

are complex numbers , and the $\alpha' s, \beta' s, \gamma' s$ and $\delta' s$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} + \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} + \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{m_k} \delta_j^{(k)}$$

$$-\tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1$ to n and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|arg z_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \text{ with } k = 1, \cdots, r, i = 1, \cdots, R \text{ , } i^{(k)} = 1, \cdots, R^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_s) = 0(|z_1|^{\alpha'_1}, \cdots, |z_s|^{\alpha'_s}) , max(|z_1|, \cdots, |z_s|) \rightarrow 0$

$\aleph(z_1, \cdots, z_s) = 0(|z_1|^{\beta'_1}, \cdots, |z_s|^{\beta'_s}) , min(|z_1|, \cdots, |z_s|) \rightarrow \infty$

where $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)} / \delta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta_k = max[Re((c_j^{(k)}-1)/\gamma_j^{(k)})], j=1,\cdots,n_k$$

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R \ ; \ V = m_1, n_1; \cdots ; m_r, n_r \tag{1.6}$$

$$\mathrm{W} = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \cdots, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.7}$$

$$A = \{(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.8}$$

$$B = \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.9}$$

$$C = \{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\}, \tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}, \cdots, \{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.10}$$

$$D = \{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}, \cdots, \{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.11}$$

The multivariable Aleph-function write :

$$\aleph(z_1, \cdots, z_r) = \aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A} : \mathrm{C} \\ \cdot & \\ \cdot & \cdots \\ \cdot & \\ \mathrm{z}_r & \mathrm{B} : \mathrm{D} \end{array}\right) \tag{1.12}$$

The binomial expansion formula :

$$(x+\mu)^\lambda = \mu^\lambda \sum_{m=0}^{\infty} \binom{\lambda}{m} \left(\frac{x}{\mu}\right)^m \ , \quad \left|\frac{x}{\mu}\right| < 1 \tag{1.15}$$

Saigo [1] defines the integral operator in terms of the Gauss's hypergeometric function as follows :

$$I_{0,x}^{\alpha,\beta,\eta} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-t/x) f(t) \mathrm{d}t \tag{1.16}$$

Where $\alpha > 0, \beta$ and η are real numbers, $f(x)$ is a real valued and $f$is a continuous function defined on the interval $(0,\infty)$ having the order $0(x^k)$ near $x=0$ with $k > max(0, \beta-\eta) - 1.$
When $\alpha > 0$, by letting in a positive integer such that : $0 < \alpha + n \leqslant 1$, he defines

$$I_{0,x}^{\alpha,\beta,\eta} f(x) = \frac{d^n}{dx^n} I_{0,x}^{\alpha+n,\beta-n,\eta-n} f(x) \tag{1.17}$$

provided that the R.H.S. has a definite meaning.

We have : $I_{0,x}^{\alpha,\beta,\eta}(x^\lambda) = \dfrac{\Gamma(1+\lambda)(\Gamma(1+\lambda-\beta+\eta)}{\Gamma(1+\lambda-\beta)\Gamma(1+\lambda+\alpha+\eta)}, Re(\lambda) + 1 - \beta > 0$ (1.18)

The generalized polynomials defined by Srivastava [4], is given in the following manner :

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1,\cdots,y_s] \ = \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \frac{(-N_1)_{M_1K_1}}{K_1!} \cdots \frac{(-N_s)_{M_sK_s}}{K_s!}$$

$$A[N_1,K_1;\cdots;N_s,K_s] y_1^{K_1} \cdots y_s^{K_s} \tag{1.19}$$

Where $M_1, \cdots, M_s$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \cdots; N_s, K_s]$ are arbitrary constants, real or complex.

2. Fractional integrals

In the present paper, we use the following notation.

$$A = \frac{(-N_1)_{M_1K_1}}{K_1!} \cdots \frac{(-N_s)_{M_sK_s}}{K_s!} A[N_1, K_1; \cdots; N_s, K_s]$$

Formula 1

$$I_{0,x}^{\alpha,\beta,\eta}[x^k(x^{v_1}+a)^{\lambda}(b-x^{v_2})^{-\delta} S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}\left(\begin{array}{c} \mathrm{e}_1 x^{k_1}(x^{v_1}+a)^{a_1}(b-x^{v_2})^{-b_1} \\ \cdots \\ \mathrm{e}_s x^{k_s}(x^{v_1}+a)^{a_s}(b-x^{v_2})^{-b_s} \end{array}\right)$$

$$\aleph\left(\begin{array}{c} \mathrm{z}_1 x^{\rho_1}(x^{v_1}+a)^{\sigma_1}(b-x^{v_2})^{-\delta_1} \\ \cdots \\ \mathrm{z}_r x^{\rho_r}(x^{v_1}+a)^{\sigma_r}(b-x^{v_2})^{-\delta_r} \end{array}\right)\Bigg]$$

$$= a^{\lambda}b^{-\delta}x^{k-\beta} \sum_{l,m=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \frac{(x^{v_1}/a)^l (x^{v_2}/b)^m}{l!m!} A e_1^{K_1} \cdots e_s^{K_s} \aleph_{U_{44}:W}^{0,\mathfrak{n}+4:V}\left(\begin{array}{c} \mathrm{z}_1 x^{\rho_1} a^{\sigma_1} b^{-\delta_1} \\ \cdots \\ \mathrm{z}_1 x^{\rho_1} a^{\sigma_1} b^{-\delta_1} \end{array}\right|$$

$$\begin{array}{ll} (\text{-}\lambda - a_1K_1 - \cdots - a_sK_s; \sigma_1, \cdots, \sigma_r), & (1\text{-}\delta - m - b_1K_1 - \cdots - b_sK_s; \delta_1, \cdots, \delta_r), \\ \cdots & \cdots \\ (\text{l-}\lambda - a_1K_1 - \cdots - a_sK_s; \sigma_1, \cdots, \sigma_r), & (1\text{-}\delta - b_1K_1 - \cdots - b_sK_s; \delta_1, \cdots, \delta_r), \end{array}$$

$$\begin{array}{c} (\text{-k+}\beta - \eta - lv_1 - mv_2 - k_1K_1 - \cdots - k_sK_s; \rho_1, \cdots, \rho_r), \\ \cdots \\ (\text{-k-}\alpha - \eta - lv_1 - mv_2 - k_1K_1 - \cdots - k_sK_s; \rho_1, \cdots, \rho_r), \end{array}$$

$$\left.\begin{array}{c} (\text{-k -l}v_1 - mv_2 - k_1K_1 - \cdots - k_sK_s; \rho_1, \cdots, \rho_r), A : C \\ \cdots \\ (\text{-k+}\beta - lv_1 - mv_2 - k_1K_1 - \cdots - k_sK_s; \rho_1, \cdots, \rho_r), B : D \end{array}\right) \tag{2.1}$$

