

Euler type double integrals involving Kampé de Fériet function, general class of polynomials and multivariable I-function defined by Nambisan

F.Y. AYANT¹

¹ Teacher in High School , France
 E-mail : fredericayant@gmail.com

ABSTRACT

The aim of the present document is to evaluate three double Euler type integrals involving Kampé de Fériet function, general class of polynomials and multivariable Aleph-function. Importance of our findings lies in the fact that they involve the multivariable I-function defined by Nambisan et al [3], which are the sufficiently general in nature and are capable of yielding a large number of simpler and useful results merely by specializing the parameters in them.

KEYWORDS : I-function of several variables, double Euler type integrals, Kampé de Fériet function, general class of polynomials, multivariable H-function, Srivastava-Daoust function.

2010 Mathematics Subject Classification. 33C45, 33C60, 26D20

1. Introduction and preliminaries.

The object of this document is to evaluate three double Eulerian integrals involving Kampé de Fériet function, general class of polynomial and the I-function of several variables defined by Nambisan et al [3]. The multivariable I-function defined by Nambisan et al [3] is an extension of the multivariable H-function defined by Srivastava et al [6]. The multiple Mellin-Barnes integral occurring in this paper will be referred to as the multivariable I-function throughout our present study and will be defined and represented as follows.

The multivariable I-function is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1, \dots, z_r) = I_{p,q;p_1,q_1;\dots;p_r,q_r}^{0,n;m_1,n_1;\dots;m_r,n_r} \left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{matrix} \middle| \begin{matrix} (a_j; \alpha_j^{(1)}, \dots, \alpha_j^{(r)}; A_j)_{1,p} : \\ \\ \\ \\ (b_j; \beta_j^{(1)}, \dots, \beta_j^{(r)}; B_j)_{1,q} : \end{matrix} \right. \tag{1.1}$$

$$\left. \begin{matrix} (c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,p_1}; \dots; (c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{1,p_r} \\ \\ (d_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{1,q_1}; \dots; (d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})_{1,q_r} \end{matrix} \right) \tag{1.1}$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \psi(s_1, \dots, s_r) \prod_{i=1}^r \theta_i(s_i) z_i^{s_i} ds_1 \dots ds_r \tag{1.2}$$

where $\phi(s_1, \dots, s_r), \theta_i(s_i), i = 1, \dots, r$ are given by :

$$\phi(s_1, \dots, s_r) = \frac{\prod_{j=1}^n \Gamma^{A_j} (1 - a_j + \sum_{i=1}^r \alpha_j^{(i)} s_j)}{\prod_{j=n+1}^p \Gamma^{A_j} (a_j - \sum_{i=1}^r \alpha_j^{(i)} s_j) \prod_{j=1}^q \Gamma^{B_j} (1 - b_j + \sum_{i=1}^r \beta_j^{(i)} s_j)} \tag{1.3}$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{n_i} \Gamma^{C_j^{(i)}} (1 - c_j^{(i)} + \gamma_j^{(i)} s_i) \prod_{j=1}^{m_i} \Gamma^{D_j^{(i)}} (d_j^{(i)} - \delta_j^{(i)} s_i)}{\prod_{j=n_i+1}^{p_i} \Gamma^{C_j^{(i)}} (c_j^{(i)} - \gamma_j^{(i)} s_i) \prod_{j=m_i+1}^{q_i} \Gamma^{D_j^{(i)}} (1 - d_j^{(i)} - \delta_j^{(i)} s_i)} \quad (1.4)$$

For more details, see Nambisan et al [3].

Following the result of Braaksma [1] the I-function of r variables is analytic if :

$$U_i = \sum_{j=1}^p A_j \alpha_j^{(i)} - \sum_{j=1}^q B_j \beta_j^{(i)} + \sum_{j=1}^{p_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=1}^{q_i} D_j^{(i)} \delta_j^{(i)} \leq 0, i = 1, \dots, r \quad (1.5)$$

The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable I-function.

The integral (2.1) converges absolutely if

$$|\arg(z_k)| < \frac{1}{2} \Delta_k \pi, k = 1, \dots, r \text{ where}$$

$$\Delta_k = - \sum_{j=n_k+1}^p A_j \alpha_j^{(k)} - \sum_{j=1}^q B_j \beta_j^{(k)} + \sum_{j=1}^{m_k} D_j^{(k)} \delta_j^{(k)} - \sum_{j=m_k+1}^{q_k} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n_k} C_j^{(k)} \gamma_j^{(k)} - \sum_{j=n_k+1}^{p_k} C_j^{(k)} \gamma_j^{(k)} > 0 \quad (1.6)$$

We will use these notations for this paper :

$$X = m_1, n_1; \dots; m_r, n_r; V = p_1, q_1; \dots; p_r, q_r \quad (1.7)$$

$$A = (a_j; \alpha_j^{(1)}, \dots, \alpha_j^{(r)}; A_j)_{1,p} \quad (1.8)$$

$$B = (b_j; \beta_j^{(1)}, \dots, \beta_j^{(r)}; B_j)_{1,q} \quad (1.9)$$

$$C = (c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,p_1}; \dots; (c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{1,p_r} \quad (1.10)$$

$$D = (d_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{1,q_1}; \dots; (d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})_{1,q_r} \quad (1.11)$$

the contracted form is

$$I_{p,q;V}^{0,n;X} \left(\begin{array}{c|c} z_1 & \mathbf{A} : \mathbf{C} \\ \cdot & \cdot \cdot \cdot \\ \cdot & \cdot \cdot \cdot \\ z_r & \mathbf{B} : \mathbf{D} \end{array} \right) \quad (1.12)$$

