Int. Jr. of Mathematical Sciences & Applications
Vol. 8, No. 1, (January-June, 2018)

ISSN No: 2230-9888
www.ijmsa.yolasite.com

SOME REDUCTION FORMULAE FOR THE I – FUNCTION OF SEVERAL VARIABLES

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ABSTRACT
In this paper we introduced some reduction formulae for the I - function of several variables. A number of known and unknown results can be obtained as special cases.

KEYWORDS : I-function of several variables, reduction of formulae, Mellin-Barnes integrals contour.

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20

1.Introduction and preliminaries.

Recently, Kumari and Nambisan [3] have studied some reduction formulae for the multivariable A-function defined by Gautam and Asgar [2]. The aim of this paper deals with certain reduction formulae for the special values of I-function of r variables defined by Prasad [4], which are of great interest and generalize many known and unknown results in literature especially the results given by Vidya [8] and Cook [1].

The multivariable I-function defined by Prasad [4] generalizes the multivariable H-function studied by Srivastava and Panda [6,7]. This function of r-variables is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1,z_2,\cdots,z_r)=I^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;\cdots;p_r,q_r:p^{(1)},q^{(1)};\cdots;p^{(r)},q^{(r)}}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};\cdots; \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};\cdots; \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj})_{1,p_r}:(a^{(1)}_j,\alpha^{(1)}_j)_{1,p^{(1)}};\cdots;(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}} \\ \\ (\mathrm{b}_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj})_{1,q_r}:(b^{(1)}_j,\beta^{(1)}_j)_{1,q^{(1)}};\cdots;(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}} \end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\prod_{i=1}^{r}\phi_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.7}$$

where

$$\phi_i(s_i)=\frac{\prod_{j=1}^{m^{(i)}}\Gamma(b^{(i)}_j-\beta^{(i)}_js_i)\prod_{j=1}^{n^{(i)}}\Gamma(1-a^{(i)}_j+\alpha^{(i)}_js_i)}{\prod_{j=m^{(i)}+1}^{q^{(i)}}\Gamma(1-b^{(i)}_j+\beta^{(i)}_js_i)\prod_{j=n^{(i)}+1}^{p^{(i)}}\Gamma(a^{(i)}_j-\alpha^{(i)}_js_i)}\ ,i=1,\cdots,r \tag{1.8}$$

and

$$\phi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n_2}\Gamma(1-a_{2j}+\sum_{i=1}^{2}\alpha^{(i)}_{2j}s_i)\prod_{j=1}^{n_3}\Gamma(1-a_{3j}+\sum_{i=1}^{3}\alpha^{(i)}_{3j}s_i)\cdots}{\prod_{j=n_2+1}^{p_2}\Gamma(a_{2j}-\sum_{i=1}^{2}\alpha^{(i)}_{2j}s_i)\prod_{j=n_3+1}^{p_3}\Gamma(a_{3j}-\sum_{i=1}^{3}\alpha^{(i)}_{3j}s_i)\cdots}$$

$$\frac{\cdots \prod_{j=1}^{n_r} \Gamma(1-a_{rj}+\sum_{i=1}^{r} \alpha_{rj}^{(i)} s_i)}{\cdots \prod_{j=n_r+1}^{p_r} \Gamma(a_{rj}-\sum_{i=1}^{r} \alpha_{rj}^{(i)} s_i) \prod_{j=1}^{q_2} \Gamma(1-b_{2j}-\sum_{i=1}^{2} \beta_{2j}^{(i)} s_i)}$$

$$\times \frac{1}{\prod_{j=1}^{q_3} \Gamma(1-b_{3j}+\sum_{i=1}^{3} \beta_{3j}^{(i)} s_i) \cdots \prod_{j=1}^{q_r} \Gamma(1-b_{rj}-\sum_{i=1}^{r} \beta_{rj}^{(i)} s_i)} \tag{1.9}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [4]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.7) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_i| < \frac{1}{2}\Omega_i \pi$, where

$$\Omega_i = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)} \right) + \cdots +$$

$$\left(\sum_{k=1}^{n_s} \alpha_{sk}^{(i)} - \sum_{k=n_s+1}^{p_s} \alpha_{sk}^{(i)} \right) - \left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_s} \beta_{sk}^{(i)} \right) \tag{1.10}$$

where $i = 1, \cdots, r$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.We may establish the the asymptotic expansion in the following convenient form :

$$I(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \to 0$$

$$I(z_1, \cdots, z_r) \ = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \to \infty$$

where $k = 1, \cdots, r : \alpha'_k = min[Re(b_j^{(k)}/\beta_j^{(k)})], j = 1, \cdots, m^{(k)}$ and

$$\beta'_k = max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \cdots, n^{(k)}$$

In this paper, we shall note

$$U = p_2, q_2; p_3, q_3; \cdots; p_{r-1}, q_{r-1} \tag{1.11}$$

$$V = 0, n_2; 0, n_3; \cdots; 0, n_{r-1} \tag{1.12}$$

$$A = (a_{2k}; \alpha_{2k}^{(1)}, \alpha_{2k}^{(2)})_{1,p_2}; \cdots; (a_{(r-1)k}; \alpha_{(r-1)k}^{(1)}, \alpha_{(r-1)k}^{(2)}, \cdots, \alpha_{(r-1)k}^{(r-1)})_{1,p_{r-1}} : (a_{rk}; \alpha_{rk}^{(1)}, \alpha_{rk}^{(2)}, \cdots, \alpha_{rk}^{(r)})_{1,p_r} : \tag{1.13}$$

$$B = (b_{2k}; \beta_{2k}^{(1)}, \beta_{2k}^{(2)})_{1,q_2}; \cdots; (b_{(r-1)k}; \beta_{(r-1)k}^{(1)}, \beta_{(r-1)k}^{(2)}, \cdots, \beta_{(r-1)k}^{(r-1)})_{1,q_{r-1}} :(b_{rk}; \beta_{rk}^{(1)}, \beta_{rk}^{(2)}, \cdots, \beta_{rk}^{(r)})_{1,q_r} : \tag{1.14}$$

2. Results used

To prove our main results we shall use the following formulae. These are obtained by using the well known relation $\Gamma(z+1) = z\Gamma(z)$. We have the two results.