Where $U_{44} = p_i + 4, q_i + 4, \tau_i; R$

The validity conditions are the following :

a) $min(v_1, v_2, \rho_i, \sigma_i, \delta_i) > 0, i = 1, \cdots, r$

b) $Re[\rho + \sum_{i=1}^{r} \rho_i \min_{1 \leqslant j \leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}] > -1$

c) $max[|arg(x^{v_1}/a)|, |arg(x^{v_2}/b)|] < \pi$

d) $|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.5)

e) The series occuring on the right-hand side of (2.1) are absolutely convergents.

Formula 2

$$I_{0,x}^{\alpha,\beta,\eta}I_{0,y}^{\alpha',\beta',\eta'}\Bigg[x^k(x^{v_1}+a)^{\sigma}(b-x^{v_2})^{-\delta}y^l(y^{v_3}+c)^h(b-y^{v_4})^{-g}$$

$$S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}\left(\begin{array}{c}\mathrm{e}_1x^{l_1}(x^{v_1}+a)^{a_1}(b-x^{v_2})^{-b_1}y^{l'_1}(y^{v_3}+c)^{a'_1}(d-y^{v_4})^{-b'_1}\\ \cdot\ \cdot\ \cdot\\ \mathrm{e}_sx^{l_s}(x^{v_1}+a)^{a_s}(b-x^{v_2})^{-b_s}y^{l'_s}(y^{v_3}+c)^{a'_s}(d-y^{v_4})^{-b'_s}\end{array}\right)$$

$$\aleph\left(\begin{array}{c}\mathrm{z}_1x^{k_1}(x^{v_1}+a)^{\sigma_1}(b-x^{v_2})^{-\delta_1}y^{k'_1}(y^{v_3}+c)^{h_1}(b-y^{v_4})^{-g_1}\\ \cdot\ \cdot\ \cdot\\ \mathrm{z}_rx^{k_r}(x^{v_1}+a)^{\sigma_r}(b-x^{v_2})^{-\delta_r}y^{k'_r}(y^{v_3}+c)^{h_r}(b-y^{v_4})^{-g_r}\end{array}\right)\Bigg]$$

$$=a^{\sigma}b^{-\delta}x^{k-\beta}c^hd^{-g}y^{\lambda-\beta'}\sum_{m,n,r,s=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(x^{v_1}/a)^m(x^{v_2}/b)^n(y^{v_3}/c)^r(y^{v_4}/d)^s}{m!n!r!s!}A$$

$$e_1^{K_1}\cdots e_s^{K_s}$$

$$\aleph_{U_{88}:W}^{0,\mathfrak{n}+8:V}\left(\begin{array}{c}\mathrm{z}_1x^{k_1}a^{\sigma_1}b^{-\delta_1}y^{\lambda_1}c^{h_1}d^{-g_1}\\ \cdot\ \cdot\ \cdot\\ \mathrm{z}_1x^{k_r}a^{\sigma_r}b^{-\delta_r}y^{\lambda_r}c^{h_r}d^{-g_r}\end{array}\right|\begin{array}{c}(\text{-}\sigma-a_1K_1-\cdots-a_sK_s;\sigma_1,\cdots,\sigma_r),\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ (\text{1-}\sigma-a_1K_1-\cdots-a_sK_s;\sigma_1,\cdots,\sigma_r),\end{array}$$

$$\begin{array}{c}(\text{-h -a'}_1K_1-\cdots-a'_sK_s;h_1,\cdots,h_r),\ (\text{1-}\delta-n-b_1K_1-\cdots-b_sK_s;\delta_1,\cdots,\delta_r),\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\qquad\qquad\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ (\text{-h+r -a'}_1K_1-\cdots-a'_sK_s;h_1,\cdots,h_r),\ (\text{1-}\delta-b_1K_1-\cdots-b_sK_s;\delta_1,\cdots,\delta_r),\end{array}$$

$$\begin{array}{c}(\text{-l - rv}_3-sv_4-l'_1K_1-\cdots-l'_sK_s;k'_1,\cdots,k'_r),\quad(\text{1-g - s- b'}_1K_1-\cdots-b'_sK_s;g_1,\cdots,g_r),\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\qquad\qquad\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ (\text{-l }+\beta'-rv_3-sv_4-l'_1K_1-\cdots-l'_sK_s;k'_1,\cdots,k'_r),\ (\text{1-g -b'}_1K_1-\cdots-b'_sK_s;g_1,\cdots,g_r),\end{array}$$

$$\begin{array}{c}(\text{-k }+\beta-\eta-mv_1-nv_2-l_1K_1-\cdots-l_sK_s;k_1,\cdots,k_r),\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ (\text{-k-1-}\eta-mv_1-nv_2-l_1K_1-\cdots-l_sK_s;k_1,\cdots,k_r),\end{array}$$

$$\begin{array}{c}(\text{-l }+\beta'-\eta'-rv_3-sv_4-l'_1K_1-\cdots-l'_sK_s;k'_1,\cdots,k'_r),\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ (\text{-l -1-}\eta'-rv_3-sv_4-l'_1K_1-\cdots-l'_sK_s;k'_1,\cdots,k'_r),\end{array}$$

$$\left.\begin{array}{c}(\text{-k - mv}_1-nv_2-l_1K_1-\cdots-l_sK_s;k_1,\cdots,k_r),A:C\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ (\text{-k }+\beta-mv_1-nv_2-l_1K_1-\cdots-l_sK_s;k_1,\cdots,k_r),B:D\end{array}\right)\qquad(2.2)$$