Srivastava and Garg [5] introduced and defined a general class of multivariable polynomials as follows

$$S_L^{h_1, \dots, h_s} [z_1, \dots, z_s] = \sum_{R_1, \dots, R_s=0}^{h_1 R_1 + \dots + h_s R_s \leq L} (-L)_{h_1 R_1 + \dots + h_s R_s} B(E; R_1, \dots, R_s) \frac{z_1^{R_1} \dots z_s^{R_s}}{R_1! \dots R_s!} \quad (1.13)$$

The coefficients $B(E; R_1, \dots, R_s)$ are arbitrary constants, real or complex.

We will note

$$B_s = \frac{(-L)_{h_1 R_1 + \dots + h_s R_s} B(E; R_1, \dots, R_s)}{R_1! \dots R_s!} \quad (1.14)$$

2 . Results required :

Lemme 1

$$\begin{aligned} & \int_0^1 x^{a-1} (1-x)^{b-1} {}_1F_1(c; f; sx) F_{G:H}^{D:E} \left[\begin{matrix} (d) : (e') ; \dots ; (e^{(n)}) \\ (g) : (h') ; \dots ; (h^{(n)}) \end{matrix} ; r_1 x, \dots, r_n x \right] dx \\ &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \sum_{m, m_1, \dots, m_n=0}^{\infty} \frac{(a, m + m_1 + \dots + m_n) ((d), m_1 + \dots + m_n) (c, m)}{(a+b, m + m_1 + \dots + m_n) ((g), m_1 + \dots + m_n) (f, m)} \\ & \frac{((e'), m_1) \dots ((e^{(n)}), m_n) s^m r_1^{m_1} \dots r_n^{m_n}}{((h'), m_1) \dots ((h^{(n)}), m_n) m! m_1! \dots m_n!} \end{aligned} \quad (2.1)$$

Lemme 2

$$\begin{aligned} & \int_0^1 x^{a-1} (1-x)^{b-1} {}_1F_1(c; f; sx) F_{G:H}^{D:E} \left[\begin{matrix} (d) : (e') ; \dots ; (e^{(n)}) \\ (g) : (h') ; \dots ; (h^{(n)}) \end{matrix} ; r_1 x, r_2, \dots, r_n \right] dx \\ &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \sum_{m, m_1, \dots, m_n=0}^{\infty} \frac{((d), m_1 + \dots + m_n) (a, m) (c, m)}{((g), m_1 + \dots + m_n) (a+b, m + m_1) (f, m)} \\ & \frac{((e'), m_1) \dots ((e^{(n)}), m_n) s^m r_1^{m_1} \dots r_n^{m_n}}{((h'), m_1) \dots ((h^{(n)}), m_n) m! m_1! \dots m_n!} \end{aligned} \quad (2.2)$$

Lemme 3

$$\begin{aligned} & \int_0^1 x^{a-1} (1-x)^{b-1} {}_1F_1(c; f; sx) F_{G:H}^{D:E} \left[\begin{matrix} (d) : (e') ; \dots ; (e^{(n)}) \\ (g) : (h') ; \dots ; (h^{(n)}) \end{matrix} ; \right. \\ & \left. r_1 x, \dots, r_k x, r_{k+1} (1-x), \dots, r_n (1-x) \right] dx \\ &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \sum_{m, m_1, \dots, m_n=0}^{\infty} \frac{((d), m_1 + \dots + m_n) (a, m + m_1 + \dots + m_k) (c, m) (b; m_{k+1} + \dots + m_n)}{(a+b, m + m_1 + \dots + m_n) ((g), m_1 + \dots + m_n) (f, m)} \\ & \frac{((e'), m_1) \dots ((e^{(n)}), m_n) s^m r_1^{m_1} \dots r_n^{m_n}}{((h'), m_1) \dots ((h^{(n)}), m_n) m! m_1! \dots m_n!} \end{aligned} \quad (2.3)$$

For more details , see Exton [2]

Lemme 4

$$\int_0^1 x^{c-1}(1-x)^{-1/2} {}_2F_1(a, b; a+b+1/2; x) dx = \frac{\pi\Gamma(c)\Gamma(a+b+1/2)\Gamma(c-a-b+1/2)}{\Gamma(a+1/2)\Gamma(b+1/2)\Gamma(c-a+1/2)\Gamma(c-b+1/2)} \quad (2.4)$$

Where $\text{Re}(c) > 0, \text{Re}(2c-a-b) > -1$, see Vyas and Rathie [7]