Formula 1

$$\frac{\Gamma(1\pm\sigma\pm\lambda s)\Gamma(\mu\sigma\pm\mu\lambda s)}{\Gamma(\sigma\pm\lambda s)} = (\sigma\pm\lambda s)\Gamma(\mu\sigma\pm\mu\lambda s) = \frac{1}{\mu}(\mu\sigma\pm\mu\lambda s)\Gamma(\mu\sigma\pm\mu\lambda s) = \frac{1}{\mu}\Gamma(1\pm\mu\sigma\pm\mu\lambda s) \tag{2.1}$$

where $\mu > 0$

Formula 2

$$\frac{\Gamma(\sigma\pm\lambda s)\Gamma(1\pm\mu\sigma\pm\mu\lambda s)}{\Gamma(1\pm\sigma\pm\lambda s)} = \frac{\Gamma(1\pm\mu\sigma\pm\mu\lambda s)}{(\pm\sigma\pm\lambda s)} = \mu\Gamma(\pm\mu\sigma\pm\lambda s) \tag{2.2}$$

where $\mu > 0$

3. Reduction formulae

Theorem 1

Let

$$X_1 = m^{(1)}+2l, n^{(1)};\cdots;m^{(r)}+2l, n^{(r)} \tag{3.1}$$

$$Y_1 = p^{(1)}+l, q^{(1)}+2l;\cdots;p^{(r)}+l, q^{(r)}+2l \tag{3.2}$$

$$A_1 = (a_k^{(1)};\alpha_k^{(1)})_{1,p^{(1)}}, (\sigma_j^{(1)};\lambda_j^{(1)})_{1,l};\cdots;(a_k^{(r)};\alpha_k^{(r)})_{1,p^{(r)}}, (\sigma_j^{(r)};\lambda_j^{(r)})_{1,l} \tag{3.3}$$

$$B_1 = (\sigma_j^{(1)}+1;\lambda_j^{(1)})_{1,l}, (\mu_j\sigma_j^{(1)};\mu_j\lambda_j^{(1)})_{1,l}, (b_k^{(1)};\beta_k^{(1)})_{1,q^{(1)}};\cdots;(\sigma_j^{(r)}+1;\lambda_j^{(r)})_{1,l}, (\mu_j\sigma_j^{(r)};\mu_j\lambda_j^{(r)})_{1,l}, (b_k^{(r)};\beta_k^{(r)})_{1,q^{(r)}} \tag{3.4}$$

$$X_1' = m^{(1)}+l, n^{(1)};\cdots;m^{(r)}+l, n^{(r)} \tag{3.5}$$

$$Y_1' = p^{(1)}, q^{(1)}+l;\cdots;p^{(r)}, q^{(r)}+l \tag{3.6}$$

$$A_1' = (a_k^{(1)};\alpha_k^{(1)})_{1,p^{(1)}};\cdots;(a_k^{(r)};\alpha_k^{(r)})_{1,p^{(r)}} \tag{3.7}$$

$$B_1' = (\mu_j\sigma_j^{(1)}+1;\mu_j\lambda_j^{(1)})_{1,l}, (b_k^{(1)};\beta_k^{(1)})_{1,q^{(1)}};\cdots;(\mu_j\sigma_j^{(r)}+1;\mu_j\lambda_j^{(r)})_{1,l}, (b_k^{(r)};\beta_k^{(r)})_{1,q^{(r)}} \tag{3.8}$$

We obtain the following formula

$$I^{V;0,n_r;X_1}_{U:p_r,q_r;Y_1}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{z}_r & \mathrm{B}:\mathrm{B}_1 \end{array}\right) = \prod_{m=1}^{l}\frac{1}{(\mu_m)^r}\, I^{V;0,n_r;X_1'}_{U:p_r,q_r;Y_1'}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}_1' \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{z}_r & \mathrm{B}:\mathrm{B}_1' \end{array}\right) \tag{3.9}$$

for $\mu_m > 0, m = 1,\cdots,l.$

Proof

In order to establish we proceed as follows. For this, denoting the left-hand side of (3.9) by L.H.S. and expressing the multivariable I-function occurring in (3.9) in terms of Mellin-Barnes integrals contour with the help of (1.7), we have :

$$\text{L.H.S.} = \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\prod_{i=1}^{r}\phi_i(s_i)z_i^{s_i}\left[\prod_{i=1}^{r}\prod_{j=1}^{l}\frac{\Gamma(1+\sigma_j^{(i)}-\lambda_j^{(i)}s_i)\Gamma(\mu_j\sigma_j^{(i)}-\mu_j\lambda_j^{(i)}s_i)}{\Gamma(\sigma_j^{(i)}-\lambda_j^{(i)}s_i)}\right]\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.10}$$

Now using the general property of gamma function in (2.1), we obtain :

$$\prod_{j=1}^{l}\frac{\Gamma(1+\sigma_j^{(i)}-\lambda_j^{(i)}s_i)\Gamma(\mu_j\sigma_j^{(i)}-\mu_j\lambda_j^{(i)}s_i)}{\Gamma(\sigma_j^{(i)}-\lambda_j^{(i)}s_i)}=\prod_{j=1}^{l}(\sigma_j^{(i)}-\lambda_j^{(i)}s_i)\Gamma(\mu_i\sigma_j^{(i)}-\mu_j\lambda_j^{(i)}s_i)$$

$$=\prod_{j=1}^{l}(\sigma_j^{(i)}-\lambda_j^{(i)}s_i)\frac{\Gamma(1+\mu_i\sigma_j^{(i)}-\mu_j\lambda_j^{(i)}s_i)}{(\mu_j)(\sigma_j^{(i)}-\lambda_j^{(i)}s_i)}=\prod_{j=1}^{l}\frac{1}{\mu_j}\Gamma(1+\mu_i\sigma_j^{(i)}-\mu_i\lambda_j^{(i)}s_i) \tag{3.11}$$

Substituting (3.11) in (3.10) and reinterpreting the resulting Mellin-Barnes integrals contour in multivariable I-function, we get the result (3.9).