Where $U_{88}=p_i+8,q_i+8,\tau_i;R$

The validity conditions are the following :

a) $min(v_1,v_2,v_3,v_4,k_i,\sigma_i,\delta_i,k'_i,g_i,h_i)>0, i=1,\cdots,r$

b) $Re[\rho+\sum_{i=1}^{r}k_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>-1, Re[\rho+\sum_{i=1}^{r}k'_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>-1$

c) $max[|arg(x^{v_1}/a)|,|arg(x^{v_2}/b)|,|arg(y^{v_3}/e)|,|arg(y^{v_4}/d)|]<\pi$

d) $|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.5)

e) The series occuring on the right-hand side of (2.2) are absolutely convergents.

Proof of (2.1)

To prove the fractional derivative formula (2.1), we first express both the general class of polynomials $S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}$ and occuring on its left-hand side in the series from given by (1.19) and replace the multivariable Aleph-function occuring therein by its Mellin-Barnes contour integral given by (1.1). Now making use of the following binomial expansions (1.15) and apply the formula (1.18). We interpret the resulting Mellin-Barnes contour integrals as a multivariable Aleph-function, we get the desired result.

Proof of (2.2)

Formula (2.2) can be proved if we apply the fractional integral formula (2.1) twice, first with respect to the variable y, and then with respect to the variable x ; x and y are independent variables.

3. Particular cases

In (2.1), replace δ by - δ and setting $\sigma_i,\delta_i\rightarrow 0\ (i=1,\cdots,r)$, we have.

Formula 1

$$I_{0,x}^{\alpha,\beta,\eta}[x^k(x^{v_1}+a)^{\lambda}(b-x^{v_2})^{-\delta}\,S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}\begin{pmatrix}\mathrm{e}_1x^{k_1}(x^{v_1}+a)^{a_1}(b-x^{v_2})^{-b_1}\\ \cdots\\ \mathrm{e}_sx^{k_s}(x^{v_1}+a)^{a_s}(b-x^{v_2})^{-b_s}\end{pmatrix}\aleph\begin{pmatrix}\mathrm{z}_1x^{\rho_1}\\ \cdots\\ \mathrm{z}_rx^{\rho_r}\end{pmatrix}\Bigg]$$

$$=a^{\lambda}b^{-\delta}x^{k-\beta}\sum_{l,m=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\binom{\lambda}{l}\binom{\delta}{m}(x^{v_1}/a)^l(x^{v_2}/b)\,Ae_1^{K_1}\cdots e_s^{K_s}\aleph_{U_{22}:W}^{0,\mathfrak{n}+2:V}\left(\begin{matrix}\mathrm{z}_1x^{\rho_1}\\ \cdots\\ \mathrm{z}_1x^{\rho_1}\end{matrix}\ \middle|\ \begin{matrix}(\text{-k}+\beta-\eta-lv_1-mv_2-k_1K_1-\cdots-k_sK_s;\rho_1,\cdots,\rho_r),\\ \cdots\cdots\\ (\text{-k-}\alpha-\eta-lv_1-mv_2-k_1K_1-\cdots-k_sK_s;\rho_1,\cdots,\rho_r),\end{matrix}\right.$$

$$\left.\begin{matrix}(\text{-k -l}v_1-mv_2-k_1K_1-\cdots-k_sK_s;\rho_1,\cdots,\rho_r),A:C\\ \cdots\cdots\\ (\text{-k}+\beta-lv_1-mv_2-k_1K_1-\cdots-k_sK_s;\rho_1,\cdots,\rho_r),B:D\end{matrix}\right)\tag{3.1}$$

Where $U_{22}=p_i+2,q_i+2,\tau_i;R$

In (2.2), replace δ,g by - δ, $-g$ respectively and setting $\sigma_i,\delta_i,h_i,g_i\rightarrow 0\ (i=1,\cdots,r)$, we get.

Formula 2

$$I_{0,x}^{\alpha,\beta,\eta}I_{0,y}^{\alpha',\beta',\eta'}\Big[x^k(x^{v_1}+a)^{\sigma}(b-x^{v_2})^{-\delta}y^l(y^{v_3}+c)^h(b-y^{v_4})^{-g}$$

$$S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}\begin{pmatrix}\mathrm{e}_1x^{l_1}(x^{v_1}+a)^{a_1}(b-x^{v_2})^{-b_1}y^{l'_1}(y^{v_3}+c)^{a'_1}(d-y^{v_4})^{-b'_1}\\ \cdots\\ \mathrm{e}_sx^{l_s}(x^{v_1}+a)^{a_s}(b-x^{v_2})^{-b_s}y^{l'_s}(y^{v_3}+c)^{a'_s}(d-y^{v_4})^{-b'_s}\end{pmatrix}\aleph\begin{pmatrix}\mathrm{z}_1x^{k_1}y^{k'_1}\\ \cdots\\ \mathrm{z}_rx^{k_r}y^{k'_r}\end{pmatrix}\Bigg]$$

$$=a^{\sigma}b^{-\delta}x^{k-\beta}c^hd^{-g}y^{\lambda-\beta'}\sum_{m,n,r,s=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\binom{\sigma}{m}\binom{\delta}{n}\binom{h}{r}\binom{g}{s}A$$

$$(x^{v_1}/a)^m(x^{v_2}/b)^n(y^{v_3}/c)^r(y^{v_4}/d)^s\ e_1^{K_1}\cdots e_s^{K_s}$$

$$\aleph_{U_{44}:W}^{0,\mathfrak{n}+4:V}\left(\begin{array}{c|c}\mathrm{z}_1x^{k_1}y^{\lambda_1} & (\text{-l - rv}_3-sv_4-l'_1K_1-\cdots-l'_sK_s;k'_1,\cdots,k'_r),\\ \cdots & \cdots\cdots\\ \mathrm{z}_1x^{k_r}y^{\lambda_r} & (\text{-l }+\beta'-rv_3-sv_4-l'_1K_1-\cdots-l'_sK_s;k'_1,\cdots,k'_r),\end{array}\right.$$