3. Main results

Theorem 1

$$\int_0^1 \int_0^1 x^{c-1}(1-x)^{-1/2} {}_2F_1(a, b; a+b+1/2; x) y^{\alpha-1}(1-y)^{\beta-1} {}_1F_1(\delta; f; s'y)$$

$$F_{G:H}^{D:E} \left[\begin{matrix} (d) : (e') ; \dots ; (e^{(n)}) ; \\ (g) : (h') ; \dots ; (h^{(n)}) ; \end{matrix} ; r_1 y, \dots, r_n y \right] S_L^{h_1, \dots, h_s} \begin{pmatrix} a_1 x^{t_1} y^{n_1} (1-y)^{\phi_1} \\ \dots \\ a_s x^{t_s} y^{n_s} (1-y)^{\phi_s} \end{pmatrix}$$

$$I \left(\begin{matrix} z_1 x^{\sigma_1} y^{\rho_1} (1-y)^{\lambda_1} \\ \dots \\ z_r x^{\sigma_r} y^{\rho_r} (1-y)^{\lambda_r} \end{matrix} \middle| \begin{matrix} A : C \\ B : D \end{matrix} \right) dx dy = \frac{\pi\Gamma(a+b+1/2)}{\Gamma(a+1/2)\Gamma(b+1/2)} \sum_{m_1, \dots, m_n=0}^{\infty}$$

$$\frac{((d), m_1 + \dots + m_n)(\delta, m)((e'), m_1) \dots ((e^{(n)}), m_n) s'^m r_1^{m_1} \dots r_n^{m_n}}{((g), m_1 + \dots + m_n)(f, m)((h'), m_1) \dots ((h^{(n)}), m_n) m! m_1! \dots m_n!} \sum_{R_1, \dots, R_s=0}^{h_1 R_1 + \dots + h_s R_s \leq L} B_s a^{R_1} \dots a_s^{R_s}$$

$$I_{p+4, q+3; W}^{0, n+4; X} \left(\begin{matrix} z_1 \\ \dots \\ z_r \end{matrix} \middle| \begin{matrix} (1-c - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r; 1), & (1/2-c + a+b - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r; 1), \\ \dots \\ (1/2-c+a - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r; 1), & (1/2-c+b - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r; 1), \end{matrix} \right)$$

$$(1 - \beta - \sum_{i=1}^s R_i \phi_i; \lambda_1, \dots, \lambda_r; 1),$$

$$(1 - \alpha - \beta - \sum_{i=1}^s R_i (n_i + \phi_i) - m - \sum_{i=1}^n m_i; \rho_1 + \lambda_1, \dots, \rho_r + \lambda_r; 1),$$

$$(1 - \alpha - m - \sum_{i=1}^n m_i - \sum_{i=1}^s R_i n_i; \rho_1, \dots, \rho_r; 1), A : C \quad (3.1)$$

$$\begin{matrix} \dots \\ B : D \end{matrix}$$

Provided that :

1) $\min\{t_i, n_i, \phi_i, \sigma_j, \rho_j, \lambda_j\} > 0; i = 1, \dots, s; j = 1, \dots, r$

2) $\text{Re}(c + \sum_{i=1}^r \sigma_i s_i + \sum_{i=1}^s t_i R_i) > 0; \text{Re}(c - a - b + \sum_{i=1}^r \sigma_i s_i + \sum_{i=1}^s t_i R_i) > -1/2$

$$3) \operatorname{Re}(\beta + \sum_{i=1}^s \phi_i R_i + \sum_{i=1}^r \lambda_i s_i) > 0; \operatorname{Re}(c - a + \sum_{i=1}^r \sigma_i s_i + \sum_{i=1}^s t_i R_i) > -1/2$$

$$4) \operatorname{Re}(c - b + \sum_{i=1}^r \sigma_i s_i + \sum_{i=1}^s t_i R_i) > -1/2; \operatorname{Re}(\alpha + m + \sum_{i=1}^n m_i + \sum_{i=1}^r \rho_i s_i + \sum_{i=1}^s R_i n_i) > 0$$

$$5) \operatorname{Re}(\alpha + \beta + m + \sum_{i=1}^n m_i + \sum_{i=1}^r (\rho_i + \lambda_i) s_i + \sum_{i=1}^s R_i (n_i + \phi_i)) > 0$$

$$6) |\arg(z_k)| < \frac{1}{2} \Delta_k \pi \text{ where } \Delta_k \text{ is given in (1.6)}$$

Theorem 2

$$\int_0^1 \int_0^1 x^{c-1} (1-x)^{-1/2} {}_2F_1(a, b; a+b+1/2; x) y^{\alpha-1} (1-y)^{\beta-1} {}_1F_1(\delta; f; s'y)$$

$$F_{G:H}^{D:E} \left[\begin{matrix} (d) : (e') ; \dots ; (e^{(n)}) ; \\ (g) : (h') ; \dots ; (h^{(n)}) ; \end{matrix} ; r_1 y, r_2 \dots, r_n \right] S_L^{h_1, \dots, h_s} \begin{pmatrix} a_1 x^{t_1} y^{n_1} (1-y)^{\phi_1} \\ \dots \\ a_s x^{t_s} y^{n_s} (1-y)^{\phi_s} \end{pmatrix}$$

$$I \left(\begin{matrix} z_1 x^{\sigma_1} y^{\rho_1} (1-y)^{\lambda_1} \\ \dots \\ z_r x^{\sigma_r} y^{\rho_r} (1-y)^{\lambda_r} \end{matrix} \middle| \begin{matrix} A : C \\ B : D \end{matrix} \right) dx dy = \frac{\pi \Gamma(a+b+1/2)}{\Gamma(a+1/2) \Gamma(b+1/2)} \sum_{m, m_1, \dots, m_n=0}^{\infty}$$