Theorem 2

Let

$$X_2=m^{(1)}+l,n^{(1)}+l;\cdots;m^{(r)}+l,n^{(r)}+l \tag{3.12}$$

$$Y_2=p^{(1)}+2l,q^{(1)}+l;\cdots;p^{(r)}+2l,q^{(r)}+l \tag{3.13}$$

$$A_2=(1+\mu_j\sigma_j^{(1)};\mu_j\lambda_j^{(1)})_{1,l},(a_k^{(1)};\alpha_k^{(1)})_{1,p^{(1)}},(\sigma_j^{(1)};\lambda_j^{(1)})_{1,l};\cdots;(1+\mu_j\sigma_j^{(r)};\mu_j\lambda_j^{(r)})_{1,l},(a_k^{(r)};\alpha_k^{(r)})_{1,p^{(r)}},(\sigma_j^{(r)};\lambda_j^{(r)})_{1,l} \tag{3.14}$$

$$B_2=(1+\sigma_j^{(1)};\lambda_j^{(1)})_{1,l},(b_k^{(1)};\beta_k^{(1)})_{1,q^{(1)}};\cdots;(1+\sigma_j^{(r)};\lambda_j^{(r)})_{1,l},(b_k^{(r)};\beta_k^{(r)})_{1,q^{(r)}} \tag{3.15}$$

$$X_2'=m^{(1)},n^{(1)}+l;\cdots;m^{(r)},n^{(r)}+l \tag{3.16}$$

$$Y_2'=p^{(1)}+l,q^{(1)};\cdots;p^{(r)}+l,q^{(r)} \tag{3.17}$$

$$A_2'=(\mu_j\sigma_j^{(1)};\mu_j\lambda_j^{(1)})_{1,k},(a_k^{(1)};\alpha_k^{(1)})_{1,p^{(1)}};\cdots;(\mu_j\sigma_j^{(r)};\mu_j\lambda_j^{(r)})_{1,k},(a_k^{(r)};\alpha_k^{(r)})_{1,p^{(r)}} \tag{3.18}$$

$$B_2'=(b_k^{(1)};\beta_k^{(1)})_{1,q^{(1)}};\cdots;(b_k^{(r)};\beta_k^{(r)})_{1,q^{(r)}} \tag{3.19}$$

We obtain the following formula

$$I_{U:p_r,q_r;Y_2}^{V;0,n_r;X_2}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{z}_r & \mathrm{B}:\mathrm{B}_2 \end{array}\right)=\prod_{m=1}^{l}\frac{1}{(-\mu_m)^r}\,I_{U:p_r,q_r;Y_2'}^{V;0,n_r;X_2'}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}_2' \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{z}_r & \mathrm{B}:\mathrm{B}_2' \end{array}\right) \tag{3.20}$$

for $\mu_m>0, m=1,\cdots,l.$

Theorem 3

Let

$$X_3=m^{(1)}+l,n^{(1)};\cdots;m^{(r)}+l,n^{(r)} \tag{3.21}$$

$$Y_3=p^{(1)}+l,q^{(1)}+2l;\cdots;p^{(r)}+l,q^{(r)}+2l \tag{3.22}$$

$$A_3 = (a_k^{(1)};\alpha_k^{(1)})_{1,p^{(1)}}, (1+\sigma_j^{(1)};\lambda_j^{(1)})_{1,l};\cdots;(a_k^{(r)};\alpha_k^{(r)})_{1,p^{(r)}}, (1+\sigma_j^{(r)};\lambda_j^{(r)})_{1,l} \tag{3.23}$$

$$B_3 = (\sigma_j^{(1)};\lambda_j^{(1)})_{1,l}, (b_k^{(1)};\beta_k^{(1)})_{1,q^{(1)}}, (1+\mu_j\sigma_j^{(1)};\mu_j\lambda_j^{(1)})_{1,l};\cdots;(\sigma_j^{(r)};\lambda_j^{(r)})_{1,l}, (b_k^{(r)};\beta_k^{(r)})_{1,q^{(r)}}, (1+\mu_j\sigma_j^{(r)};\mu_j\lambda_j^{(r)})_{1,l} \tag{3.24}$$

$$X_3' = m^{(1)}, n^{(1)};\cdots;m^{(r)}, n^{(r)} \tag{3.25}$$

$$Y_3' = p^{(1)}, q^{(1)}+l;\cdots;p^{(r)}, q^{(r)}+l \tag{3.26}$$

$$A_3' = (a_k^{(1)};\alpha_k^{(1)})_{1,p^{(1)}};\cdots;(a_k^{(r)};\alpha_k^{(r)})_{1,p^{(r)}} \tag{3.27}$$

$$B_3' = (b_k^{(1)},\beta_k^{(1)})_{1,q^{(1)}}, (\mu_j\sigma_j^{(1)};\mu_j\lambda_j^{(1)})_{1,l};\cdots;(b_k^{(r)},\beta_k^{(r)})_{1,q^{(r)}}, (\mu_j\sigma_j^{(l)};\mu_j\lambda_j^{(r)})_{1,l} \tag{3.28}$$

We obtain the following formula

$$I_{U:p_r,q_r;Y_3}^{V;0,n_r;X_3}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}_3 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{z}_r & \mathrm{B}:\mathrm{B}_3 \end{array}\right) = \prod_{m=1}^{l}\frac{1}{(-\mu_m)^r}\, I_{U:p_r,q_r;Y_3'}^{V;0,n_r;X_3'}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}_3' \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{z}_r & \mathrm{B}:\mathrm{B}_3' \end{array}\right) \tag{3.29}$$

for $\mu_m > 0, m = 1,\cdots,l$.

Theorem 4

Let

$$X_4 = m^{(1)}+l, n^{(1)};\cdots;m^{(r)}+l, n^{(r)} \tag{3.30}$$

$$Y_4 = p^{(1)}+2l, q^{(1)}+l;\cdots;p^{(r)}+2l, q^{(r)}+l \tag{3.31}$$

$$A_4 = (a_k^{(1)};\alpha_k^{(1)})_{1,p^{(1)}}, (1+\sigma_j^{(1)};\lambda_j^{(1)})_{1,l}, (\mu_j\sigma_j^{(1)};\mu_j\lambda_j^{(1)})_{1,l};\cdots;(a_k^{(r)};\alpha_k^{(r)})_{1,p^{(r)}}, (1+\sigma_j^{(r)};\lambda_j^{(r)})_{1,l}, (\mu_j\sigma_j^{(r)};\mu_j\lambda_j^{(r)})_{1,l} \tag{3.32}$$

$$B_4 = (\sigma_j^{(1)};\lambda_j^{(1)})_{1,l}, (b_k^{(1)};\beta_k^{(1)})_{1,q^{(1)}};\cdots;(\sigma_j^{(r)};\lambda_j^{(r)})_{1,l}, (b_k^{(r)};\beta_k^{(r)})_{1,q^{(r)}} \tag{3.33}$$

$$X_4' = m^{(1)}, n^{(1)};\cdots;m^{(r)}, n^{(r)} \tag{3.34}$$

$$Y_4' = p^{(1)}+l, q^{(1)};\cdots;p^{(r)}+l, q^{(r)} \tag{3.35}$$

$$A_4' = (a_k^{(1)};\alpha_k^{(1)})_{1,p^{(1)}}, (1+\mu_j\sigma_j^{(1)};\mu_j\lambda_j^{(1)})_{1,l};\cdots;(a_k^{(r)};\alpha_k^{(r)})_{1,p^{(r)}}, (1+\mu_j\sigma_j^{(r)};\mu_j\lambda_j^{(r)})_{1,l} \tag{3.36}$$

$$B_4' = (b_k^{(1)},\beta_k^{(1)})_{1,q^{(1)}};\cdots;(b_k^{(r)},\beta_k^{(r)})_{1,q^{(r)}} \tag{3.37}$$

We obtain the following formula

$$I_{U:p_r,q_r;Y_4}^{V;0,n_r;X_4}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}_4 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{z}_r & \mathrm{B}:\mathrm{B}_4 \end{array}\right) = \prod_{m=1}^{l}\frac{1}{(\mu_m)^r}\, I_{U:p_r,q_r;Y_4'}^{V;0,n_r;X_4'}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}_4' \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{z}_r & \mathrm{B}:\mathrm{B}_4' \end{array}\right) \tag{3.38}$$

for $\mu_m > 0, m = 1, \cdots, l$.