$$(\text{-k }+\beta-\eta-mv_1-nv_2-l_1K_1-\cdots-l_sK_s;k_1,\cdots,k_r),$$
$$\cdots\cdots$$
$$(\text{-k-1-}\eta-mv_1-nv_2-l_1K_1-\cdots-l_sK_s;k_1,\cdots,k_r),$$

$$(\text{-k - mv}_1-nv_2-l_1K_1-\cdots-l_sK_s;k_1,\cdots,k_r),$$
$$\cdots\cdots$$
$$(\text{-k }+\beta-mv_1-nv_2-l_1K_1-\cdots-l_sK_s;k_1,\cdots,k_r),$$

$$\left.\begin{array}{c}(\text{-l }+\beta'-\eta'-rv_3-sv_4-l'_1K_1-\cdots-l'_sK_s;k'_1,\cdots,k'_r),A:C\\ \cdots\cdots\\ (\text{-l-1-}\eta'-rv_3-sv_4-l'_1K_1-\cdots-l'_sK_s;k'_1,\cdots,k'_r),B:D\end{array}\right) \tag{3.2}$$

Where $U_{44}=p_i+4,q_i+4,\tau_i;R$

If $e_1=e$ and $e_2=\cdots=e_s=0$, then the class of polynomials $S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\,(e_1,\cdots,e_s)$ defined of (1.19) degenere in the class of polynomial $S_N^M\,(e)$ defined by Srivastava [3]. We have.

Formula 1

$$I_{0,x}^{\alpha,\beta,\eta}[x^k(x^{v_1}+a)^{\lambda}(b-x^{v_2})^{-\delta}\,S_N^M(\,ex^{k'}(x^{v_1}+a)^A(b-x^{v_2})^{-B})$$

$$\aleph\begin{pmatrix}\mathrm{z}_1x^{\rho_1}(x^{v_1}+a)^{\sigma_1}(b-x^{v_2})^{-\delta_1}\\ \cdots\\ \mathrm{z}_rx^{\rho_r}(x^{v_1}+a)^{\sigma_r}(b-x^{v_2})^{-\delta_r}\end{pmatrix}\Bigg]$$

$$= a^{\lambda} b^{-\delta} x^{k-\beta} \sum_{l,m=0}^{\infty} \sum_{K=0}^{[N/M]} \frac{(x^{v_1}/a)^l (x^{v_2}/b)^m}{l!m!} A_K e^K \aleph_{U_{44}:W}^{0,\mathfrak{n}+4:V} \left(\begin{array}{c} \mathrm{z}_1 x^{\rho_1} a^{\sigma_1} b^{-\delta_1} \\ \cdots \\ \mathrm{z}_1 x^{\rho_1} a^{\sigma_1} b^{-\delta_1} \end{array} \middle| \begin{array}{l} (\text{-}\lambda - AK; \sigma_1, \cdots, \sigma_r),\ (1\text{-}\delta - m - BK; \delta_1, \cdots, \delta_r), \\ \qquad \cdots\cdots \qquad\qquad \cdots\cdots \\ (\text{l-}\lambda - AK; \sigma_1, \cdots, \sigma_r), \quad (1\text{-}\delta - BK; \delta_1, \cdots, \delta_r), \end{array} \right.$$

$$\begin{array}{c} (\text{-k}+\beta - \eta - lv_1 - mv_2 - k'K; \rho_1, \cdots, \rho_r), \\ \cdots\cdots \\ (\text{-k-}\alpha - \eta - lv_1 - mv_2 - k'K - \cdots - k_s K_s; \rho_1, \cdots, \rho_r), \end{array}$$

$$\left. \begin{array}{c} (\text{-k -lv}_1 - mv_2 - k'K; \rho_1, \cdots, \rho_r), A : C \\ \cdots\cdots \\ (\text{-k}+\beta - lv_1 - mv_2 - k'K; \rho_1, \cdots, \rho_r), B : D \end{array} \right) \tag{3.3}$$

Where $U_{44} = p_i + 4, q_i + 4, \tau_i; R$

Formula 2

$$I_{0,x}^{\alpha,\beta,\eta} I_{0,y}^{\alpha',\beta',\eta'} \Big[x^k (x^{v_1} + a)^{\sigma} (b - x^{v_2})^{-\delta} y^l (y^{v_3} + c)^h (b - y^{v_4})^{-g}$$

$$S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r} (\ e x^l (x^{v_1} + a)^A (b - x^{v_2})^{-B} y^{l'} (y^{v_3} + c)^{A'} (d - y^{v_4})^{-B'}\ )$$

$$\aleph \left(\begin{array}{c} \mathrm{z}_1 x^{k_1} (x^{v_1} + a)^{\sigma_1} (b - x^{v_2})^{-\delta_1} y^{k'_1} (y^{v_3} + c)^{h_1} (b - y^{v_4})^{-g_1} \\ \cdots \\ \mathrm{z}_r x^{k_r} (x^{v_1} + a)^{\sigma_r} (b - x^{v_2})^{-\delta_r} y^{k'_r} (y^{v_3} + c)^{h_r} (b - y^{v_4})^{-g_r} \end{array} \right) \Bigg]$$

$$= a^{\sigma} b^{-\delta} x^{k-\beta} c^h d^{-g} y^{\lambda-\beta'} \sum_{m,n,r,s=0}^{\infty} \sum_{K=0}^{[N/M]} \frac{(x^{v_1}/a)^m (x^{v_2}/b)^n (y^{v_3}/c)^r (y^{v_4}/d)^s}{m!n!r!s!} \ A_K e^K$$