$$\frac{((d), m_1 + \dots + m_n)(\delta, m)((e'), m_1) \dots ((e^{(n)}), m_n) s'^m r_1^{m_1} \dots r_n^{m_n}}{((g), m_1 + \dots + m_n)(f, m)((h'), m_1) \dots ((h^{(n)}), m_n) m! m_1! \dots m_n!} \sum_{R_1, \dots, R_s=0}^{h_1 R_1 + \dots + h_s R_s \leq L} B_s a^{R_1} \dots a_s^{R_s}$$

$$I_{p+4, q+3; W}^{0, n+4; X} \left(\begin{matrix} z_1 \\ \dots \\ z_r \end{matrix} \middle| \begin{matrix} (1-c - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r; 1), & (1/2-c + a+b - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r; 1), \\ \dots & \dots \\ (1/2-c + a - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r; 1), & (1/2-c + b - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r; 1), \end{matrix} \right)$$

$$(1 - \beta - m_1 - \sum_{i=1}^s R_i \phi_i; \lambda_1, \dots, \lambda_r; 1),$$

$$(1 - \alpha - \beta - \sum_{i=1}^s R_i (n_i + \phi_i) - m - \sum_{i=1}^n m_i; \rho_1 + \lambda_1, \dots, \rho_r + \lambda_r; 1),$$

$$(1 - \alpha - m - \sum_{i=1}^s R_i n_i; \rho_1, \dots, \rho_r; 1), A : C \Bigg) \tag{3.2}$$

$$\begin{matrix} \dots \\ B : D \end{matrix}$$

Provided that :

$$1) \min\{t_i, n_i, \phi_i, \sigma_j, \rho_j, \lambda_j\} > 0; i = 1, \dots, s; j = 1, \dots, r$$

$$2) \operatorname{Re}(c + \sum_{i=1}^r \sigma_i s_i + \sum_{i=1}^s t_i R_i) > 0; \operatorname{Re}(c - a - b + \sum_{i=1}^r \sigma_i s_i + \sum_{i=1}^s t_i R_i) > -1/2$$

$$3) \operatorname{Re}(\beta + m_1 + \sum_{i=1}^s \phi_i R_i + \sum_{i=1}^r \lambda_i s_i) > 0; \operatorname{Re}(c - a + \sum_{i=1}^r \sigma_i s_i + \sum_{i=1}^s t_i R_i) > -1/2$$

$$4) \operatorname{Re}(c - b + \sum_{i=1}^r \sigma_i s_i + \sum_{i=1}^s t_i R_i) > -1/2; \operatorname{Re}(\alpha + m + \sum_{i=1}^r \rho_i s_i + \sum_{i=1}^s R_i n_i) > 0$$

$$5) \operatorname{Re}(\alpha + \beta + m + m_1 + \sum_{i=1}^r (\rho_i + \lambda_i) s_i + \sum_{i=1}^s R_i (n_i + \phi_i)) > 0$$

$$6) |\arg(z_k)| < \frac{1}{2} \Delta_k \pi \text{ where } \Delta_k \text{ is given in (1.6)}$$

Theorem 3

$$\int_0^1 \int_0^1 x^{c-1} (1-x)^{-1/2} {}_2F_1(a, b; a+b+1/2; x) y^{\alpha-1} (1-y)^{\beta-1} {}_1F_1(\delta; f; s'y)$$

$$F_{G:H}^{D:E} \left[\begin{matrix} (d) : (e'); \dots; (e^{(n)}); \\ (g) : (h'); \dots; (h^{(n)}); \end{matrix} ; r_1 x, \dots, r_k x, r_{k+1}(1-x), \dots, r_n(1-x) \right]$$

$$S_L^{h_1, \dots, h_s} \left(\begin{matrix} a_1 x^{t_1} y^{n_1} (1-y)^{\phi_1} \\ \dots \\ a_s x^{t_s} y^{n_s} (1-y)^{\phi_s} \end{matrix} \middle| \begin{matrix} z_1 x^{\sigma_1} y^{\rho_1} (1-y)^{\lambda_1} \\ \dots \\ z_r x^{\sigma_r} y^{\rho_r} (1-y)^{\lambda_r} \end{matrix} \middle| \begin{matrix} A : C \\ B : D \end{matrix} \right) dx dy$$

$$= \frac{\pi \Gamma(a+b+1/2)}{\Gamma(a+1/2) \Gamma(b+1/2)} \sum_{m, m_1, \dots, m_n=0}^{\infty} \frac{((d), m_1 + \dots + m_n)(\delta, m)}{((g), m_1 + \dots + m_n)(f, m)}$$

$$\sum_{R_1, \dots, R_s=0}^{h_1 R_1 + \dots + h_s R_s \leq L} B_s a^{R_1} \dots a_s^{R_s} \frac{((e'), m_1) \dots ((e^{(n)}), m_n) s'^m r_1^{m_1} \dots r_n^{m_n}}{((h'), m_1) \dots ((h^{(n)}), m_n) m! m_1! \dots m_n!}$$

