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Analysis of Single Server Bulk Queue with Two Choices of Service with Compulsory Vacation, Balking and Re-service

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Abstract

This paper deals with customers arriving in bulk or group in a single server queueing system in Poisson distribution which provides two types of general services in bulk of fixed size $M(\geq 1)$ or $a\min(n, M)$ in first come first served basis arbitrarily. Once the customer is serviced he leaves the system. In case of the server, the server takes compulsory arbitrary vacation after completion of the service. If the required bulk of customers is not available on the return of the server, the server again goes for vacation or remains in the system till the bulk is reached. The arriving batch balks during the period when the server is busy or when the server is on vacation or other constraints. This may reflect customer's impatient behavior. After the completion of any one kind of service the customer may leave the system or join the system for re-service. We obtain the time dependent probability generating functions and from it the corresponding steady state results are obtained. Also the various performance measures of service system calculated. Certain cases are briefed along with the numerical example.

Key words: bulk arrival, bulk service, vacation, balking, re-service

2010 Mathematics Subject Classification: 60K20, 60K25, 68M20, 90B22

1. Introduction

Many researchers like Chaudhry and Templeton [3], Gross and Harris [7] and Medhi [13] have worked on bulk queues. Vacation queue has been surveyed extensively by Doshi [5]. Chang and Takine [2] have studied Factorization and Stochastic decomposition properties in bulk queues with generalized vacation. Madan, Ab-Rawi and Al-Nasser [10] and Madan and Gautam Choudhury [11] have talked the single server queue with two types of general services and vacations. Madan [9] and Thangaraj and Vanitha [17] have discussed the compulsory sever vacation. Punniyamoorthy and Uma [19] have analyzed single server bulk queue with two choices of service and compulsory vacation. Monita Baruah, Madan and Tillal Eldabi [14] and Rajadurai, Saravanarajan and Chandrasekaran [15] and have detailed the balking and re-service in a vacation queue with batch arrival and two types of heterogeneous service. Jeyakumar and Arumuganathan [8] have analyzed the request for re-service in non-Markovian bulk queue.

This paper explains the queueing system with bulk arrival and bulk service with compulsory server vacation, balking and re-service. The arrival is under Poisson distribution and the services and vacation are arbitrary. Each batch of customers is provided with two types of services (heterogeneous) out of which they may choose. The bulk of customers is served under first come first served basis. The arriving batch balks when the server is busy or when the server is on vacation or other constraints. This may reflect customer's impatient behavior.After the completion of any one kind of service the customer may leave the system or join the system for re-service.

Bulk queues have been discussed extensively in the field of queueing theory. Many researchers have shown interest in its applications. Various works have been produced on bulk queues. In this concept, the arrivals or departures or both happen in batches of fixed or variable size. In many real time situations, customers served in batches. So, customer served singly is far from truth. We see concept of this paper has many real-time applications like emergency services in hospitals dealing with serious treatments, telecommunications, traffic signal systems, inventory and production, postal service, banking and so on.

The electricity is generated in the generating station and through the transmission lines,it reaches the substation as it cannot be directly distributed. From the substation, it is distributed to the primary customers who are the users of higher voltage and to the secondary customers who are the users of lower voltage. The substation is the server and the service of primary customers is type 1 service whereas that of the secondary customers is type 2 service.

Re-service is a real life phenomenon. In this phenomenon, the customers who got service may come in need of re-service for the already done service. This may be applied to the concept of electricity; type 1and type 2 service may come for re-service as required.

This paper is organized as follows: The mathematical model is briefed in section 2. Definitions and Notations are explained in section 3. Equations governing the system are explained in section 4. The time dependent solutions have been derived in section 5 and corresponding steady state results have been calculated clearly in section 6. The performance measures of the system are computed in section 7. Some particular cases dealt with in section 8. Numerical illustrations and conclusion are presented in section 9 and section 10 respectively.

2. The Mathematical Model

We assume the following to describe the queueing model of our study:

(a) Customers (units) arrive at the system in batches of variable size in a compound Poisson process.

(b) Let $\lambda\pi_i dt$ $(i = 1,2,3, \dots)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t + dt]$, where $0 \leq \pi_i \leq 1$, $\sum_{i=1}^{\infty} \pi_i = 1$, $\lambda > 0$ is the mean arrival rate of batches.

(c) We consider the case when there is single server providing parallel service of two types on a first come first served basis (FCFS). At the start of the service, each batch of customers has the choice of choosing either first service with probability θ_1 or the second service with probability $\theta_2$and $\theta_1 + \theta_2 = 1$.

(d) The service of customers (units) is rendered in batches of fixed size $M(\geq 1)$or $\min(n, M)$, where n is the number of customers in the queue.

(e) We assume that the random variable of service time $S_j (j = 1,2)$ of the j^{th} kind of service follows a general probability law with distribution function $G_j(s_j)$, $g_j(s_j)$ is the probability density function and $E(S_j^k)$ is the k^{th} moment $(k = 1,2, \dots)$ of service time $j = 1,2$.

(f) Let $\mu_j(x)$ be the conditional probability of type j service completion during the period $(x, x + dx]$, given that elapsed service time is x, so that

$$\mu_j(x) = \frac{g_j(x)}{1 - G_j(x)}, j = 1,2 \tag{1}$$

and therefore

$$g_j(s_j) = \mu_j(s_j) e^{-\int_0^{s_j} \mu_j(x) dx}, j = 1,2 \tag{2}$$

(g) After completion of continuous service to the batches of fixed size $M(\geq 1)$, the server will go for compulsory vacation.

(h) We further assume that the random variable of vacation time Y follows a general probability law with distribution function $V(y)$, $v(y)$ is the probability density function and $E(Y^k)$ is the k^{th} moment $(k = 1,2, \dots)$ of vacation time.

(i) Let $\alpha(x)$ be the conditional probability of completion of a vacation period during the interval $(x, x + dx]$, given that the elapsed vacation time is x, so that

$$\alpha(x) = \frac{v(x)}{1-V(x)} \tag{3}$$

and therefore

$$v(y) = \alpha(y)e^{-\int_0^y \alpha(x)dx} \tag{4}$$

(j) On returning from vacation the server instantly starts the service if there is a batch of size M or he remains idle in the system.

(k) We assume that $(1-\eta_1)(0 \leq \eta_1 \leq 1)$ is the probability that an arriving batch balks during the period when the server is busy (available on the system) and $(1-\eta_2)(0 \leq \eta_2 \leq 1)$ is the probability that an arriving batch balks during the period when the server is on vacation.

(l) As soon as service (of any one kind) is complete he has the option to leave the system or join the system for re-service, if necessary. We assume that probability of repeating type j service as ε_j and leaving the system without re-service as $(1-\varepsilon_j), j = 1,2$. We consider either service may be repeated only once.

(m) All arriving batches are allowed to join the system at all times.

(n) Finally, it is assumed that the inter-arrival times of the customers, the service times of each kind of service and vacation times of the server, all these stochastic processes involved in the system are independent of each other.

3. Definitions and Notations

We define:

$P_{n,j}(x,t)$: Probability that at time t, the server is active providing and there are n $(n \geq 0)$ customers in the queue, excluding a batch of M customers in type j service, $j = 1,2$ and the elapsed service time of this customer is x. Accordingly, $P_{n,j}(t) = \int_0^\infty P_{n,j}(x,t)dx$ denotes the probability that there are n customers in the queue excluding a batch of M customers in type j service, $j = 1,2$ irrespective of the elapsed service time x.