$$\aleph_{U_{88}:W}^{0,\mathfrak{n}+8:V} \left(\begin{array}{c} \mathrm{z}_1 x^{k_1} a^{\sigma_1} b^{-\delta_1} y^{\lambda_1} c^{h_1} d^{-g_1} \\ \cdots \\ \mathrm{z}_1 x^{k_r} a^{\sigma_r} b^{-\delta_r} y^{\lambda_r} c^{h_r} d^{-g_r} \end{array} \middle| \begin{array}{l} (-\sigma - AK; \sigma_1, \cdots, \sigma_r),\ \ (\text{-h -A'K ;h}_1, \cdots, h_r), \\ \qquad \cdots\cdots \qquad\qquad\qquad \cdots\cdots \\ (\text{l-}\sigma - AK; \sigma_1, \cdots, \sigma_r), (\text{-h+r -A'K ;h}_1, \cdots, h_r), \end{array} \right.$$

$$\begin{array}{l} (1\text{-}\delta - n - BK; \delta_1, \cdots, \delta_r), (\text{1-g - s- B'K ;g}_1, \cdots, g_r), \quad (\text{-l - rv}_3 - sv_4 - l'K; k'_1, \cdots, k'_r), \\ \qquad \cdots\cdots \qquad\qquad\qquad \cdots\cdots \qquad\qquad\qquad \cdots\cdots \\ \ \ (1\text{-}\delta - BK; \delta_1, \cdots, \delta_r), \quad (\text{1-g -B'K;g}_1, \cdots, g_r), \quad (\text{-l} + \beta' - rv_3 - sv_4 - l'K; k'_1, \cdots, k'_r), \end{array}$$

$$\begin{array}{l} (\text{-k} + \beta - \eta - mv_1 - nv_2 - lK; k_1, \cdots, k_r), (\text{-l} + \beta' - \eta' - rv_3 - sv_4 - l'K; k'_1, \cdots, k'_r), \\ \qquad \cdots\cdots \qquad\qquad\qquad\qquad\qquad \cdots\cdots \\ \ \ (\text{-k-1-}\eta - mv_1 - nv_2 - lK; k_1, \cdots, k_r), \quad (\text{-l-1-}\eta' - rv_3 - sv_4 - l'K; k'_1, \cdots, k'_r), \end{array}$$

$$\left.\begin{array}{c}(\text{-k - mv}_1 - nv_2 - lK; k_1, \cdots, k_r), A : C \\ \cdots\cdots \\ (\text{-1} + \beta - mv_1 - nv_2 - lK; k_1, \cdots, k_r), B : D\end{array}\right) \tag{3.4}$$

Where $U_{88} = p_i + 8, q_i + 8, \tau_i; R$

4. Multivariable I-function

If $\tau_i, \tau_{i^{(1)}}, \cdots, \tau_{i^{(r)}} \rightarrow 1$, then the Aleph-function of several variables degenere in the I-function of several variables defined by Sharma and Ahmad [2]. We get the following formulas.

Formula 1

$$I_{0,x}^{\alpha,\beta,\eta}[x^k(x^{v_1}+a)^{\lambda}(b-x^{v_2})^{-\delta}\, S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}\left(\begin{array}{c}\text{e}_1 x^{k_1}(x^{v_1}+a)^{a_1}(b-x^{v_2})^{-b_1} \\ \cdots \\ \text{e}_s x^{k_s}(x^{v_1}+a)^{a_s}(b-x^{v_2})^{-b_s}\end{array}\right)$$

$$I\left(\begin{array}{c}\text{z}_1 x^{\rho_1}(x^{v_1}+a)^{\sigma_1}(b-x^{v_2})^{-\delta_1} \\ \cdots \\ \text{z}_r x^{\rho_r}(x^{v_1}+a)^{\sigma_r}(b-x^{v_2})^{-\delta_r}\end{array}\right)\Bigg]$$

$$= a^{\lambda}b^{-\delta}x^{k-\beta}\sum_{l,m=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(x^{v_1}/a)^l(x^{v_2}/b)^m}{l!m!}Ae_1^{K_1}\cdots e_s^{K_s}I_{U_{44}:W}^{0,\text{n}+4:V}\left(\begin{array}{c}\text{z}_1 x^{\rho_1}a^{\sigma_1}b^{-\delta_1} \\ \cdots \\ \text{z}_1 x^{\rho_1}a^{\sigma_1}b^{-\delta_1}\end{array}\right|$$

$$\begin{array}{cc}(\text{-}\lambda - a_1K_1 - \cdots - a_sK_s; \sigma_1, \cdots, \sigma_r), & (\text{1-}\delta - m - b_1K_1 - \cdots - b_sK_s; \delta_1, \cdots, \delta_r), \\ \cdots\cdots & \cdots\cdots \\ (\text{l-}\lambda - a_1K_1 - \cdots - a_sK_s; \sigma_1, \cdots, \sigma_r), & (\text{1-}\delta - b_1K_1 - \cdots - b_sK_s; \delta_1, \cdots, \delta_r),\end{array}$$

$$\begin{array}{c}(\text{-k+}\beta - \eta - lv_1 - mv_2 - k_1K_1 - \cdots - k_sK_s; \rho_1, \cdots, \rho_r), \\ \cdots\cdots \\ (\text{-k-}\alpha - \eta - lv_1 - mv_2 - k_1K_1 - \cdots - k_sK_s; \rho_1, \cdots, \rho_r),\end{array}$$

$$\left.\begin{array}{c}(\text{-k -lv}_1 - mv_2 - k_1K_1 - \cdots - k_sK_s; \rho_1, \cdots, \rho_r), A' : C' \\ \cdots\cdots \\ (\text{-k+}\beta - lv_1 - mv_2 - k_1K_1 - \cdots - k_sK_s; \rho_1, \cdots, \rho_r), B' : D'\end{array}\right) \tag{4.1}$$

Where $U_{44} = p_i + 4, q_i + 4; R$

The validity conditions are the following :

a) $min(v_1, v_2, \rho_i, \sigma_i, \delta_i) > 0, i = 1, \cdots, r$

b) $Re[\rho + \sum_{i=1}^{r}\rho_i \min_{1 \leqslant j \leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}] > -1$

c) $max[|arg(x^{v_1}/a)|,|arg(x^{v_2}/b)|] < \pi$

d) $|argz_k| < \frac{1}{2}A_i'^{(k)}\pi$, where : $A_i'^{(k)} = \sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)} - \sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)} - \sum_{j=1}^{q_i}\beta_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} -$

$\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)} + \sum_{j=1}^{m_k}\delta_j^{(k)} - \sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} > 0$,

with $k = 1\cdots,r$, $i = 1$ to $i = 1,\cdots,R$, $i^{(k)} = 1,\cdots,R^{(k)}$

e) The series occuring on the right-hand side of (2.1) are absolutely convergents.