$$I_{p+4, q+3; W}^{0, n+4; X} \left(\begin{matrix} z_1 \\ \dots \\ z_r \end{matrix} \middle| \begin{matrix} (1-c - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r; 1), & (1/2-c + a+b - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r; 1), \\ \dots & \dots \\ (1/2-c + a - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r; 1), & (1/2-c + b - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r; 1), \end{matrix} \right)$$

$$(1 - \beta - \sum_{i=1}^{k+1} m_i - \sum_{i=1}^s R_i \phi_i; \lambda_1, \dots, \lambda_r; 1),$$

$$(1 - \alpha - \beta - \sum_{i=1}^s R_i (n_i + \phi_i) - m - \sum_{i=1}^n m_i; \rho_1 + \lambda_1, \dots, \rho_r + \lambda_r; 1),$$

$$\left(\begin{array}{l} (1 - \alpha - m - \sum_{i=1}^k m_i - \sum_{i=1}^s R_i n_i; \rho_1, \dots, \rho_r; 1), A : C \\ \vdots \\ B : D \end{array} \right) \quad (3.3)$$

Provided that :

- 1) $\min\{t_i, n_i, \phi_i, \sigma_j, \rho_j, \lambda_j\} > 0; i = 1, \dots, s; j = 1, \dots, r$
- 2) $Re(c + \sum_{i=1}^r \sigma_i s_i + \sum_{i=1}^s t_i R_i) > 0; Re(c - a - b + \sum_{i=1}^r \sigma_i s_i + \sum_{i=1}^s t_i R_i) > -1/2$
- 3) $Re(\alpha + m + \sum_{i=1}^k m_i + \sum_{i=1}^s \phi_i R_i + \sum_{i=1}^r \lambda_i s_i) > 0; Re(c - a + \sum_{i=1}^r \sigma_i s_i + \sum_{i=1}^s t_i R_i) > -1/2$
- 4) $Re(c - b + \sum_{i=1}^r \sigma_i s_i + \sum_{i=1}^s t_i R_i) > -1/2; Re(\beta + \sum_{i=k+1}^r m_i + \sum_{i=1}^r \lambda_i s_i + \sum_{i=1}^s R_i n_i) > 0$
- 5) $Re(\alpha + \beta + m + \sum_{i=1}^n m_i + \sum_{i=1}^r (\rho_i + \lambda_i) s_i + \sum_{i=1}^s R_i (n_i + \phi_i)) > 0$
- 6) $|arg(z_k)| < \frac{1}{2} \Delta_k \pi$ where Δ_k is given in (1.6)

Proof of (3.1): The equation (3.1) writes

$$\int_0^1 x^c (1-x)^{-1/2} {}_2F_1(a, b; a+b+1/2; x) \left[\int_0^1 y^{\alpha-1} (1-y)^{\beta-1} {}_1F_1(\delta; f; s'y) \right. \\ \left. F_{G:H}^{D:E} \left[\begin{array}{l} (d) : (e'); \dots; (e^{(n)}); \\ (g) : (h'); \dots; (h^{(n)}); \end{array} ; r_1 y, \dots, r_n y \right] \sum_{R_1, \dots, R_s=0}^{h_1 R_1 + \dots + h_s R_s \leq L} B_s [a_1 x^{t_1} y^{n_1} (1-y)^{\phi_1}]^{R_1} \right. \\ \left. \dots [a_s x^{t_s} y^{n_s} (1-y)^{\phi_s}]^{R_s} I \left(\begin{array}{l} z_1 x^{\sigma_1} y^{\rho_1} (1-y)^{\lambda_1} \\ \vdots \\ z_r x^{\sigma_r} y^{\rho_r} (1-y)^{\lambda_r} \end{array} \middle| \begin{array}{l} A : C \\ B : D \end{array} \right) dy \right] dx \quad (3.4)$$

We first express the multivariable Aleph-function involving in L.H.S. of (3.4) in terms of Mellin-Barnes contour integral with the help of (1.1) and then interchange the order of integration. We find that L.H.S. of (3.4)

$$\frac{1}{(2\pi\omega)^r} \left(\int_{L_1} \dots \int_{L_r} \psi(s_1, \dots, s_r) \prod_{k=1}^r \theta_k(s_k) z_k^{s_k} \sum_{R_1, \dots, R_s=0}^{h_1 R_1 + \dots + h_s R_s \leq L} B_s a^{R_1} \dots a_s^{R_s} \right. \\ \left. \left(\int_0^1 x^{c+\sigma_1 s_1 + \dots + \sigma_r s_r} (1-x)^{-1/2} {}_2F_1(a, b; a+b+1/2; x) dx \right) \right)$$

$$\left[\int_0^1 y^{a+\rho_1 s_1+\dots+\rho_r s_r+n_1 R_1+\dots+n_s R_s-1} (1-y)^{\lambda_1 s_1+\dots+\lambda_r s_r+\phi_1 R_1+\dots+\phi_s R_s-1} {}_1F_1(\delta; f; s'y) \right. \\ \left. F_{G:H}^{D:E} \left[\begin{matrix} (d) : (e') ; \dots ; (e^{(n)}) ; \\ (g) : (h') ; \dots ; (h^{(n)}) ; \end{matrix} ; r_1 y, \dots, r_n y \right] dy \right] ds_1 \dots ds_r \quad (3.6)$$

Now using the result (2.1) to evaluate the y-integral and (2.4) to evaluate the x-integral and reinterpreting the multiple contour integral so obtained with the help of (3.1), we arrive at the R.H.S. of (3.1). The results (3.2) and (3.3) can be proved similarly with the help of results given by (2.2), (2.3) and (2.4).