$R_{n,j}(x,t)$: Probability that at time t, the server is active providing and there are n $(n \geq 0)$ customers in the queue, excluding a batch of M customers in repeating type j service, $j = 1,2$ and the elapsed service time of this customer is x. Accordingly, $R_{n,j}(t) = \int_0^\infty P_{n,j}(x,t)dx$ denotes the probability that there are n customers in the queue excluding a batch of M customers in repeating type j service, $j = 1,2$ irrespective of the elapsed service time x.

$V_n(x,t)$: Probability that at time t, there are n $(n \geq 0)$ customers in the queue and the server is on vacation with the elapsed vacation time x. Accordingly, $V_n(t) = \int_0^\infty V_n(x,t)dx$ denotes the probability that there are n customers in the queue and the server is on vacation irrespective of the value of x.

$Q(t)$: Probability that at time t, there are less than M customers in the system and the server is idle but available in the system.

4. Equations Governing the System

According to the Mathematical model mentioned above, the system has the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_{n,1}(x,t) + \frac{\partial}{\partial t}P_{n,1}(x,t) + \big(\lambda + \mu_1(x)\big)P_{n,1}(x,t) = \lambda\eta_1 \sum_{k=1}^{n} \pi_k P_{n-k,1}(x,t) + \lambda(1-\eta_1)P_{n,1}(x,t) \tag{5}$$

$$\frac{\partial}{\partial x}P_{0,1}(x,t) + \frac{\partial}{\partial t}P_{0,1}(x,t) + \big(\lambda + \mu_1(x)\big)P_{0,1}(x,t) = \lambda(1-\eta_1)P_{0,1}(x,t) \tag{6}$$

$$\frac{\partial}{\partial x}P_{n,2}(x,t) + \frac{\partial}{\partial t}P_{n,2}(x,t) + \big(\lambda + \mu_2(x)\big)P_{n,2}(x,t) = \lambda\eta_1 \sum_{k=1}^{n} \pi_k P_{n-k,2}(x,t) + \lambda(1-\eta_1)P_{n,2}(x,t) \tag{7}$$

$$\frac{\partial}{\partial x}P_{0,2}(x,t) + \frac{\partial}{\partial t}P_{0,2}(x,t) + \big(\lambda + \mu_2(x)\big)P_{0,2}(x,t) = \lambda(1-\eta_1)P_{0,2}(x,t) \tag{8}$$

$$\frac{\partial}{\partial x}R_{n,1}(x,t)+\frac{\partial}{\partial t}R_{n,1}(x,t)+\big(\lambda+\mu_1(x)\big)R_{n,1}(x,t)=\lambda\eta_1\sum_{k=1}^{n}\pi_k R_{n-k,1}(x,t)+\lambda(1-\eta_1)R_{n,1}(x,t) \quad (9)$$
$$\frac{\partial}{\partial x}R_{0,1}(x,t)+\frac{\partial}{\partial t}R_{0,1}(x,t)+\big(\lambda+\mu_1(x)\big)R_{0,1}(x,t)=\lambda(1-\eta_1)R_{0,1}(x,t) \quad (10)$$
$$\frac{\partial}{\partial x}R_{n,2}(x,t)+\frac{\partial}{\partial t}R_{n,2}(x,t)+\big(\lambda+\mu_2(x)\big)R_{n,2}(x,t)=\lambda\eta_1\sum_{k=1}^{n}\pi_k R_{n-k,2}(x,t)+\lambda(1-\eta_1)R_{n,2}(x,t) \quad (11)$$
$$\frac{\partial}{\partial x}R_{0,2}(x,t)+\frac{\partial}{\partial t}R_{0,2}(x,t)+\big(\lambda+\mu_2(x)\big)R_{0,2}(x,t)=\lambda(1-\eta_1)R_{0,2}(x,t) \quad (12)$$
$$\frac{\partial}{\partial x}V_n(x,t)+\frac{\partial}{\partial t}V_n(x,t)+\big(\lambda+\alpha(x)\big)V_n(x,t)=\lambda\eta_2\sum_{k=1}^{n}\pi_k V_{n-k}(x,t)+\lambda(1-\eta_2)V_n(x,t) \quad (13)$$
$$\frac{\partial}{\partial x}V_0(x,t)+\frac{\partial}{\partial t}V_0(x,t)+\big(\lambda+\alpha(x)\big)V_0(x,t)=\lambda(1-\eta_2)V_0(x,t) \quad (14)$$
$$\frac{d}{dt}Q(t)+\lambda Q(t)=\lambda(1-\eta_1)Q(t)+\int_0^\infty V_0(x,t)\alpha(x)dx \quad (15)$$

Equations (5) – (15) are to be solved subject to the following boundary conditions:

$$P_{n,1}(0,t)=\theta_1\int_0^\infty V_{n+M}(x,t)\alpha(x)dx \quad (16)$$
$$P_{0,1}(0,t)=\theta_1\sum_{b=1}^{M}\int_0^\infty V_b(x,t)\alpha(x)dx+\theta_1\lambda\eta_1 Q(t) \quad (17)$$
$$P_{n,2}(0,t)=\theta_2\int_0^\infty V_{n+M}(x,t)\alpha(x)dx \quad (18)$$
$$P_{0,2}(0,t)=\theta_2\sum_{b=1}^{M}\int_0^\infty V_b(x,t)\alpha(x)dx+\theta_2\lambda\eta_1 Q(t) \quad (19)$$
$$R_{n,1}(0,t)=\varepsilon_1\int_0^\infty P_{n,1}(x,t)\mu_1(x)dx \quad (20)$$
$$R_{n,2}(0,t)=\varepsilon_2\int_0^\infty P_{n,2}(x,t)\mu_2(x)dx \quad (21)$$
$$V_n(0,t)=(1-\varepsilon_1)\int_0^\infty P_{n,1}(x,t)\mu_1(x)dx+(1-\varepsilon_2)\int_0^\infty P_{n,2}(x,t)\mu_2(x)dx \quad (22)$$

We assume that initially the server is available but idle because of less than M customers so that the initial conditions are

$V_n(0)=0;\ V_0(0)=0;\ Q(0)=1; P_{n,j}(0)=0, R_{n,j}(0)=0,$ for $n=0,1,2,\ldots$ & $j=1,2.$ (23)

5. Probability Generating Function of the Queue Size: The Transient Solution

We define the following probability generating functions:

$$\left.\begin{aligned}
P_j(x,z,t)&=\sum_{n=0}^{\infty}P_{n,j}(x,t)z^n,\ \ j=1,2\\
P_j(z,t)&=\sum_{n=0}^{\infty}P_{n,j}(t)z^n,\ \ j=1,2\\
R_j(x,z,t)&=\sum_{n=0}^{\infty}R_{n,j}(x,t)z^n,\ \ j=1,2\\
R_j(z,t)&=\sum_{n=0}^{\infty}R_{n,j}(t)z^n,\ \ j=1,2\\
V(x,z,t)&=\sum_{n=0}^{\infty}V_n(x,t)z^n\\
V(z,t)&=\sum_{n=0}^{\infty}V_n(t)z^n\\
\pi(z)&=\sum_{n=1}^{\infty}\pi_n z^n
\end{aligned}\right\} \quad (24)$$