Theorem 5

Let

$$X_5 = m^{(1)}, n^{(1)} + 2l; \cdots ; m^{(r)}, n^{(r)} + 2l \tag{3.39}$$

$$Y_5 = p^{(1)} + 2l, q^{(1)} + l; \cdots ; p^{(r)} + 2l, q^{(r)} + l \tag{3.40}$$

$$A_5 = (a_k^{(1)}; \alpha_k^{(1)})_{1,p^{(1)}}, (\sigma_j^{(1)}; \lambda_j^{(1)})_{1,l}, (1 + \mu_j \sigma_j^{(1)}; \mu_j \lambda_j^{(1)})_{1,l}; \cdots ; (a_k^{(r)}; \alpha_k^{(r)})_{1,p^{(r)}}, (\sigma_j^{(r)}; \lambda_j^{(r)})_{1,l}, (1 + \mu_j \sigma_j^{(r)}; \mu_j \lambda_j^{(r)})_{1,l} \tag{3.41}$$

$$B_5 = (b_k^{(1)}; \beta_k^{(1)})_{1,q^{(1)}}, (1 + \sigma_j^{(r)}; \lambda_j^{(1)})_{1,l}; \cdots ; (b_k^{(r)}; \beta_k^{(r)})_{1,q^{(r)}}, (1 + \sigma_j^{(r)}; \lambda_j^{(r)})_{1,l} \tag{3.42}$$

$$X_5' = m^{(1)}, n^{(1)} + l; \cdots ; m^{(r)}, n^{(r)} + l \tag{3.43}$$

$$Y_5' = p^{(1)} + l, q^{(1)}; \cdots ; p^{(r)} + l, q^{(r)} \tag{3.44}$$

$$A_5' = (\mu_j \sigma_j^{(1)}; \mu_j \lambda_j^{(1)})_{1,k}, (a_k^{(1)}; \alpha_k^{(1)})_{1,p^{(1)}}; \cdots ; (\mu_j \sigma_j^{(r)}; \mu_j \lambda_j^{(r)})_{1,k}, (a_k^{(r)}; \alpha_k^{(r)})_{1,p^{(r)}} \tag{3.45}$$

$$B_5' = (b_k^{(1)}, \beta_k^{(1)})_{1,q^{(1)}}; \cdots ; (b_k^{(r)}, \beta_k^{(r)})_{1,q^{(r)}} \tag{3.46}$$

We obtain the following formula

$$I_{U:p_r,q_r;Y_5}^{V;0,n_r;X_5}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A} : \mathrm{A}_5 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{z}_r & \mathrm{B} : \mathrm{B}_5 \end{array}\right) = \prod_{m=1}^{l} \frac{1}{(\mu_m)^r} I_{U:p_r,q_r;Y_5'}^{V;0,n_r;X_5'}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A} : \mathrm{A}_5' \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{z}_r & \mathrm{B} : \mathrm{B}_5' \end{array}\right) \tag{3.47}$$

for $\mu_m > 0, m = 1, \cdots, l$.

Theorem 6

Let

$$X_6 = m^{(1)} + l, n^{(1)} + l; \cdots ; m^{(r)} + l, n^{(r)} + l \tag{3.48}$$

$$Y_6 = p^{(1)} + l, q^{(1)} + 2l; \cdots ; p^{(r)} + l, q^{(r)} + 2l \tag{3.49}$$

$$A_6 = (\sigma_j^{(1)}; \lambda_j^{(1)})_{1,k}, (a_k^{(1)}; \alpha_k^{(1)})_{1,p^{(1)}}; \cdots ; (\sigma_j^{(r)}; \lambda_j^{(r)})_{1,k}, (a_k^{(r)}; \alpha_k^{(r)})_{1,p^{(r)}} \tag{3.50}$$

$$B_6 = (\mu_j \sigma_j^{(1)}; \mu_j \lambda_j^{(1)})_{1,l}, (b_k^{(1)}; \beta_k^{(1)})_{1,q^{(1)}}, (1 + \sigma_j^{(1)}; \lambda_j^{(1)}); \cdots ; (\mu_j \sigma_j^{(r)}; \mu_j \lambda_j^{(r)})_{1,l}, (b_k^{(r)}; \beta_k^{(r)})_{1,q^{(r)}}, (1 + \sigma_j^{(r)}; \lambda_j^{(r)})_{1,l} \tag{3.51}$$

$$X_6' = m^{(1)} + l, n^{(1)}; \cdots ; m^{(r)} + l, n^{(r)} \tag{3.52}$$

$$Y_6' = p^{(1)}, q^{(1)} + l; \cdots ; p^{(r)}, q^{(r)} + l \tag{3.53}$$

$$A_6' = (a_k^{(1)}; \alpha_k^{(1)})_{1,p^{(1)}}; \cdots ; (a_k^{(r)}; \alpha_k^{(r)})_{1,p^{(r)}} \tag{3.54}$$

$$B'_6=(1+\mu_j\sigma_j^{(1)};\mu_j\lambda_j^{(1)})_{1,l},(b_k^{(1)};\beta_k^{(1)})_{1,q^{(1)}};\cdots;(1+\mu_j\sigma_j^{(r)};\mu_j\lambda_j^{(r)})_{1,l},(b_k^{(r)};\beta_k^{(r)})_{1,q^{(r)}} \tag{3.55}$$

We obtain the following formula

$$I_{U:p_r,q_r;Y_6}^{V;0,n_r;X_6}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}_6 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}:\mathrm{B}_6 \\ \mathrm{z}_r & \end{array}\right)=\prod_{m=1}^{l}\frac{1}{(\mu_m)^r}\,I_{U:p_r,q_r;Y'_6}^{V;0,n_r;X'_6}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}'_6 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}:\mathrm{B}'_6 \\ \mathrm{z}_r & \end{array}\right) \tag{3.56}$$

for $\mu_m>0, m=1,\cdots,l$.