4. Multivariable I-function

If $A'_j = B'_j = C'_j{}^{(i)} = D'_j{}^{(i)} = 1$, the multivariable I-function defined by Nambisan et al [3] reduces to multivariable H-function defined by Srivastava et al [6] and we obtain the following multiple integral transform

Corollary 1

$$\int_0^1 \int_0^1 x^{c-1} (1-x)^{-1/2} {}_2F_1(a, b; a+b+1/2; x) y^{\alpha-1} (1-y)^{\beta-1} {}_1F_1(\delta; f; s'y)$$

$$F_{G:H}^{D:E} \left[\begin{matrix} (d) : (e') ; \dots ; (e^{(n)}) ; \\ (g) : (h') ; \dots ; (h^{(n)}) ; \end{matrix} ; r_1 y, \dots, r_n y \right] S_L^{h_1, \dots, h_s} \begin{pmatrix} a_1 x^{t_1} y^{n_1} (1-y)^{\phi_1} \\ \dots \\ a_s x^{t_s} y^{n_s} (1-y)^{\phi_s} \end{pmatrix}$$

$$H \left(\begin{matrix} z_1 x^{\sigma_1} y^{\rho_1} (1-y)^{\lambda_1} \\ \dots \\ z_r x^{\sigma_r} y^{\rho_r} (1-y)^{\lambda_r} \end{matrix} \middle| \begin{matrix} A : C \\ B : D \end{matrix} \right) dx dy = \frac{\pi \Gamma(a+b+1/2)}{\Gamma(a+1/2) \Gamma(b+1/2)} \sum_{m, m_1, \dots, m_n=0}^{\infty}$$

$$\frac{((d), m_1 + \dots + m_n)(\delta, m)((e'), m_1) \dots ((e^{(n)}), m_n) s'^m r_1^{m_1} \dots r_n^{m_n}}{((g), m_1 + \dots + m_n)(f, m)((h'), m_1) \dots ((h^{(n)}), m_n) m! m_1! \dots m_n!} \sum_{R_1, \dots, R_s=0}^{h_1 R_1 + \dots + h_s R_s \leq L} B_s a^{R_1} \dots a_s^{R_s}$$

$$H_{p+4, q+3; W}^{0, n+4; X} \left(\begin{matrix} z_1 \\ \dots \\ z_r \end{matrix} \middle| \begin{matrix} (1-c - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r), & (1/2-c + a+b - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r), \\ \dots \\ (1/2-c+a - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r), & (1/2-c+b - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r), \end{matrix} \right)$$

$$(1 - \beta - \sum_{i=1}^s R_i \phi_i; \lambda_1, \dots, \lambda_r),$$

$$(1 - \alpha - \beta - \sum_{i=1}^s R_i (n_i + \phi_i) - m - \sum_{i=1}^n m_i; \rho_1 + \lambda_1, \dots, \rho_r + \lambda_r),$$

$$(1 - \alpha - m - \sum_{i=1}^n m_i - \sum_{i=1}^s R_i n_i; \rho_1, \dots, \rho_r), A : C \\ \dots \\ B : D \quad (4.1)$$

under the same condition and notations that (3.1) with $A'_j = B'_j = C'_j{}^{(i)} = D'_j{}^{(i)} = 1$

Corollary 2

$$\int_0^1 \int_0^1 x^{c-1} (1-x)^{-1/2} {}_2F_1(a, b; a+b+1/2; x) y^{\alpha-1} (1-y)^{\beta-1} {}_1F_1(\delta; f; s'y)$$

$$F_{G:H}^{D:E} \left[\begin{matrix} (d) : (e') ; \dots ; (e^{(n)}) \\ (g) : (h') ; \dots ; (h^{(n)}) \end{matrix} ; r_1 y, r_2 \dots, r_n \right] S_L^{h_1, \dots, h_s} \begin{pmatrix} a_1 x^{t_1} y^{n_1} (1-y)^{\phi_1} \\ \vdots \\ a_s x^{t_s} y^{n_s} (1-y)^{\phi_s} \end{pmatrix}$$

$$H \left(\begin{matrix} z_1 x^{\sigma_1} y^{\rho_1} (1-y)^{\lambda_1} \\ \vdots \\ z_r x^{\sigma_r} y^{\rho_r} (1-y)^{\lambda_r} \end{matrix} \middle| \begin{matrix} A : C \\ B : D \end{matrix} \right) dx dy = \frac{\pi \Gamma(a+b+1/2)}{\Gamma(a+1/2) \Gamma(b+1/2)} \sum_{m, m_1, \dots, m_n=0}^{\infty}$$

$$\frac{((d), m_1 + \dots + m_n)(\delta, m) ((e'), m_1) \dots ((e^{(n)}), m_n) s'^m r_1^{m_1} \dots r_n^{m_n}}{((g), m_1 + \dots + m_n)(f, m) ((h'), m_1) \dots ((h^{(n)}), m_n) m! m_1! \dots m_n!} \sum_{R_1, \dots, R_s=0}^{h_1 R_1 + \dots + h_s R_s \leq L} B_s a^{R_1} \dots a_s^{R_s}$$