Define the Laplace-Stieltjes Transform of a function $f(t)$ as follows:

$$\bar{f}(s)=L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt \quad (25)$$

Taking Laplace Transform of equations (5) – (22) and using (23), we get,

$$\frac{\partial}{\partial x}\bar{P}_{n,1}(x,s)+\big(s+\lambda+\mu_1(x)\big)\bar{P}_{n,1}(x,s)=\lambda\eta_1\sum_{k=1}^{n}\pi_k\bar{P}_{n-k,1}(x,s)+\lambda(1-\eta_1)\bar{P}_{n,1}(x,s) \quad (26)$$
$$\frac{\partial}{\partial x}\bar{P}_{0,1}(x,s)+\big(s+\lambda+\mu_1(x)\big)\bar{P}_{0,1}(x,s)=\lambda(1-\eta_1)\bar{P}_{0,1}(x,s) \quad (27)$$
$$\frac{\partial}{\partial x}\bar{P}_{n,2}(x,s)+\big(s+\lambda+\mu_2(x)\big)\bar{P}_{n,2}(x,s)=\lambda\eta_1\sum_{k=1}^{n}\pi_k\bar{P}_{n-k,2}(x,s)+\lambda(1-\eta_1)\bar{P}_{n,2}(x,s) \quad (28)$$
$$\frac{\partial}{\partial x}\bar{P}_{0,2}(x,s)+\big(s+\lambda+\mu_2(x)\big)\bar{P}_{0,2}(x,s)=\lambda(1-\eta_1)\bar{P}_{0,2}(x,s) \quad (29)$$
$$\frac{\partial}{\partial x}\bar{R}_{n,1}(x,s)+\big(s+\lambda+\mu_1(x)\big)\bar{R}_{n,1}(x,s)=\lambda\eta_1\sum_{k=1}^{n}\pi_k\bar{R}_{n-k,1}(x,s)+\lambda(1-\eta_1)\bar{R}_{n,1}(x,s) \quad (30)$$
$$\frac{\partial}{\partial x}\bar{R}_{0,1}(x,s)+\big(s+\lambda+\mu_1(x)\big)\bar{R}_{0,1}(x,s)=\lambda(1-\eta_1)\bar{R}_{0,1}(x,s) \quad (31)$$
$$\frac{\partial}{\partial x}\bar{R}_{n,2}(x,s)+\big(s+\lambda+\mu_2(x)\big)\bar{R}_{n,2}(x,s)=\lambda\eta_1\sum_{k=1}^{n}\pi_k\bar{R}_{n-k,2}(x,s)+\lambda(1-\eta_1)\bar{R}_{n,2}(x,s) \quad (32)$$
$$\frac{\partial}{\partial x}\bar{R}_{0,2}(x,s)+\big(s+\lambda+\mu_2(x)\big)\bar{R}_{0,2}(x,s)=\lambda(1-\eta_1)\bar{R}_{0,2}(x,s) \quad (33)$$
$$\frac{\partial}{\partial x}\bar{V}_n(x,s)+\big(s+\lambda+\alpha(x)\big)\bar{V}_n(x,s)=\lambda\eta_2\sum_{k=1}^{n}\pi_k\bar{V}_{n-k}(x,s)+\lambda(1-\eta_2)\bar{V}_n(x,s) \quad (34)$$
$$\frac{\partial}{\partial x}\bar{V}_0(x,s)+\big(s+\lambda+\alpha(x)\big)\bar{V}_0(x,s)=\lambda(1-\eta_2)\bar{V}_0(x,s) \quad (35)$$

$(s+\lambda\eta_1)\bar{Q}(s) = 1 + \int_0^\infty \bar{V}_0(x,s)\alpha(x)dx$ (36)

$\bar{P}_{n,1}(0,s) = \theta_1 \int_0^\infty \bar{V}_{n+M}(x,s)\alpha(x)dx$ (37)

$\bar{P}_{0,1}(0,s) = \theta_1 \sum_{b=1}^{M} \int_0^\infty \bar{V}_b(x,s)\alpha(x)dx + \theta_1\lambda\eta_1\bar{Q}(s)$ (38)

$\bar{P}_{n,2}(0,s) = \theta_2 \int_0^\infty \bar{V}_{n+M}(x,s)\alpha(x)dx$ (39)

$\bar{P}_{0,2}(0,s) = \theta_2 \sum_{b=1}^{M} \int_0^\infty \bar{V}_b(x,s)\alpha(x)dx + \theta_2\lambda\eta_1\bar{Q}(s)$ (40)

$\bar{R}_{n,1}(0,s) = \varepsilon_1 \int_0^\infty \bar{P}_{n,1}(x,s)\mu_1(x)dx$ (41)

$\bar{R}_{n,2}(0,s) = \varepsilon_2 \int_0^\infty \bar{P}_{n,2}(x,s)\mu_2(x)dx$ (42)

$\bar{V}_n(0,s) = (1-\varepsilon_1)\int_0^\infty \bar{P}_{n,1}(x,s)\mu_1(x)dx + (1-\varepsilon_2)\int_0^\infty \bar{P}_{n,2}(x,s)\mu_2(x)dx$ (43)

Multiplying the equation (26) by z^n and summing over n from 1to ∞, adding equation (27) and using the generating functions defined in (24), we obtain,

$\frac{\partial}{\partial x}\bar{P}_1(x,z,s) + \{s+\lambda\eta_1(1-\pi(z)) + \mu_1(x)\}\bar{P}_1(x,z,s) = 0$ (44)

Performing similar operations on equations (28) – (35), we obtain,

$\frac{\partial}{\partial x}\bar{P}_2(x,z,s) + \{s+\lambda\eta_1(1-\pi(z)) + \mu_2(x)\}\bar{P}_2(x,z,s) = 0$ (45)

$\frac{\partial}{\partial x}\bar{R}_1(x,z,s) + \{s+\lambda\eta_1(1-\pi(z)) + \mu_1(x)\}\bar{R}_1(x,z,s) = 0$ (46)

$\frac{\partial}{\partial x}\bar{R}_2(x,z,s) + \{s+\lambda\eta_1(1-\pi(z)) + \mu_2(x)\}\bar{R}_2(x,z,s) = 0$ (47)

$\frac{\partial}{\partial x}\bar{V}(x,z,s) + \{s+\lambda\eta_2(1-\pi(z)) + \alpha(x)\}\bar{V}(x,z,s) = 0$ (48)

Multiplying the equation (37) by z^{n+M} and summing over n from 1 to ∞ and adding, multiplying the equation (38) by z^M, and using the generating functions defined in (24), we obtain,

$$z^M\bar{P}_1(0,z,s) = \theta_1\int_0^\infty \bar{V}(x,z,s)\alpha(x)dx - \theta_1\int_0^\infty \bar{V}_0(x,s)\alpha(x)dx - \theta_1\sum_{b=1}^{M-1}\int_0^\infty \bar{V}_b(x,s)\alpha(x)dx\, z^b$$

$$+\theta_1\sum_{b=1}^{M-1}\int_0^\infty \bar{V}_b(x,s)\alpha(x)dx\, z^M + \theta_1\lambda\eta_1\bar{Q}(s)z^M \quad (49)$$