Theorem 6

Let

$$X_7=m^{(1)}+l,n^{(1)};\cdots;m^{(r)}+l,n^{(r)} \tag{3.57}$$

$$Y_7=p^{(1)}+2l,q^{(1)}+l;\cdots;p^{(r)}+2l,q^{(r)}+l \tag{3.58}$$

$$A_7=(a_k^{(1)};\alpha_k^{(1)})_{1,p^{(1)}},(\sigma_j^{(1)};\lambda_j^{(1)})_{1,l},(1+\mu_j\sigma_j^{(1)};\mu_j\lambda_j^{(1)})_{1,l};\cdots;(a_k^{(r)};\alpha_k^{(r)})_{1,p^{(r)}},(\sigma_j^{(r)};\lambda_j^{(r)})_{1,l},(1+\mu_j\sigma_j^{(r)};\mu_j\lambda_j^{(r)})_{1,l} \tag{3.59}$$

$$B_7=(1+\sigma_j^{(1)};\lambda_j^{(1)})_{1,l},(b_k^{(1)};\beta_k^{(1)})_{1,q^{(1)}};\cdots;(1+\sigma_j^{(r)};\lambda_j^{(r)})_{1,l},(b_k^{(r)};\beta_k^{(r)})_{1,q^{(r)}} \tag{3.60}$$

$$X'_7=m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)} \tag{3.61}$$

$$Y'_7=p^{(1)},q^{(1)}+l;\cdots;p^{(r)},q^{(r)}+l \tag{3.62}$$

$$A'_7=(a_k^{(1)};\alpha_k^{(1)})_{1,p^{(1)}},(1+\mu_j\sigma_j^{(1)};\mu_j\lambda_j^{(1)})_{1,l};\cdots;(a_k^{(r)};\alpha_k^{(r)})_{1,p^{(r)}},(1+\mu_j\sigma_j^{(r)};\mu_j\lambda_j^{(r)})_{1,l} \tag{3.63}$$

$$B'_7=(b_k^{(1)};\beta_k^{(1)})_{1,q^{(1)}};(b_k^{(r)};\beta_k^{(r)})_{1,q^{(r)}} \tag{3.64}$$

We obtain the following formula

$$I_{U:p_r,q_r;Y_7}^{V;0,n_r;X_7}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}_7 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}:\mathrm{B}_7 \\ \mathrm{z}_r & \end{array}\right)=\prod_{m=1}^{l}\frac{1}{(\mu_m)^r}\,I_{U:p_r,q_r;Y'_7}^{V;0,n_r;X'_7}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}'_7 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}:\mathrm{B}'_7 \\ \mathrm{z}_r & \end{array}\right) \tag{3.65}$$

for $\mu_m>0, m=1,\cdots,l$.

Theorem 8

Let

$$X_8=m^{(1)},n^{(1)}+l;\cdots;m^{(r)},n^{(r)}+l \tag{3.66}$$

$$Y_8=p^{(1)}+l,q^{(1)}+2l;\cdots;p^{(r)}+l,q^{(r)}+2l \tag{3.67}$$

$$A_8=(1+\sigma_j^{(1)};\lambda_j^{(1)})_{1,k},(a_k^{(1)};\alpha_k^{(1)})_{1,p^{(1)}};\cdots;(1+\sigma_j^{(r)};\lambda_j^{(r)})_{1,k},(a_k^{(r)};\alpha_k^{(r)})_{1,p^{(r)}} \tag{3.68}$$

$$B_8 = (b_k^{(1)};\beta_k^{(1)})_{1,q^{(1)}},(1+\mu_j\sigma_j^{(1)};\mu_j\lambda_j^{(1)})_{1,l},(\sigma_j^{(1)};\lambda_j^{(1)})_{1,l};\cdots;(b_k^{(r)};\beta_k^{(r)})_{1,q^{(r)}},(1+\mu_j\sigma_j^{(r)}+1;\mu_j\lambda_j^{(r)})_{1,l},(\sigma_j^{(r)};\lambda_j^{(r)})_{1,l} \tag{3.69}$$

$$X_8' = m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)} \tag{3.70}$$

$$Y_8' = p^{(1)},q^{(1)}+l;\cdots;p^{(r)},q^{(r)}+l \tag{3.71}$$

$$A_8' = (a_k^{(1)};\alpha_k^{(1)})_{1,p^{(1)}};\cdots;(a_k^{(r)};\alpha_k^{(r)})_{1,p^{(r)}} \tag{3.72}$$

$$B_8' = (b_k^{(1)};\beta_k^{(1)})_{1,q^{(1)}},(\mu_j\sigma_j^{(1)};\mu_j\lambda_j^{(1)})_{1,l};\cdots;(b_k^{(r)};\beta_k^{(r)})_{1,q^{(r)}},(\mu_j\sigma_j^{(r)};\mu_j\lambda_j^{(r)})_{1,l} \tag{3.73}$$

We obtain the following formula

$$I_{U:p_r,q_r;Y_8}^{V;0,n_r;X_8}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}_8 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}:\mathrm{B}_8 \\ \mathrm{z}_r & \end{array}\right) = \prod_{m=1}^{l}\frac{1}{(\mu_m)^r}\, I_{U:p_r,q_r;Y_8'}^{V;0,n_r;X_8'}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}_8' \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}:\mathrm{B}_8' \\ \mathrm{z}_r & \end{array}\right) \tag{3.74}$$

for $\mu_m > 0, m = 1,\cdots,l.$

Theorem 9

Let

$$X_9 = m^{(1)},n^{(1)}+l;\cdots;m^{(r)},n^{(r)}+l \tag{3.75}$$

$$Y_9 = p^{(1)}+2l,q^{(1)}+l;\cdots;p^{(r)}+2l,q^{(r)}+l \tag{3.76}$$

$$A_9 = (1+\sigma_j^{(1)};\lambda_j^{(1)})_{1,l},(a_k^{(1)};\alpha_k^{(1)})_{1,p^{(1)}},(\mu_j\sigma_j^{(1)};\mu_j\lambda_j^{(1)})_{1,l};\cdots;(1+\sigma_j^{(r)};\lambda_j^{(r)})_{1,l},(a_k^{(r)};\alpha_k^{(r)})_{1,p^{(r)}},(\mu_j\sigma_j^{(r)};\mu_j\lambda_j^{(r)})_{1,l} \tag{3.77}$$