$$H_{p+4, q+3; W}^{0, n+4; X} \left(\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (1-c - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r), & (1/2-c + a+b - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r), \\ \vdots \\ (1/2-c+a - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r), & (1/2-c+b - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r), \end{matrix} \right.$$

$$(1 - \beta - m_1 - \sum_{i=1}^s R_i \phi_i; \lambda_1, \dots, \lambda_r),$$

$$(1 - \alpha - \beta - \sum_{i=1}^s R_i (n_i + \phi_i) - m - \sum_{i=1}^n m_i; \rho_1 + \lambda_1, \dots, \rho_r + \lambda_r),$$

$$(1 - \alpha - m - \sum_{i=1}^s R_i n_i; \rho_1, \dots, \rho_r), A : C$$

$$\left. \begin{matrix} \vdots \\ B : D \end{matrix} \right) \tag{4.2}$$

under the same condition and notations that (3.2) with $A'_j = B'_j = C'_j{}^{(i)} = D'_j{}^{(i)} = 1$

Corollary 3

$$\int_0^1 \int_0^1 x^{c-1} (1-x)^{-1/2} {}_2F_1(a, b; a+b+1/2; x) y^{\alpha-1} (1-y)^{\beta-1} {}_1F_1(\delta; f; s'y)$$

$$F_{G:H}^{D:E} \left[\begin{matrix} (d) : (e') ; \dots ; (e^{(n)}) \\ (g) : (h') ; \dots ; (h^{(n)}) \end{matrix} ; r_1 x, \dots, r_k x, r_{k+1}(1-x), \dots, r_n(1-x) \right]$$

and we have the following formulas

Corollary 4

$$\int_0^1 \int_0^1 x^{c-1} (1-x)^{-1/2} {}_2F_1(a, b; a+b+1/2; x) y^{\alpha-1} (1-y)^{\beta-1} {}_1F_1(\delta; f; s'y)$$

$$F_{G:H}^{D:E} \left[\begin{matrix} (d) : (e') ; \dots ; (e^{(n)}) \\ (g) : (h') ; \dots ; (h^{(n)}) \end{matrix} ; r_1 y, \dots, r_n y \right] F_{\bar{C}:D'; \dots ; D^{(s)}}^{1+\bar{A}:B'; \dots ; B^{(s)}} \left(\begin{matrix} a_1 x^{t_1} y^{n_1} (1-y)^{\phi_1} \\ \dots \\ a_s x^{t_s} y^{n_s} (1-y)^{\phi_s} \end{matrix} \right)$$

$$\left[\begin{matrix} (-L); R_1, \dots, R_s \\ [(a); \theta', \dots, \theta^{(s)}] : [(b'); \phi']; \dots ; [(b^{(s)}); \phi^{(s)}] \\ [(c); \psi', \dots, \psi^{(s)}] : [(d'); \delta']; \dots ; [(d^{(s)}); \delta^{(s)}] \end{matrix} \right)$$

$$I \left(\begin{matrix} z_1 x^{\sigma_1} y^{\rho_1} (1-y)^{\lambda_1} \\ \dots \\ z_r x^{\sigma_r} y^{\rho_r} (1-y)^{\lambda_r} \end{matrix} \middle| \begin{matrix} A : C \\ B : D \end{matrix} \right) dx dy = \frac{\pi \Gamma(a+b+1/2)}{\Gamma(a+1/2) \Gamma(b+1/2)} \sum_{m, m_1, \dots, m_n=0}^{\infty}$$

$$\frac{((d), m_1 + \dots + m_n)(\delta, m) ((e'), m_1) \dots ((e^{(n)}), m_n) s'^m r_1^{m_1} \dots r_n^{m_n}}{((g), m_1 + \dots + m_n)(f, m) ((h'), m_1) \dots ((h^{(n)}), m_n) m! m_1! \dots m_n!} \sum_{R_1, \dots, R_s=0}^{h_1 R_1 + \dots + h_s R_s \leq L} B'_s a^{R_1} \dots a_s^{R_s}$$

$$I_{p+4, q+3; W}^{0, n+4; X} \left(\begin{matrix} z_1 \\ \dots \\ z_r \end{matrix} \middle| \begin{matrix} (1-c - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r; 1), & (1/2-c + a+b - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r; 1), \\ \dots & \dots \\ (1/2-c + a - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r; 1), & (1/2-c + b - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r; 1), \end{matrix} \right)$$

$$(1 - \beta - \sum_{i=1}^s R_i \phi_i; \lambda_1, \dots, \lambda_r; 1),$$

$$(1 - \alpha - \beta - \sum_{i=1}^s R_i (n_i + \phi_i) - m - \sum_{i=1}^n m_i; \rho_1 + \lambda_1, \dots, \rho_r + \lambda_r; 1),$$

$$(1 - \alpha - m - \sum_{i=1}^n m_i - \sum_{i=1}^s R_i n_i; \rho_1, \dots, \rho_r; 1), A : C$$

$$\left. \begin{matrix} \dots \\ B : D \end{matrix} \right) \tag{5.3}$$

under the same condition and notations that (3.1)

where $B'_s = \frac{(-L)_{h_1 R_1 + \dots + h_s R_s} B(E; R_1, \dots, R_s)}{R_1! \dots R_s!}$; $B(L; R_1, \dots, R_s)$ is defined by (5.1).