Using (36), we obtain,

$\bar{P}_1(0,z,s) = \theta_1 z^{-M}\left[\int_0^\infty \bar{V}(x,z,s)\alpha(x)dx + U_1 + U_2\bar{Q}(s) + 1\right]$ (50)

Where $U_1 = \sum_{b=1}^{M-1}(z^M - z^b)\int_0^\infty \bar{V}_b(x,s)\alpha(x)dx$ and $U_2 = \lambda\eta_1(z^M-1) - s$

Performing similar operations on equations (39) and (40), we obtain,

$\bar{P}_2(0,z,s) = \theta_2 z^{-M}\left[\int_0^\infty \bar{V}(x,z,s)\alpha(x)dx + U_1 + U_2\bar{Q}(s) + 1\right]$ (51)

Multiplying the equation (41) by z^n and summing over n from 0 to ∞ and using the generating functions defined in (24), we obtain,

$\bar{R}_1(0,z,s) = \varepsilon_1\int_0^\infty \bar{P}_1(x,z,s)\mu_1(x)dx$ (52)

Performing similar operations on equations (42) and (43), we obtain,

$\bar{R}_2(0,z,s) = \varepsilon_2\int_0^\infty \bar{P}_2(x,z,s)\mu_2(x)dx$ (53)

$\bar{V}(0,z,s) = (1-\varepsilon_1)\int_0^\infty \bar{P}_1(x,z,s)\mu_1(x)dx + (1-\varepsilon_2)\int_0^\infty \bar{P}_2(x,z,s)\mu_2(x)dx$ (54)

We now integrate equations (44) – (48) between the limits 0 and x and obtain,

$\bar{P}_1(x,z,s) = \bar{P}_1(0,z,s)e^{-T_1x-\int_0^x \mu_1(x)dx}$ (55)

$\bar{P}_2(x,z,s) = \bar{P}_2(0,z,s)e^{-T_1x-\int_0^x \mu_2(x)dx}$ (56)

$\bar{R}_1(x,z,s) = \bar{R}_1(0,z,s)e^{-T_1x-\int_0^x \mu_1(x)dx}$ (57)

$\bar{R}_2(x,z,s) = \bar{R}_2(0,z,s)e^{-T_1x-\int_0^x \mu_2(x)dx}$ (58)

$\bar{V}(x,z,s) = \bar{V}(0,z,s)e^{-T_2x-\int_0^x \alpha(x)dx}$ (59)

Where $T_1 = s+\lambda\eta_1(1-\pi(z))$ and $T_2 = s+\lambda\eta_2(1-\pi(z))$

Integrating equations (55) – (59) by parts, with respect to x, we get,

$\bar{P}_1(z,s) = \bar{P}_1(0,z,s)\left[\frac{1-\bar{G}_1(T_1)}{T_1}\right]$ (60)

$\bar{P}_2(z,s) = \bar{P}_2(0,z,s)\left[\frac{1-\bar{G}_2(T_1)}{T_1}\right]$ (61)

$\bar{R}_1(z,s) = \bar{R}_1(0,z,s)\left[\frac{1-\bar{G}_1(T_1)}{T_1}\right]$ (62)

$\bar{R}_2(z,s) = \bar{R}_2(0,z,s)\left[\frac{1-\bar{G}_2(T_1)}{T_1}\right]$ (63)

$\bar{V}(z,s) = \bar{V}(0,z,s)\left[\frac{1-\bar{V}(T_2)}{T_2}\right]$ (64)

Where $\bar{G}_j(T_1) = \int_0^\infty e^{-T_1 x} dG_j(x)$, is the Laplace Transform of j^{th} type of service, $j = 1,2$.

$\bar{V}(T_2) = \int_0^\infty e^{-T_2 x} dV(x)$ is the Laplace Transform of vacation time.

Multiplying the equations (55) and (57), (56) and (58) and (59) by with $\mu_1(x)$, $\mu_2(x)$ and $\alpha(x)$ integrating by parts, with respect to x, we get,

$\int_0^\infty \bar{P}_1(x,z,s)\mu_1(x)dx = \bar{P}_1(0,z,s)\bar{G}_1(T_1)$ (65)

$\int_0^\infty \bar{P}_2(x,z,s)\mu_2(x)dx = \bar{P}_2(0,z,s)\bar{G}_2(T_1)$ (66)

$\int_0^\infty \bar{R}_1(x,z,s)\mu_1(x)dx = \bar{R}_1(0,z,s)\bar{G}_1(T_1)$ (67)

$\int_0^\infty \bar{R}_2(x,z,s)\mu_2(x)dx = \bar{R}_2(0,z,s)\bar{G}_2(T_1)$ (68)

$\int_0^\infty \bar{V}(x,z,s)\alpha(x)dx = \bar{V}(0,z,s)\bar{V}(T_2)$ (69)

Substituting (69) in (50) we get,

$\bar{P}_1(0,z,s) = \theta_1 z^{-M}[\bar{V}(0,z,s)\bar{V}(T_2) + U_1 + U_2\bar{Q}(s) + 1]$ (70)

Substituting (65), (66) in (52), (53) and (54) we get,

$\bar{R}_1(0,z,s) = \varepsilon_1\bar{P}_1(0,z,s)\bar{G}_1(T_1)$ (71)

$\bar{R}_2(0,z,s) = \varepsilon_2\bar{P}_2(0,z,s)\bar{G}_2(T_1)$ (72)

$\bar{V}(0,z,s) = (1-\varepsilon_1)\bar{P}_1(0,z,s)\bar{G}_1(T_1) + (1-\varepsilon_2)\bar{P}_2(0,z,s)\bar{G}_2(T_1)$ (73)

Substituting (73) in (70) we get,

$\bar{P}_1(0,z,s) = \theta_1 z^{-M}\left[\begin{Bmatrix} (1-\varepsilon_1)\bar{P}_1(0,z,s)\bar{G}_1(T_1) \\ +(1-\varepsilon_2)\bar{P}_2(0,z,s)\bar{G}_2(T_1)\end{Bmatrix}\bar{V}(T_2) + U_1 + U_2\bar{Q}(s) + 1\right]$ (74)

Performing similar operations on (51), we obtain,

$\bar{P}_2(0,z,s) = \theta_2 z^{-M}\left[\begin{Bmatrix} (1-\varepsilon_1)\bar{P}_1(0,z,s)\bar{G}_1(T_1) \\ +(1-\varepsilon_2)\bar{P}_2(0,z,s)\bar{G}_2(T_1)\end{Bmatrix}\bar{V}(T_2) + U_1 + U_2\bar{Q}(s) + 1\right]$ (75)

Solving (74) and (75), we get,

$\bar{P}_1(0,z,s) = \frac{\theta_1\{U_1+U_2\bar{Q}(s)+1\}}{z^M - \bar{V}(T_2)\big(\theta_1(1-\varepsilon_1)\bar{G}_1(T_1)+\theta_2(1-\varepsilon_2)\bar{G}_2(T_1)\big)}$ (76)