Formula 2

$$I_{0,x}^{\alpha,\beta,\eta}I_{0,y}^{\alpha',\beta',\eta'}\Bigg[x^k(x^{v_1}+a)^\sigma(b-x^{v_2})^{-\delta}y^l(y^{v_3}+c)^h(b-y^{v_4})^{-g}$$

$$S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}\begin{pmatrix}\mathrm{e}_1x^{l_1}(x^{v_1}+a)^{a_1}(b-x^{v_2})^{-b_1}y^{l_1'}(y^{v_3}+c)^{a_1'}(d-y^{v_4})^{-b_1'}\\ \cdot\cdot\cdot\\ \mathrm{e}_sx^{l_s}(x^{v_1}+a)^{a_s}(b-x^{v_2})^{-b_s}y^{l_s'}(y^{v_3}+c)^{a_s'}(d-y^{v_4})^{-b_s'}\end{pmatrix}$$

$$I\begin{pmatrix}\mathrm{z}_1x^{k_1}(x^{v_1}+a)^{\sigma_1}(b-x^{v_2})^{-\delta_1}y^{k_1'}(y^{v_3}+c)^{h_1}(b-y^{v_4})^{-g_1}\\ \cdot\cdot\cdot\\ \mathrm{z}_rx^{k_r}(x^{v_1}+a)^{\sigma_r}(b-x^{v_2})^{-\delta_r}y^{k_r'}(y^{v_3}+c)^{h_r}(b-y^{v_4})^{-g_r}\end{pmatrix}\Bigg]$$

$$=a^\sigma b^{-\delta}x^{k-\beta}c^hd^{-g}y^{\lambda-\beta'}\sum_{m,n,r,s=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(x^{v_1}/a)^m(x^{v_2}/b)^n(y^{v_3}/c)^r(y^{v_4}/d)^s}{m!n!r!s!}A$$

$$e_1^{K_1}\cdots e_s^{K_s}\,I_{U_{88}:W}^{0,\mathfrak{n}+8:V}\left(\begin{array}{c|c}\mathrm{z}_1x^{k_1}a^{\sigma_1}b^{-\delta_1}y^{\lambda_1}c^{h_1}d^{-g_1} & (\text{-}\sigma-a_1K_1-\cdots-a_sK_s;\sigma_1,\cdots,\sigma_r),\\ \cdot\cdot\cdot & \cdot\cdot\cdot\cdot\cdot\cdot\\ \mathrm{z}_1x^{k_r}a^{\sigma_r}b^{-\delta_r}y^{\lambda_r}c^{h_r}d^{-g_r} & (\text{l-}\sigma-a_1K_1-\cdots-a_sK_s;\sigma_1,\cdots,\sigma_r),\end{array}\right.$$

$$(\text{-h -a'}_1K_1-\cdots-a_s'K_s;h_1,\cdots,h_r),\ (1\text{-}\delta-n-b_1K_1-\cdots-b_sK_s;\delta_1,\cdots,\delta_r),$$
$$\cdot\cdot\cdot\cdot\cdot\cdot \qquad \cdot\cdot\cdot\cdot\cdot\cdot$$
$$(\text{-h+r -a'}_1K_1-\cdots-a_s'K_s;h_1,\cdots,h_r),\ (1\text{-}\delta-b_1K_1-\cdots-b_sK_s;\delta_1,\cdots,\delta_r),$$

$$(1\text{-g - s- b'}_1K_1-\cdots-b_s'K_s;g_1,\cdots,g_r),\ (\text{-l - rv}_3-sv_4-l_1'K_1-\cdots-l_s'K_s;k_1',\cdots,k_r'),$$
$$\cdot\cdot\cdot\cdot\cdot\cdot \qquad \cdot\cdot\cdot\cdot\cdot\cdot$$
$$(1\text{-g -b'}_1K_1-\cdots-b_s'K_s;g_1,\cdots,g_r),\ (\text{-l }+\beta'-rv_3-sv_4-l_1'K_1-\cdots-l_s'K_s;k_1',\cdots,k_r'),$$

$$(\text{-k }+\beta-\eta-mv_1-nv_2-l_1K_1-\cdots-l_sK_s;k_1,\cdots,k_r),$$
$$\cdot\cdot\cdot\cdot\cdot\cdot$$
$$(\text{-k-1-}\eta-mv_1-nv_2-l_1K_1-\cdots-l_sK_s;k_1,\cdots,k_r),$$

$$(\text{-l }+\beta'-\eta'-rv_3-sv_4-l_1'K_1-\cdots-l_s'K_s;k_1',\cdots,k_r'),$$
$$\cdot\cdot\cdot\cdot\cdot\cdot$$
$$(\text{-l-1-}\eta'-rv_3-sv_4-l_1'K_1-\cdots-l_s'K_s;k_1',\cdots,k_r'),$$

$$\left.\begin{array}{c}\text{(-k - mv}_1 - nv_2 - l_1K_1 - \cdots - l_sK_s; k_1, \cdots, k_r), A' : C' \\ \cdots\cdots \\ \text{(-k} + \beta - mv_1 - nv_2 - l_1K_1 - \cdots - l_sK_s; k_1, \cdots, k_r), B' : D'\end{array}\right) \tag{4;2}$$