Corollary 5

$$\int_0^1 \int_0^1 x^{c-1} (1-x)^{-1/2} {}_2F_1(a, b; a+b+1/2; x) y^{\alpha-1} (1-y)^{\beta-1} {}_1F_1(\delta; f; s'y)$$

$$\begin{aligned}
& F_{\bar{C}:D';\dots;D^{(s)}}^{1+\bar{A}:B';\dots;B^{(s)}} \left(\begin{array}{c} a_1 x^{t_1} y^{n_1} (1-y)^{\phi_1} \\ \vdots \\ a_s x^{t_s} y^{n_s} (1-y)^{\phi_s} \end{array} \right) \\
& [(-L); R_1, \dots, R_s] [(a); \theta', \dots, \theta^{(s)}] : [(b^{(s)}); \phi^{(s)}] \left. \begin{array}{c} [(b'); \phi'] ; \dots ; [(b^{(s)}); \phi^{(s)}] \\ [(c); \psi', \dots, \psi^{(s)}] : [(d'); \delta'] ; \dots ; [(d^{(s)}); \delta^{(s)}] \end{array} \right) I \left(\begin{array}{c} z_1 x^{\sigma_1} y^{\rho_1} (1-y)^{\lambda_1} \\ \vdots \\ z_r x^{\sigma_r} y^{\rho_r} (1-y)^{\lambda_r} \end{array} \right) \\
& \left. \begin{array}{c} A : C \\ B : D \end{array} \right) dx dy \\
& = \frac{\pi \Gamma(a+b+1/2)}{\Gamma(a+1/2) \Gamma(b+1/2)} \sum_{m, m_1, \dots, m_n=0}^{\infty} \frac{((d), m_1 + \dots + m_n)(\delta, m)}{((g), m_1 + \dots + m_n)(f, m)} \\
& \sum_{R_1, \dots, R_s=0}^{h_1 R_1 + \dots + h_s R_s \leq L} B'_s a^{R_1} \dots a_s^{R_s} \frac{((e'), m_1) \dots ((e^{(n)}), m_n) s'^m r_1^{m_1} \dots r_n^{m_n}}{((h'), m_1) \dots ((h^{(n)}), m_n) m! m_1! \dots m_n!} \\
& I_{p+4, q+3; W}^{0, n+4; X} \left(\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array} \left| \begin{array}{c} (1-c - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r; 1), \quad (1/2-c + a+b - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r; 1), \\ \vdots \\ (1/2-c+a - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r; 1), \quad (1/2-c+b - \sum_{i=1}^s R_i t_i; \sigma_1, \dots, \sigma_r; 1), \end{array} \right. \right. \\
& \quad (1 - \beta - \sum_{i=1}^{k+1} m_i - \sum_{i=1}^s R_i \phi_i; \lambda_1, \dots, \lambda_r; 1), \\
& \quad (1 - \alpha - \beta - \sum_{i=1}^s R_i (n_i + \phi_i) - m - \sum_{i=1}^n m_i; \rho_1 + \lambda_1, \dots, \rho_r + \lambda_r; 1), \\
& \quad \left. (1 - \alpha - m - \sum_{i=1}^k m_i - \sum_{i=1}^s R_i n_i; \rho_1, \dots, \rho_r; 1), A : C \right) \\
& \quad \left. \begin{array}{c} \vdots \\ B : D \end{array} \right) \tag{5.5}
\end{aligned}$$

under the same condition and notations that (3.3)

where $B'_s = \frac{(-L)_{h_1 R_1 + \dots + h_s R_s} B(E; R_1, \dots, R_s)}{R_1! \dots R_s!}$; $B(L; R_1, \dots, R_s)$ is defined by (5.1).

6. Conclusion

The I-function of several variables defined by Nambisan et al [3] is quite basic in nature. Therefore, on specializing the parameters of this function, we may obtain various other special functions such as H-function of several variables defined by Srivastava et al [6]

References :

- [1] B. L. J. Braaksma, "Asymptotic expansions and analytic continuations for a class of Barnes integrals," *Compositio Mathematica*, vol. 15, pp. 239–341, 1964.
- [2] Exton, H, *Handbook of hypergeometric integrals*, Ellis Horwood Ltd, Chichester (1978)
- [3] Prathima J. Nambisan V. and Kurumujji S.K. A Study of I-function of Several Complex Variables, *International Journal of Engineering Mathematics* Vol(2014) , 2014 page 1-12
- [4] Srivastava H.M. and Daoust M.C. Certain generalized Neumann expansions associated with Kampé de Fériet function. *Nederl. Akad. Wetensch. Proc. Ser A72 = Indag Math* 31(1969) page 449-457.
- [5] Srivastava H.M. And Garg M. Some integral involving a general class of polynomials and multivariable H-function. *Rev. Roumaine Phys.* 32(1987), page 685-692.
- [6] H.M. Srivastava And R.Panda. Some expansion theorems and generating relations for the H-function of several complex variables. *Comment. Math. Univ. St. Paul.* 24(1975), p.119-137.
- [7] Vyas V.M. and Rathie K., An integral involving hypergeometric function. *The mathematics education* 31(1997) page33