$\bar{P}_2(0,z,s) = \frac{\theta_2\{U_1+U_2\bar{Q}(s)+1\}}{z^M - \bar{V}(T_2)\big(\theta_1(1-\varepsilon_1)\bar{G}_1(T_1)+\theta_2(1-\varepsilon_2)\bar{G}_2(T_1)\big)}$ (77)

Substituting (76) and (77) in (71) and (72) respectively, we get,

$\bar{R}_1(0,z,s) = \frac{\varepsilon_1\theta_1\{U_1+U_2\bar{Q}(s)+1\}\bar{G}_1(T_1)}{z^M - \bar{V}(T_2)\big(\theta_1(1-\varepsilon_1)\bar{G}_1(T_1)+\theta_2(1-\varepsilon_2)\bar{G}_2(T_1)\big)}$ (78)

$\bar{R}_2(0,z,s) = \frac{\varepsilon_2\theta_2\{U_1+U_2\bar{Q}(s)+1\}\bar{G}_2(T_1)}{z^M - \bar{V}(T_2)\big(\theta_1(1-\varepsilon_1)\bar{G}_1(T_1)+\theta_2(1-\varepsilon_2)\bar{G}_2(T_1)\big)}$ (79)

Substituting (76) and (77) in (73), we get,

$\bar{V}(0,z,s) = \frac{\{\theta_1(1-\varepsilon_1)\bar{G}_1(T_1)+\theta_2(1-\varepsilon_2)\bar{G}_2(T_1)\}\{U_1+U_2\bar{Q}(s)+1\}}{z^M - \bar{V}(T_2)\big(\theta_1(1-\varepsilon_1)\bar{G}_1(T_1)+\theta_2(1-\varepsilon_2)\bar{G}_2(T_1)\big)}$ (80)

Substituting (76) – (80) in (60) – (64) respectively, we get,

$\bar{P}_1(z,s) = \left[\frac{\theta_1\{U_1+U_2\bar{Q}(s)+1\}}{z^M - \bar{V}(T_2)\big(\theta_1(1-\varepsilon_1)\bar{G}_1(T_1)+\theta_2(1-\varepsilon_2)\bar{G}_2(T_1)\big)}\right]\left[\frac{1-\bar{G}_1(T_1)}{T_1}\right]$ (81)

$\bar{P}_2(z,s) = \left[\frac{\theta_2\{U_1+U_2\bar{Q}(s)+1\}}{z^M - \bar{V}(T_2)\big(\theta_1(1-\varepsilon_1)\bar{G}_1(T_1)+\theta_2(1-\varepsilon_2)\bar{G}_2(T_1)\big)}\right]\left[\frac{1-\bar{G}_2(T_1)}{T_1}\right]$ (82)

$\bar{R}_1(z,s) = \left[\frac{\varepsilon_1\theta_1\{U_1+U_2\bar{Q}(s)+1\}\bar{G}_1(T_1)}{z^M - \bar{V}(T_2)\big(\theta_1(1-\varepsilon_1)\bar{G}_1(T_1)+\theta_2(1-\varepsilon_2)\bar{G}_2(T_1)\big)}\right]\left[\frac{1-\bar{G}_1(T_1)}{T_1}\right]$ (83)

$\bar{R}_2(z,s) = \left[\frac{\varepsilon_2\theta_2\{U_1+U_2\bar{Q}(s)+1\}\bar{G}_2(T_1)}{z^M - \bar{V}(T_2)\big(\theta_1(1-\varepsilon_1)\bar{G}_1(T_1)+\theta_2(1-\varepsilon_2)\bar{G}_2(T_1)\big)}\right]\left[\frac{1-\bar{G}_2(T_1)}{T_1}\right]$ (84)

$$\bar{V}(z,s)=\left[\frac{\left\{\begin{matrix}\theta_1(1-\varepsilon_1)\bar{G}_1(T_1)\\+\theta_2(1-\varepsilon_2)\bar{G}_2(T_1)\end{matrix}\right\}\{U_1+U_2\bar{Q}(s)+1\}}{z^M-\bar{V}(T_2)\big(\theta_1(1-\varepsilon_1)\bar{G}_1(T_1)+\theta_2(1-\varepsilon_2)\bar{G}_2(T_1)\big)}\right]\left[\frac{1-\bar{V}(T_2)}{T_2}\right] \tag{85}$$

We note that there are M unknowns, $\bar{Q}(s)$and $\bar{V}_b(x,s), b=1,2,\ldots,M-1$ appearing in equations (81) – (85).

Now (81) – (85) give the probability generating function of the service system with M unknowns. By Rouche's theorem of complex variables, it can be proved that $z^M-\bar{V}(T_2)\big(\theta_1(1-\varepsilon_1)\bar{G}_1(T_1)+\theta_2(1-\varepsilon_2)\bar{G}_2(T_1)\big)$ has M zeroes inside the contour $|z|=1$. Since $\bar{P}_1(z,s),\bar{P}_2(z,s),\bar{R}_1(z,s),\bar{R}_2(z,s)$ and $\bar{V}(z,s)$ are analytic inside the unit circle $|z|=1$, the numerator in the right hand side of equations (81) – (85) must vanish at these points, which gives rise to a set of M linear equations which are sufficient to determine Munknowns.

6. The Steady State Results

To define the steady state probabilities and corresponding generating functions, we drop the argument t, and for that matter the argument s wherever it appears in the time-dependent analysis up to this point. Then the corresponding steady state results can be obtained by using the well-known Tauberian Property

$$\lim_{s\to 0} s\bar{f}(s)=\lim_{t\to\infty} f(t) \tag{86}$$

if the limit on the right exists.

Now (81) – (85) we have,

$$P_1(z)=\frac{\theta_1\{U_1^*+U_2^*Q\}\left\{\frac{1-\bar{G}_1(f_1(z))}{f_1(z)}\right\}}{z^M-\bar{V}(f_2(z))\big(\theta_1(1-\varepsilon_1)\bar{G}_1(f_1(z))+\theta_2(1-\varepsilon_2)\bar{G}_2(f_1(z))\big)} \tag{87}$$

$$P_2(z)=\frac{\theta_2\{U_1^*+U_2^*Q\}\left\{\frac{1-\bar{G}_2(f_1(z))}{f_1(z)}\right\}}{z^M-\bar{V}(f_2(z))\big(\theta_1(1-\varepsilon_1)\bar{G}_1(f_1(z))+\theta_2(1-\varepsilon_2)\bar{G}_2(f_1(z))\big)} \tag{88}$$

$$R_1(z)=\frac{\varepsilon_1\theta_1\{U_1^*+U_2^*Q\}\bar{G}_1(f_1(z))\left\{\frac{1-\bar{G}_1(f_1(z))}{f_1(z)}\right\}}{z^M-\bar{V}(f_2(z))\big(\theta_1(1-\varepsilon_1)\bar{G}_1(f_1(z))+\theta_2(1-\varepsilon_2)\bar{G}_2(f_1(z))\big)} \tag{89}$$

$$R_2(z)=\frac{\varepsilon_2\theta_2\{U_1^*+U_2^*Q\}\bar{G}_2(f_1(z))\left\{\frac{1-\bar{G}_2(f_1(z))}{f_1(z)}\right\}}{z^M-\bar{V}(f_2(z))\big(\theta_1(1-\varepsilon_1)\bar{G}_1(f_1(z))+\theta_2(1-\varepsilon_2)\bar{G}_2(f_1(z))\big)} \tag{90}$$

$$V(z)=\frac{\left\{\begin{matrix}\theta_1(1-\varepsilon_1)\bar{G}_1(f_1(z))\\+\theta_2(1-\varepsilon_2)\bar{G}_2(f_1(z))\end{matrix}\right\}\{U_1^*+U_2^*Q\}\left\{\frac{1-\bar{V}(f_2(z))}{f_2(z)}\right\}}{z^M-\bar{V}(f_2(z))\big(\theta_1(1-\varepsilon_1)\bar{G}_1(f_1(z))+\theta_2(1-\varepsilon_2)\bar{G}_2(f_1(z))\big)} \tag{91}$$