Where $U_{88} = p_i + 8, q_i + 8; R$

The validity conditions are the following :

a) $min(v_1, v_2, v_3, v_4, k_i, \sigma_i, \delta_i, k'_i, g_i, h_i) > 0, i = 1, \cdots, r$

b) $Re[\rho + \sum_{i=1}^{r} k_i \min_{1\leqslant j\leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}] > -1, Re[\rho + \sum_{i=1}^{r} k'_i \min_{1\leqslant j\leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}] > -1$

c) $max[|arg(x^{v_1}/a)|, |arg(x^{v_2}/b)|, |arg(y^{v_3}/e)|, |arg(y^{v_4}/d)|] < \pi$

d) $|argz_k| < \frac{1}{2} A_i'^{(k)} \pi$, where : $A_i'^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} -$
$\sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} + \sum_{j=1}^{m_k} \delta_j^{(k)} - \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0$,

with $k = 1 \cdots, r$, $i = 1$ to $i = 1, \cdots, R$, $i^{(k)} = 1, \cdots, R^{(k)}$

e) The series occuring on the right-hand side of (2.2) are absolutely convergents.

5. Multivariable H-function

If $R = R^{(1)} =, \cdots, R^{(r)} = 1$, the multivariable I-function degenere in the multivariable H-function defined by Srivastava et al [5] and we have the following results :

Formula 1

$$I_{0,x}^{\alpha,\beta,\eta}[x^k(x^{v_1}+a)^{\lambda}(b-x^{v_2})^{-\delta}\, S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}\left(\begin{array}{c}\text{e}_1 x^{k_1}(x^{v_1}+a)^{a_1}(b-x^{v_2})^{-b_1} \\ \cdots \\ \text{e}_s x^{k_s}(x^{v_1}+a)^{a_s}(b-x^{v_2})^{-b_s}\end{array}\right)$$

$$H\left(\begin{array}{c}\text{z}_1 x^{\rho_1}(x^{v_1}+a)^{\sigma_1}(b-x^{v_2})^{-\delta_1} \\ \cdots \\ \text{z}_r x^{\rho_r}(x^{v_1}+a)^{\sigma_r}(b-x^{v_2})^{-\delta_r}\end{array}\right)\Bigg]$$

$$=a^{\lambda}b^{-\delta}x^{k-\beta}\sum_{l,m=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(x^{v_1}/a)^l(x^{v_2}/b)^m}{l!m!}\,Ae_1^{K_1}\cdots e_s^{K_s}H_{p+4,q+4:W}^{0,\mathfrak{n}+4:V}\left(\begin{array}{c}\text{z}_1 x^{\rho_1}a^{\sigma_1}b^{-\delta_1} \\ \cdots \\ \text{z}_1 x^{\rho_1}a^{\sigma_1}b^{-\delta_1}\end{array}\right|$$

$$\begin{array}{cc}\text{(-}\lambda - a_1K_1 - \cdots - a_sK_s; \sigma_1, \cdots, \sigma_r), & \text{(1-}\delta - m - b_1K_1 - \cdots - b_sK_s; \delta_1, \cdots, \delta_r), \\ \cdots\cdots & \cdots\cdots \\ \text{(l-}\lambda - a_1K_1 - \cdots - a_sK_s; \sigma_1, \cdots, \sigma_r), & \text{(1-}\delta - b_1K_1 - \cdots - b_sK_s; \delta_1, \cdots, \delta_r),\end{array}$$

$$(\text{-k}+\beta-\eta-lv_1-mv_2-k_1K_1-\cdots-k_sK_s;\rho_1,\cdots,\rho_r),$$
$$\cdots\cdots$$
$$(\text{-k-}\alpha-\eta-lv_1-mv_2-k_1K_1-\cdots-k_sK_s;\rho_1,\cdots,\rho_r),$$

$$\left.\begin{matrix}(\text{-k -lv}_1-mv_2-k_1K_1-\cdots-k_sK_s;\rho_1,\cdots,\rho_r),A'':C''\\ \cdots\cdots\\ (\text{-k}+\beta-lv_1-mv_2-k_1K_1-\cdots-k_sK_s;\rho_1,\cdots,\rho_r),B'':D''\end{matrix}\right) \tag{5.1}$$

The validity conditions are the following :

a) $min(v_1,v_2,\rho_i,\sigma_i,\delta_i)>0, i=1,\cdots,r$

b) $Re[\rho+\sum_{i=1}^{r}\rho_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>-1$

c) $max[|arg(x^{v_1}/a)|,|arg(x^{v_2}/b)|]<\pi$

d) $|argz_i|<\frac{1}{2}A_i\pi$, where : $A_i=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(i)}-\sum_{j=\mathfrak{n}+1}^{p}\alpha_j^{(i)}-\sum_{j=1}^{q}\beta_j^{(i)}+\sum_{j=1}^{n_i}\gamma_j^{(i)}-\sum_{j=n_i+1}^{p_i}\gamma_j^{(i)}$
$+\sum_{j=1}^{m_i}\delta_j^{(i)}-\sum_{j=m_i+1}^{q_i}\delta_j^{(i)}>0$, with $i=1,\cdots,r$

e) The series occuring on the right-hand side of (2.1) are absolutely convergents.

Formula 2

$$I_{0,x}^{\alpha,\beta,\eta}I_{0,y}^{\alpha',\beta',\eta'}\Bigg[x^k(x^{v_1}+a)^{\sigma}(b-x^{v_2})^{-\delta}y^l(y^{v_3}+c)^h(b-y^{v_4})^{-g}$$

$$S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}\begin{pmatrix}\text{e}_1x^{l_1}(x^{v_1}+a)^{a_1}(b-x^{v_2})^{-b_1}y^{l'_1}(y^{v_3}+c)^{a'_1}(d-y^{v_4})^{-b'_1}\\ \cdots\\ \text{e}_sx^{l_s}(x^{v_1}+a)^{a_s}(b-x^{v_2})^{-b_s}y^{l'_s}(y^{v_3}+c)^{a'_s}(d-y^{v_4})^{-b'_s}\end{pmatrix}$$