$$B_9 = (b_k^{(1)};\beta_k^{(1)})_{1,q^{(1)}},(\sigma_j^{(1)};\lambda_j^{(1)})_{1,l};\cdots;(b_k^{(r)};\beta_k^{(r)})_{1,q^{(r)}},(\sigma_j^{(r)};\lambda_j^{(r)})_{1,l} \tag{3.78}$$

$$X_9' = m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)} \tag{3.79}$$

$$Y_9' = p^{(1)}+l,q^{(1)};\cdots;p^{(r)}+l,q^{(r)} \tag{3.80}$$

$$A_9' = (a_k^{(1)};\alpha_k^{(1)})_{1,p^{(1)}},(1+\mu_j\sigma_j^{(1)};\mu_j\lambda_j^{(1)})_{1,l};\cdots;(a_k^{(r)};\alpha_k^{(r)})_{1,p^{(r)}},(1+\mu_j\sigma_j^{(r)};\mu_j\lambda_j^{(r)})_{1,l} \tag{3.81}$$

$$B_9' = (b_k^{(1)};\beta_k^{(1)})_{1,q^{(1)}};\cdots;(b_k^{(r)};\beta_k^{(r)})_{1,q^{(r)}} \tag{3.82}$$

We obtain the following formula

$$I_{U:p_r,q_r;Y_9}^{V;0,n_r;X_9}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}_9 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}:\mathrm{B}_9 \\ \mathrm{z}_r & \end{array}\right) = \prod_{m=1}^{l}\frac{1}{(-\mu_m)^r}\, I_{U:p_r,q_r;Y_9'}^{V;0,n_r;X_9'}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}_9' \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}:\mathrm{B}_9' \\ \mathrm{z}_r & \end{array}\right) \tag{3.83}$$

for $\mu_m > 0, m = 1,\cdots,l.$

Theorem 10

Let

$$X_{10} = m^{(1)} + 2l, n^{(1)}; \cdots ; m^{(r)} + 2l, n^{(r)} \tag{3.84}$$

$$Y_{10} = p^{(1)} + l, q^{(1)} + 2l; \cdots ; p^{(r)} + l, q^{(r)} + 2l \tag{3.85}$$

$$A_{10} = (a_k^{(1)}; \alpha_k^{(1)})_{1,p^{(1)}}, (1 + \sigma_j^{(1)}; \lambda_j^{(1)})_{1,k}; \cdots ; (a_k^{(r)}; \alpha_k^{(r)})_{1,p^{(r)}}, (1 + \sigma_j^{(r)}; \lambda_j^{(r)})_{1,k} \tag{3.86}$$

$$B_{10} = (\sigma_j^{(1)}; \lambda_j^{(1)})_{1,l}, (1 + \mu_j \sigma_j^{(1)}; \mu_j \lambda_j^{(1)})_{1,l}, (b_k^{(1)}; \beta_k^{(1)})_{1,q^{(1)}}; \cdots ; (\sigma_j^{(r)}; \lambda_j^{(r)})_{1,l}, (1 + \mu_j \sigma_j^{(r)}; \mu_j \lambda_j^{(r)})_{1,l}, (b_k^{(r)}; \beta_k^{(r)})_{1,q^{(r)}} \tag{3.87}$$

$$X'_{10} = m^{(1)} + l, n^{(1)}; \cdots ; m^{(r)} + l, n^{(r)} \tag{3.88}$$

$$Y'_{10} = p^{(1)}, q^{(1)} + l; \cdots ; p^{(r)}, q^{(r)} + l \tag{3.89}$$

$$A'_{10} = (a_k^{(1)}; \alpha_k^{(1)})_{1,p^{(1)}}; \cdots ; (a_k^{(r)}; \alpha_k^{(r)})_{1,p^{(r)}} \tag{3.90}$$

$$B'_{10} = (\mu_j \sigma_j^{(1)}; \mu_j \lambda_j^{(1)})_{1,l}, (b_k^{(1)}; \beta_k^{(1)})_{1,q^{(1)}}; \cdots ; (\mu_j \sigma_j^{(r)}; \mu_j \lambda_j^{(r)})_{1,l}, (b_k^{(r)}; \beta_k^{(r)})_{1,q^{(r)}} \tag{3.91}$$

We obtain the following formula

$$I_{U:p_r,q_r;Y_{10}}^{V;0,n_r;X_{10}} \left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A} : \mathrm{A}_{10} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B} : \mathrm{B}_{10} \\ \mathrm{z}_r & \end{array} \right) = \prod_{m=1}^{l} \frac{1}{(\mu_m)^r} \, I_{U:p_r,q_r;Y'_{10}}^{V;0,n_r;X'_{10}} \left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A} : \mathrm{A}'_{10} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B} : \mathrm{B}'_{10} \\ \mathrm{z}_r & \end{array} \right) \tag{3.92}$$

for $\mu_m > 0, m = 1, \cdots, l.$

Theorem 11

Let

$$X_{11} = m^{(1)} + l, n^{(1)} + l; \cdots ; m^{(r)} + l, n^{(r)} + l \tag{3.93}$$

$$Y_{11} = p^{(1)} + 2l, q^{(1)} + l; \cdots ; p^{(r)} + 2l, q^{(r)} + l \tag{3.94}$$

$$A_{11} = (\mu_j \sigma_j^{(1)}; \mu_j \lambda_j^{(1)})_{1,l}, (a_k^{(1)}; \alpha_k^{(1)})_{1,p^{(1)}}, (1 + \sigma_j^{(1)}; \lambda_j^{(1)})_{1,l}; \cdots ; (\mu_j \sigma_j^{(r)}; \mu_j \lambda_j^{(r)})_{1,l}, (a_k^{(r)}; \alpha_k^{(r)})_{1,p^{(r)}}, (1 + \sigma_j^{(r)}; \lambda_j^{(r)})_{1,l} \tag{3.95}$$

$$B_{11} = (\sigma_j^{(1)}; \lambda_j^{(1)})_{1,l}, (b_k^{(1)}; \beta_k^{(1)})_{1,q^{(1)}}; \cdots ; (\sigma_j^{(r)}; \lambda_j^{(r)})_{1,l}, (b_k^{(r)}; \beta_k^{(r)})_{1,q^{(r)}} \tag{3.96}$$

$$X'_{11} = m^{(1)}, n^{(1)} + l; \cdots ; m^{(r)}, n^{(r)} + l \tag{3.97}$$

$$Y'_{11} = p^{(1)} + l, q^{(1)}; \cdots ; p^{(r)} + l, q^{(r)} \tag{3.98}$$

$$A'_{11} = (1 + \mu_j \sigma_j^{(1)}; \mu_j \lambda_j^{(1)})_{1,l}, (a_k^{(1)}; \alpha_k^{(1)})_{1,p^{(1)}}; \cdots ; (1 + \mu_j \sigma_j^{(r)}; \mu_j \lambda_j^{(r)})_{1,l}, (a_k^{(r)}; \alpha_k^{(r)})_{1,p^{(r)}} \tag{3.99}$$

$$B'_{11} = (b_k^{(1)};\beta_k^{(1)})_{1,q^{(1)}};\cdots;(b_k^{(r)};\beta_k^{(r)})_{1,q^{(r)}} \tag{3.100}$$