Where $f_1(z)=\lambda\eta_1\big(1-\pi(z)\big),\ f_2(z)=\lambda\eta_2\big(1-\pi(z)\big)U_1^*=\sum_{b=1}^{M-1}(z^M-z^b)\int_0^\infty V_b(x)\alpha(x)dx$ and $U_2^*=\lambda\eta_1(z^M-1)$

The M unknowns, Q and $\int_0^\infty V_b(x)\alpha(x)dx\,, b=1,2,\ldots,M-1$can be determined as before.

Let $A_q(z)$ denote the probability generating function of the queue size irrespective of the state of the system.

$$\text{i.e.}, A_q(z)=P_1(z)+P_2(z)+R_1(z)+R_2(z)+V(z)$$

Then adding equations (87) – (91), we obtain,

$$A_q(z)=\frac{[U_1^*+U_2^*Q]\left[\begin{matrix}\theta_1[1+\varepsilon_1\bar{G}_1(f_1(z))]\left\{\frac{1-\bar{G}_1(f_1(z))}{f_1(z)}\right\}+\theta_2[1+\varepsilon_2\bar{G}_2(f_1(z))]\left\{\frac{1-\bar{G}_2(f_1(z))}{f_1(z)}\right\}\\+\left\{\begin{matrix}\theta_1(1-\varepsilon_1)\bar{G}_1(f_1(z))\\+\theta_2(1-\varepsilon_2)\bar{G}_2(f_1(z))\end{matrix}\right\}\left\{\frac{1-\bar{V}(f_2(z))}{f_2(z)}\right\}\end{matrix}\right]}{z^M-\bar{V}(f_2(z))\big(\theta_1(1-\varepsilon_1)\bar{G}_1(f_1(z))+\theta_2(1-\varepsilon_2)\bar{G}_2(f_1(z))\big)} \tag{92}$$

In order to find Q, we use the normalization condition

$$A_q(1)+Q=1$$

Note that for $z=1$, $A_q(1)$is indeterminate of $\frac{0}{0}$ form.

Therefore, we apply L'Hôpital's Rule on (92), we get,

$$A_q(1)=\frac{[E_1^*+E(V)\{1-\delta\}]\left[U_1^{*'}+\lambda\eta_1 MQ\right]}{M-\lambda E(I)[\eta_1E_2^*+\eta_2E(V)\{1-\delta\}]} \tag{93}$$

Where $U_1^{*'} = \sum_{b=1}^{M-1}(M-b)\int_0^\infty V_b(x)\alpha(x)dx$, $E_1^* = \theta_1(1+\varepsilon_1)E(S_1) + \theta_2(1+\varepsilon_2)E(S_2)$,
$E_2^* = \theta_1(1-\varepsilon_1)E(S_1) + \theta_2(1-\varepsilon_2)E(S_2)and\delta = \theta_1\varepsilon_1 + \theta_2\varepsilon_2$
We use $\bar{G}_j(0) = 1,\ j = 1,2,\ \bar{V}(0) = 1,\ \pi'(1) = E(I)$, where$I$ denotes the number of customers in an arriving batch and therefore, $E(I)$ is the mean of the batch size of the arriving customers.
Also where $E(S_1)$, $E(S_2)$ and $E(V)$are the mean service times of type 1, type 2 services and vacation time, respectively.
Therefore, adding Q to equation (93) and equating to 1 and simplifying we get,

$$Q = 1 - \frac{[E_1^* + E(V)\{1-\delta\}]\left(M\lambda\eta_1 + U_1^{*'}\right)}{M + M\lambda\eta_1[E_1^* + E(V)\{1-\delta\}] - \lambda E(I)[\eta_1 E_2^* + \eta_2 E(V)\{1-\delta\}]} \tag{94}$$

Equation (94) gives the probability that the server is idle.
From equation (94) the utilization factor, ρ of the system is given by

$$\rho = \frac{[E_1^* + E(V)\{1-\delta\}]\left(M\lambda\eta_1 + U_1^{*'}\right)}{M + M\lambda\eta_1[E_1^* + E(V)\{1-\delta\}] - \lambda E(I)[\eta_1 E_2^* + \eta_2 E(V)\{1-\delta\}]} \tag{95}$$

Where $\rho < 1$ is the stability condition under which the steady state exists. Substituting for Q from (94) into (92), we have completely and explicitly determined $A_q(z)$, the probability generating function of the queue size.

7. The Performance Measures of the System

7.1. The average queue size and the system size

Let $L_q$denote the mean number of customers in the queue under the steady state. Then

$$L_q = \frac{d}{dz}A_q(z)\Big|_{z=1} \tag{96}$$

Since the formula gives $\frac{0}{0}$ form, then we write $A_q(z)$ given in (92) as$A_q(z) = \frac{N(z)}{D(z)}$
Where $N(z)$ and $D(z)$ are the numerator and denominator of the right hand side of (92) respectively.
Then using L'Hôpital's Rule twice we obtain,

$$L_q = \lim_{z\to 1}\frac{D'(z)N''(z) - N'(z)D''(z)}{2\left(D'(z)\right)^2} \tag{97}$$

Where primes and double primes in (97) denote first and second derivatives at $z = 1$, respectively. Carrying out the derivatives at $z = 1$we have,

$$N'(1) = [E_1^* + \{1-\delta\}E(V)]\left[U_1^{*'} + \lambda\eta_1 MQ\right] \tag{98}$$

$$D'(1) = M - \lambda E(I)[\eta_1 E_2^* + \eta_2 E(V)\{1-\delta\}] \tag{99}$$

$$\begin{aligned} N''(1) = {} & \lambda E(I)\left[U_1^{*'} + \lambda\eta_1 MQ\right][\eta_1 E^* + 2\eta_1 E^{***} + \{1-\delta\}\eta_2 E(V^2) + 2E(V)\eta_1 E_2^*] \\ & +[U^{**} + \lambda\eta_1 M(M-1)Q][E_1^* + \{1-\delta\}E(V)] \end{aligned} \tag{100}$$

$$\begin{aligned} D''(1) = {} & M(M-1) - 2\lambda^2\eta_1\eta_2\left(E(I)\right)^2 E(V)E_2^* - \lambda^2\left(E(I)\right)^2[\eta_1^2 E^{**} + \eta_2^2 E(V^2)\{1-\delta\}] \\ & -\lambda E\left(I(I-1)\right)[\eta_1 E_2^* + \eta_2 E(V)\{1-\delta\}] \end{aligned} \tag{101}$$