$$H\begin{pmatrix}\text{z}_1x^{k_1}(x^{v_1}+a)^{\sigma_1}(b-x^{v_2})^{-\delta_1}y^{k'_1}(y^{v_3}+c)^{h_1}(b-y^{v_4})^{-g_1}\\ \cdots\\ \text{z}_rx^{k_r}(x^{v_1}+a)^{\sigma_r}(b-x^{v_2})^{-\delta_r}y^{k'_r}(y^{v_3}+c)^{h_r}(b-y^{v_4})^{-g_r}\end{pmatrix}\Bigg]$$

$$=a^{\sigma}b^{-\delta}x^{k-\beta}c^hd^{-g}y^{\lambda-\beta'}\sum_{m,n,r,s=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(x^{v_1}/a)^m(x^{v_2}/b)^n(y^{v_3}/c)^r(y^{v_4}/d)^s}{m!n!r!s!}A$$
$$e_1^{K_1}\cdots e_s^{K_s}$$

$$H^{0,\mathfrak{n}+8:V}_{p+8,q+8:W}\left(\begin{array}{c|c} \mathrm{z}_1 x^{k_1} a^{\sigma_1} b^{-\delta_1} y^{\lambda_1} c^{h_1} d^{-g_1} & (\text{-}\sigma - a_1K_1 - \cdots - a_sK_s;\sigma_1,\cdots,\sigma_r), \\ \cdots & \cdots\cdots \\ \mathrm{z}_1 x^{k_r} a^{\sigma_r} b^{-\delta_r} y^{\lambda_r} c^{h_r} d^{-g_r} & (\text{l-}\sigma - a_1K_1 - \cdots - a_sK_s;\sigma_1,\cdots,\sigma_r), \end{array}\right.$$

$$(\text{-h -a'}_1K_1 - \cdots - a'_sK_s;h_1,\cdots,h_r),\ (1\text{-}\delta - n - b_1K_1 - \cdots - b_sK_s;\delta_1,\cdots,\delta_r),$$
$$\cdots\cdots \qquad\qquad \cdots\cdots$$
$$(\text{-h+r -a'}_1K_1 - \cdots - a'_sK_s;h_1,\cdots,h_r),\ (1\text{-}\delta - b_1K_1 - \cdots - b_sK_s;\delta_1,\cdots,\delta_r),$$

$$(1\text{-g - s- b'}_1K_1 - \cdots - b'_sK_s;g_1,\cdots,g_r),\ (\text{-l - rv}_3 - sv_4 - l'_1K_1 - \cdots - l'_sK_s;k'_1,\cdots,k'_r),$$
$$\cdots\cdots \qquad\qquad \cdots\cdots$$
$$(1\text{-g -b'}_1K_1 - \cdots - b'_sK_s;g_1,\cdots,g_r),\ (\text{-l }+\beta' - rv_3 - sv_4 - l'_1K_1 - \cdots - l'_sK_s;k'_1,\cdots,k'_r),$$

$$(\text{-k }+\beta - \eta - mv_1 - nv_2 - l_1K_1 - \cdots - l_sK_s;k_1,\cdots,k_r),$$
$$\cdots\cdots$$
$$(\text{-k-1-}\eta - mv_1 - nv_2 - l_1K_1 - \cdots - l_sK_s;k_1,\cdots,k_r),$$

$$(\text{-l }+\beta' - \eta' - rv_3 - sv_4 - l'_1K_1 - \cdots - l'_sK_s;k'_1,\cdots,k'_r),$$
$$\cdots\cdots$$
$$(\text{-l-1-}\eta' - rv_3 - sv_4 - l'_1K_1 - \cdots - l'_sK_s;k'_1,\cdots,k'_r),$$

$$\left.\begin{array}{c} (\text{-k - mv}_1 - nv_2 - l_1K_1 - \cdots - l_sK_s;k_1,\cdots,k_r), A : C \\ \cdots\cdots \\ (\text{-k }+\beta - mv_1 - nv_2 - l_1K_1 - \cdots - l_sK_s;k_1,\cdots,k_r), B : D \end{array}\right) \tag{5.2}$$

The validity conditions are the following :

a) $min(v_1, v_2, v_3, v_4, k_i, \sigma_i, \delta_i, k'_i, g_i, h_i) > 0, i = 1,\cdots,r$

b) $Re[\rho + \sum_{i=1}^{r} k_i \min_{1\leqslant j\leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}] > -1, Re[\rho + \sum_{i=1}^{r} k'_i \min_{1\leqslant j\leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}] > -1$

c) $max[|arg(x^{v_1}/a)|, |arg(x^{v_2}/b)|, |arg(y^{v_3}/e)|, |arg(y^{v_4}/d)|] < \pi$

d) $|argz_i| < \frac{1}{2}A_i\pi$, where : $A_i = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(i)} - \sum_{j=\mathfrak{n}+1}^{p} \alpha_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{n_i} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \gamma_j^{(i)}$
$+ \sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} \delta_j^{(i)} > 0$, with $i = 1,\cdots,r$

e) The series occuring on the right-hand side of (2.2) are absolutely convergents.

6. Conclusion

The aleph-function of several variables presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions such as I-function of several variables defined by Sharma and Ahmad [2], multivariable H-function, see Srivastava et al [5], and the h-function of two variables , see Srivastava et a[5].

References :

[1] M. Saigo. A remark of integral operators involving the Gauss hypergeometric function. Math. Rep. College. General.Ed. Kyushu. Univ. vol11 (1978) p 135-143.

[2] C.K. Sharma and S.S.Ahmad : On the multivariable I-function. Acta ciencia Indica Math , 1992 vol 19 , page 113-116

[3] Srivastava H.M., A contour integral involving Fox's H-function. Indian J.Math. 14(1972), page1-6.

[4] Srivastava H.M. A multilinear generating function for the Konhauser set of biorthogonal polynomials suggested by Laguerre polynomial, Pacific. J. Math. Vol 77(1985), page183-191.

[5] Srivastava H.M., Gupta K.C. and Goyal S.P., the H-function of one and two variables with applications, South Asian Publications, NewDelhi (1982).