We obtain the following formula

$$I_{U:p_r,q_r;Y_{11}}^{V;0,n_r;X_{11}}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}_{11} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}:\mathrm{B}_{11} \\ \mathrm{z}_r & \end{array}\right) = \prod_{m=1}^{l}\frac{1}{(\mu_m)^r}\, I_{U:p_r,q_r;Y'_{11}}^{V;0,n_r;X'_{11}}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}'_{11} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}:\mathrm{B}'_{11} \\ \mathrm{z}_r & \end{array}\right) \tag{3.101}$$

for $\mu_m > 0, m = 1,\cdots,l.$

Theorem 12

Let

$$X_{12} = m^{(1)} + l, n^{(1)};\cdots;m^{(r)} + l, n^{(r)} \tag{3.102}$$

$$Y_{12} = p^{(1)} + 2l, q^{(1)} + l;\cdots;p^{(r)} + 2l, q^{(r)} + l \tag{3.103}$$

$$A_{12} = (a_k^{(1)};\alpha_k^{(1)})_{1,p^{(1)}},(\sigma_j^{(1)};\lambda_j^{(1)})_{1,l},(1+\mu_j\sigma_j^{(1)};\mu_j\lambda_j^{(1)})_{1,l};\cdots;(a_k^{(r)};\alpha_k^{(r)})_{1,p^{(r)}},(\sigma_j^{(r)};\lambda_j^{(r)})_{1,l},(1+\mu_j\sigma_j^{(r)};\mu_j\lambda_j^{(r)})_{1,l} \tag{3.104}$$

$$B_{12} = (1+\sigma_j^{(1)};\lambda_j^{(1)})_{1,l},(b_k^{(1)};\beta_k^{(1)})_{1,q^{(1)}};\cdots;(1+\sigma_j^{(r)};\lambda_j^{(r)})_{1,l},(b_k^{(r)};\beta_k^{(r)})_{1,q^{(r)}} \tag{3.105}$$

$$X'_{12} = m^{(1)}n^{(1)};\cdots;m^{(r)}, n^{(r)} \tag{3.106}$$

$$Y'_{12} = p^{(1)}, q^{(1)} + l;\cdots;p^{(r)}, q^{(r)} + l \tag{3.107}$$

$$A'_{12} = (a_k^{(1)};\alpha_k^{(1)})_{1,p^{(1)}},(\mu_j\sigma_j^{(1)};\mu_j\lambda_j^{(1)})_{1,l};\cdots;(a_k^{(r)};\alpha_k^{(r)})_{1,p^{(r)}},(\mu_j\sigma_j^{(r)};\mu_j\lambda_j^{(r)})_{1,l} \tag{3.108}$$

$$B'_{12} = (b_k^{(1)};\beta_k^{(1)})_{1,q^{(1)}};\cdots;(b_k^{(r)};\beta_k^{(r)})_{1,q^{(r)}} \tag{3.109}$$

We obtain the following formula

$$I_{U:p_r,q_r;Y_{12}}^{V;0,n_r;X_{12}}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}_{12} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}:\mathrm{B}_{12} \\ \mathrm{z}_r & \end{array}\right) = \prod_{m=1}^{l}\frac{1}{(\mu_m)^r}\, I_{U:p_r,q_r;Y'_{12}}^{V;0,n_r;X'_{12}}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}'_{12} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}:\mathrm{B}'_{12} \\ \mathrm{z}_r & \end{array}\right) \tag{3.110}$$

for $\mu_m > 0, m = 1,\cdots,l.$

Theorem 13

Let

$$X_{13} = m^{(1)} + l, n^{(1)};\cdots;m^{(r)} + l, n^{(r)} \tag{3.111}$$

$$Y_{13} = p^{(1)} + l, q^{(1)} + 2l;\cdots;p^{(r)} + l, q^{(r)} + 2l \tag{3.112}$$

$$A_{13} = (a_k^{(1)};\alpha_k^{(1)})_{1,p^{(1)}},(\sigma_j^{(1)};\lambda_j^{(1)})_{1,l};\cdots;(a_k^{(r)};\alpha_k^{(r)})_{1,p^{(r)}},(\sigma_j^{(r)};\lambda_j^{(r)})_{1,l} \tag{3.113}$$

$$B_{13}=(1+\sigma_j^{(1)};\lambda_j^{(1)})_{1,l},(b_k^{(1)};\beta_k^{(1)})_{1,q^{(1)}},(\mu\sigma_j^{(1)};\mu\lambda_j^{(1)});\cdots;(1+\sigma_j^{(r)};\lambda_j^{(r)})_{1,l},(b_k^{(r)};\beta_k^{(r)})_{1,q^{(r)}},(\mu\sigma_j^{(r)};\mu\lambda_j^{(r)})_{1,l} \tag{3.114}$$

$$X'_{13}=m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)} \tag{3.115}$$

$$Y'_{13}=p^{(1)}+l,q^{(1)}+l;\cdots;p^{(r)}+l,q^{(r)}+l \tag{3.116}$$

$$A'_{13}=(a_k^{(1)};\alpha_k^{(1)})_{1,p^{(1)}};\cdots;(a_k^{(r)};\alpha_k^{(r)})_{1,p^{(r)}} \tag{3.117}$$

$$B'_{13}=(b_k^{(1)};\beta_k^{(1)})_{1,q^{(1)}},(1+\mu\sigma_j^{(1)};\mu\lambda_j^{(1)});\cdots;(b_k^{(r)};\beta_k^{(r)})_{1,q^{(r)}},(1+\mu\sigma_j^{(r)};\mu\lambda_j^{(r)})_{1,l} \tag{3.118}$$