$$L_q = \frac{\begin{array}{c}\{M - \lambda E(I)[\eta_1 E_2^* + \eta_2 E(V)\{1-\delta\}]\} \times \left\{\begin{array}{c}\lambda E(I)\left[U_1^{*'} + \lambda\eta_1 MQ\right]\left[\begin{array}{c}\eta_1 E^* + 2\eta_1 E^{***} \\ +\{1-\delta\}\eta_2 E(V^2) + 2E(V)\eta_1 E_2^*\end{array}\right] \\ +[U^{**} + \lambda\eta_1 M(M-1)Q][E_1^* + \{1-\delta\}E(V)]\end{array}\right\} \\ -[E_1^* + \{1-\delta\}E(V)]\left[U_1^{*'} + \lambda\eta_1 MQ\right] \times \left\{\begin{array}{c}M(M-1) - 2\lambda^2\eta_1\eta_2\left(E(I)\right)^2 E(V)E_2^* \\ -\lambda^2\left(E(I)\right)^2[\eta_1^2 E^{**} + \eta_2^2 E(V^2)\{1-\delta\}] \\ -\lambda E\left(I(I-1)\right)[\eta_1 E_2^* + \eta_2 E(V)\{1-\delta\}]\end{array}\right\}\end{array}}{2\left[M - \lambda E(I)[\eta_1 E_2^* + \eta_2 E(V)\{1-\delta\}]\right]^2} \tag{102}$$

Where $U^{**} = \sum_{b=1}^{M-1}\left(M(M-1) - b(b-1)\right)\int_0^\infty V_b(x)\alpha(x)dx$ and
$E^* = \theta_1(1+\varepsilon_1)E(S_1^2) + \theta_2(1+\varepsilon_2)E(S_2^2)$, $E^{**} = \theta_1(1-\varepsilon_1)E(S_1^2) + \theta_2(1-\varepsilon_2)E(S_2^2)$ and $E^{***} = \theta_1\varepsilon_1\left(E(S_1)\right)^2 + \theta_2\varepsilon_2\left(E(S_2)\right)^2$
Where $E\left(I(I-1)\right)$ is the second factorial moment of the batch size of the arriving customers. Similarly, $E(S_1^2), E(S_2^2)$ are the second moments of the service times of type 1, type 2 services, respectively. $E(V^2)$is the second moment of the vacation time and Q has been obtained in (94).
Further, we can also find L_s, the average number of customers in the system as

$$L_s = L_q + \rho \tag{103}$$

Where L_q and ρ have been found in (102) and (95) respectively.

7.2. The average waiting time in the queue and the system

Let W_q and W_s respectively denote the waiting time in the queue and the system.

Then, the mean waiting time of a customer could be found using Little's Law formula

$$W_q = \frac{L_q}{\lambda} \tag{104}$$

Where L_q has been found in (102) and λ is the arrival rate into the system.

Finally, we can also find W_s as

$$W_s = \frac{L_s}{\lambda} \tag{105}$$

Where L_s has been found in (103) and λ is the arrival rate into the system.

8. Special Cases:

Case 1. Re-service but no balking

In this case, we let $\eta_1 = \eta_2 = 1$ in the equations (94), (95) and (102)

$$Q = 1 - \frac{[E_1^* + E(V)\{1-\delta\}]\left(M\lambda + U_1^{*\prime}\right)}{M + M\lambda[E_1^* + E(V)\{1-\delta\}] - \lambda E(I)[E_2^* + E(V)\{1-\delta\}]} \tag{106}$$

$$\rho = \frac{[E_1^* + E(V)\{1-\delta\}]\left(M\lambda + U_1^{*\prime}\right)}{M + M\lambda[E_1^* + E(V)\{1-\delta\}] - \lambda E(I)[E_2^* + E(V)\{1-\delta\}]} \tag{107}$$

$$L_q = \frac{\{M - \lambda E(I)[E_2^* + E(V)\{1-\delta\}]\} \times \left\{\begin{array}{c} \lambda E(I)\left[U_1^{*\prime} + \lambda MQ\right]\left[\begin{array}{c} E^* + 2E^{***} \\ +\{1-\delta\}E(V^2) + 2E(V)E_2^* \end{array}\right] \\ +[U^{**} + \lambda M(M-1)Q][E_1^* + \{1-\delta\}E(V)] \end{array}\right\} - [E_1^* + \{1-\delta\}E(V)]\left[U_1^{*\prime} + \lambda MQ\right] \times \left\{\begin{array}{c} M(M-1) - 2\lambda^2 (E(I))^2 E(V)E_2^* \\ -\lambda^2 (E(I))^2 [E^{**} + E(V^2)\{1-\delta\}] \\ -\lambda E(I(I-1))[E_2^* + E(V)\{1-\delta\}] \end{array}\right\}}{2\left[M - \lambda E(I)[E_2^* + E(V)\{1-\delta\}]\right]^2} \tag{108}$$

Case 2.Balking but no re-service

In this case, we let $\varepsilon_1 = \varepsilon_2 = 0$ in the equations (94), (95) and (102)

$$Q = 1 - \frac{[\theta_1 E(S_1) + \theta_2 E(S_2) + E(V)]\left(M\lambda\eta_1 + U_1^{*\prime}\right)}{M + M\lambda\eta_1[\theta_1 E(S_1) + \theta_2 E(S_2) + E(V)] - \lambda E(I)[\eta_1\{\theta_1 E(S_1) + \theta_2 E(S_2)\} + \eta_2 E(V)]} \tag{109}$$

$$\rho = \frac{[\theta_1 E(S_1) + \theta_2 E(S_2) + E(V)]\left(M\lambda\eta_1 + U_1^{*\prime}\right)}{M + M\lambda\eta_1[\theta_1 E(S_1) + \theta_2 E(S_2) + E(V)] - \lambda E(I)[\eta_1\{\theta_1 E(S_1) + \theta_2 E(S_2)\} + \eta_2 E(V)]} \tag{110}$$

$$L_q = \frac{\{M - \lambda E(I)[\eta_1\{\theta_1 E(S_1) + \theta_2 E(S_2)\} + \eta_2 E(V)]\} \times \left\{\begin{array}{c} \lambda E(I)\left[U_1^{*\prime} + \lambda\eta_1 MQ\right]\left[\begin{array}{c} \eta_1\{\theta_1 E(S_1^2) + \theta_2 E(S_2^2)\} + \eta_2 E(V^2) \\ +2E(V)\eta_1\{\theta_1 E(S_1) + \theta_2 E(S_2)\} \end{array}\right] \\ +[U^{**} + \lambda\eta_1 M(M-1)Q][\theta_1 E(S_1) + \theta_2 E(S_2) + E(V)] \end{array}\right\} - [\theta_1 E(S_1) + \theta_2 E(S_2) + E(V)]\left[U_1^{*\prime} + \lambda\eta_1 MQ\right] \times \left\{\begin{array}{c} M(M-1) - 2\lambda^2\eta_1\eta_2 (E(I))^2 E(V)\{\theta_1 E(S_1) + \theta_2 E(S_2)\} \\ -\lambda^2 (E(I))^2 [\eta_1^2\{\theta_1 E(S_1^2) + \theta_2 E(S_2^2)\} + \eta_2^2 E(V^2)] \\ -\lambda E(I(I-1))[\eta_1\{\theta_1 E(S_1) + \theta_2 E(S_2)\} + \eta_2 E(V)] \end{array}\right\}}{2[M - \lambda E(I)[\eta_1\{\theta_1 E(S_1) + \theta_2 E(S_2)\} + \eta_2 E(V)]]^2} \tag{111}$$

Case 3.No balking, no re-service

In this case, we let $\eta_1 = \eta_2 = 1$ in the equations (109), (110) and (111)

$$Q = 1 - \frac{[\theta_1 E(S_1) + \theta_2 E(S_2) + E(V)]\left(M\lambda + U_1^{*\prime}\right)}{M + \lambda\{M - E(I)\}[\theta_1 E(S_1) + \theta_2 E(S_2) + E(V)]} \tag{112}$$

$$\rho = \frac{[\theta_1 E(S_1) + \theta_2 E(S_2) + E(V)]\left(M\lambda + U_1^{*\prime}\right)}{M + \lambda\{M - E(I)\}[\theta_1 E(S_1) + \theta_2 E(S_2) + E(V)]} \tag{113}$$

$$L_q = \frac{\{M - \lambda E(I)[\theta_1 E(S_1) + \theta_2 E(S_2) + E(V)]\} \times \left\{\begin{array}{c} \lambda E(I)\left[U_1^{*\prime} + \lambda MQ\right]\left[\begin{array}{c} \{\theta_1 E(S_1^2) + \theta_2 E(S_2^2) + E(V^2)\} \\ +2E(V)\{\theta_1 E(S_1) + \theta_2 E(S_2)\} \end{array}\right] \\ +[U^{**} + \lambda M(M-1)Q][\theta_1 E(S_1) + \theta_2 E(S_2) + E(V)] \end{array}\right\} - [\theta_1 E(S_1) + \theta_2 E(S_2) + E(V)]\left[U_1^{*\prime} + \lambda MQ\right] \times \left\{\begin{array}{c} M(M-1) - 2\lambda^2 (E(I))^2 E(V)\{\theta_1 E(S_1) + \theta_2 E(S_2)\} \\ -\lambda^2 (E(I))^2 [\theta_1 E(S_1^2) + \theta_2 E(S_2^2) + E(V^2)] \\ -\lambda E(I(I-1))[\theta_1 E(S_1) + \theta_2 E(S_2) + E(V)] \end{array}\right\}}{2[M - \lambda E(I)\{\theta_1 E(S_1) + \theta_2 E(S_2) + E(V)\}]^2} \tag{114}$$

Case 4.Balking and re-service with single arrival and one by one general services

In this case, we let $M = 1;\ E(I) = 1$ and $E(I(I-1)) = 0$ in the equations (94), (95) and (102)

$$Q = 1 - \frac{[E_1^* + E(V)\{1-\delta\}]\lambda\eta_1}{1 + \lambda\eta_1[E_1^* + E(V)\{1-\delta\}] - \lambda[\eta_1 E_2^* + \eta_2 E(V)\{1-\delta\}]} \tag{115}$$

$$\rho = \frac{[E_1^* + E(V)\{1-\delta\}]\lambda\eta_1}{1 + \lambda\eta_1[E_1^* + E(V)\{1-\delta\}] - \lambda[\eta_1 E_2^* + \eta_2 E(V)\{1-\delta\}]} \tag{116}$$

$$L_q = \frac{\begin{array}{c}\{1-\lambda[\eta_1 E_2^* + \eta_2 E(V)\{1-\delta\}]\} \times \left\{\lambda^2 \eta_1 Q \left[\begin{array}{c}\eta_1 E^* + 2\eta_1 E^{***} \\ + \{1-\delta\}\eta_2 E(V^2) + 2E(V)\eta_1 E_2^*\end{array}\right]\right\} \\ + [E_1^* + \{1-\delta\}E(V)]\lambda\eta_1 Q \times \{2\lambda^2 \eta_1 \eta_2 E(V) E_2^* + \lambda^2[\eta_1^2 E^{**} + \eta_2^2 E(V^2)\{1-\delta\}]\}\end{array}}{2\left[1 - \lambda[\eta_1 E_2^* + \eta_2 E(V)\{1-\delta\}]\right]^2} \tag{117}$$

Case5.Single arrival and one by one general services, no Balking, no re-service and no vacation

In this case, we let $\eta_1 = \eta_2 = 1; \varepsilon_1 = \varepsilon_2 = 0; E(V) \longrightarrow 0$and $E(V^2) \longrightarrow 0$ in the equations (115), (116) and (117)

$$Q = 1 - \lambda[\theta_1 E(S_1) + \theta_2 E(S_2)] \tag{118}$$

$$\rho = \lambda[\theta_1 E(S_1) + \theta_2 E(S_2)] \tag{119}$$

$$L_q = \frac{\{\theta_1 E(S_1^2) + \theta_2 E(S_2^2)\}\lambda^2 Q}{2[1 - \lambda\{\theta_1 E(S_1) + \theta_2 E(S_2)\}]^2} \tag{120}$$

Case 6. Only one kind of general service and no vacation

In this case, we let $\theta_1 = 1$ and$\theta_2 = 0$ in the equations (118), (119) and (120)

$$Q = 1 - \lambda E(S_1) \tag{121}$$

$$\rho = \lambda E(S_1) \tag{122}$$

$$L_q = \frac{\lambda^2 E(S_1^2)}{2\{1 - \lambda E(S_1)\}} \tag{123}$$

Note that (121) and (122) are known results from Medhi [13]. Again (123) is the well-known Pollaczek-Khinchine formula.

9. Numerical Example

In order to execute numerical illustration, following arbitrary values are chosen:

$\lambda = 10, E(S_1) = 0.3, E(S_2) = 0.4, E(V) = 0.25, \theta_1 = 0.4, \theta_2 = 0.6, \eta_1 = 0.7, \eta_2 = 0.3,$ $E(I) = 0.2, E\big(I(I-1)\big) = 0.01$but the values are varied as below: $\varepsilon_1 = 0.3, 0.2, 0.1$, $\varepsilon_2 = 0.7, 0.8, 0.9$ and $M = 10, 7, 5$. All the values are chosen in such a way that the steady state condition is satisfied.

Table 1. Computed values of the various characteristics of the system

ε_1	ε_2	M	ρ	Q	L_q
0.3	0.7	10	0.8533	0.1467	0.1842
		7	0.8545	0.1455	0.1960
		5	0.8556	0.1444	0.2077
0.2	0.8	10	0.8537	0.1463	0.1841
		7	0.8547	0.1453	0.1955
		5	0.8558	0.1442	0.2067
0.1	0.9	10	0.8542	0.1458	0.1840
		7	0.8551	0.1449	0.1949
		5	0.8559	0.1441	0.2055

The table 1 shows the computed values of various states of the server; the utilization factor, the proportion of the idle time, average number of customers in the queue. It also shows that as long as, for the different values of $\varepsilon_1$and ε_2, the server idle time is decreased, the utilization factor and the average number of customers in the queue of our model are increased.

10. Conclusion

This paper assists us in deriving the numerical results of the queueing system which constitutes the time dependent solution and steady state results. The performance measure and effectiveness are also measured.

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