We obtain the following formula

$$I_{U:p_r,q_r;Y_{13}}^{V;0,n_r;X_{13}}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}_{13} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}:\mathrm{B}_{13} \\ \mathrm{z}_r & \end{array}\right)=\prod_{m=1}^{l}\frac{1}{(\mu_m)^r}\,I_{U:p_r,q_r;Y'_{13}}^{V;0,n_r;X'_{13}}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}'_{13} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}:\mathrm{B}'_{13} \\ \mathrm{z}_r & \end{array}\right) \tag{3.119}$$

for $\mu_m>0, m=1,\cdots,l.$

Theorem 14

Let

$$X_{14}=m^{(1)},n^{(1)}+2l;\cdots;m^{(r)},n^{(r)}+2l \tag{3.120}$$

$$Y_{14}=p^{(1)}+2l,q^{(1)}+l;\cdots;p^{(r)}+2l,q^{(r)}+l \tag{3.121}$$

$$A_{14}=(1+\sigma_j^{(1)};\lambda_j^{(1)})_{1,l},(\mu_j\sigma_j^{(1)};\mu_j\lambda_j^{(1)})_{1,l},(a_k^{(1)};\alpha_k^{(1)})_{1,p^{(1)}};\cdots;(1+\sigma_j^{(r)};\lambda_j^{(r)})_{1,l},(\mu_j\sigma_j^{(r)};\mu_j\lambda_j^{(r)})_{1,l},(a_k^{(r)};\alpha_k^{(r)})_{1,p^{(r)}} \tag{3.122}$$

$$B_{14}=(b_k^{(1)};\beta_k^{(1)})_{1,q^{(1)}},(\sigma_j^{(1)};\lambda_j^{(1)})_{1,l};\cdots;(b_k^{(r)};\beta_k^{(r)})_{1,q^{(r)}},(\sigma_j^{(r)};\lambda_j^{(r)})_{1,l} \tag{3.123}$$

$$X'_{14}=m^{(1)},n^{(1)}+l;\cdots;m^{(r)},n^{(r)}+l \tag{3.124}$$

$$Y'_{14}=p^{(1)}+l,q^{(1)};\cdots;p^{(r)}+l,q^{(r)} \tag{3.125}$$

$$A'_{14}=(1+\mu_j\sigma_j^{(1)};\mu_j\lambda_j^{(1)})_{1,l},(a_k^{(1)};\alpha_k^{(1)})_{1,p^{(1)}};\cdots;(1+\mu_j\sigma_j^{(r)};\mu_j\lambda_j^{(r)})_{1,l},(a_k^{(r)};\alpha_k^{(r)})_{1,p^{(r)}} \tag{3.126}$$

$$B'_{14}=(b_k^{(1)};\beta_k^{(1)})_{1,q^{(1)}};\cdots;(b_k^{(r)};\beta_k^{(r)})_{1,q^{(r)}} \tag{3.127}$$

We obtain the following formula

$$I_{U:p_r,q_r;Y_{14}}^{V;0,n_r;X_{14}}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}_{14} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}:\mathrm{B}_{14} \\ \mathrm{z}_r & \end{array}\right)=\prod_{m=1}^{l}\frac{1}{(\mu_m)^r}\,I_{U:p_r,q_r;Y'_{14}}^{V;0,n_r;X'_{14}}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}'_{14} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}:\mathrm{B}'_{14} \\ \mathrm{z}_r & \end{array}\right) \tag{3.128}$$

for $\mu_m>0, m=1,\cdots,l.$

Theorem 15

Let

$$X_{15} = m^{(1)} + l, n^{(1)} + l; \cdots ; m^{(r)} + l, n^{(r)} + l \tag{3.129}$$

$$Y_{15} = p^{(1)} + l, q^{(1)} + 2l; \cdots ; p^{(r)} + l, q^{(r)} + 2l \tag{3.130}$$

$$A_{15} = (1 + \sigma_j^{(1)}; \lambda_j^{(1)})_{1,l}, (a_k^{(1)}; \alpha_k^{(1)})_{1,p^{(1)}}; \cdots ; (1 + \sigma_j^{(r)}; \lambda_j^{(r)})_{1,l}, (a_k^{(r)}; \alpha_k^{(r)})_{1,p^{(r)}} \tag{3.131}$$

$$B_{15} = (1 + \mu\sigma_j^{(1)}; \mu\lambda_j^{(1)})_{1,l}, (b_k^{(1)}; \beta_k^{(1)})_{1,q^{(1)}}, (\sigma_j^{(1)}; \lambda_j^{(1)}); \cdots ; (1 + \mu\sigma_j^{(r)}; \mu\lambda_j^{(r)})_{1,l}, (b_k^{(r)}; \beta_k^{(r)})_{1,q^{(r)}}, (\sigma_j^{(r)}; \lambda_j^{(r)})_{1,l} \tag{3.132}$$

$$X'_{15} = m^{(1)} + l, n^{(1)}; \cdots ; m^{(r)} + l, n^{(r)} \tag{3.133}$$

$$Y'_{15} = p^{(1)}, q^{(1)} + l; \cdots ; p^{(r)}, q^{(r)} + l \tag{3.134}$$

$$A'_{15} = (a_k^{(1)}; \alpha_k^{(1)})_{1,p^{(1)}}; \cdots ; (a_k^{(r)}; \alpha_k^{(r)})_{1,p^{(r)}} \tag{3.135}$$

$$B'_{15} = (\mu_j\sigma_j^{(1)}; \mu_j\lambda_j^{(1)})_{1,l}, (b_k^{(1)}; \beta_k^{(1)})_{1,q^{(1)}}; \cdots ; (\mu_j\sigma_j^{(r)}; \mu_j\lambda_j^{(r)})_{1,l}, (b_k^{(r)}; \beta_k^{(r)})_{1,q^{(r)}} \tag{3.136}$$

We obtain the following formula

$$I_{U:p_r,q_r;Y_{15}}^{V;0,n_r;X_{15}}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}_{15} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}:\mathrm{B}_{15} \\ \mathrm{z}_r & \end{array}\right) = \prod_{m=1}^{l} \frac{1}{(-\mu_m)^r}\, I_{U:p_r,q_r;Y'_{15}}^{V;0,n_r;X'_{15}}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}'_{15} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}:\mathrm{B}'_{15} \\ \mathrm{z}_r & \end{array}\right) \tag{3.137}$$

for $\mu_m > 0, m = 1, \cdots, l.$

Theorem 16

Let

$$X_{16} = m^{(1)}, n^{(1)} + l; \cdots ; m^{(r)}, n^{(r)} + l \tag{3.138}$$

$$Y_{16} = p^{(1)} + 2l, q^{(1)} + l; \cdots ; p^{(r)} + 2l, q^{(r)} + l \tag{3.139}$$

$$A_{16} = (\sigma_j^{(1)}; \lambda_j^{(1)})_{1,l}, (a_k^{(1)}; \alpha_k^{(1)})_{1,p^{(1)}}, (1 + \mu\sigma_j^{(1)}; \mu\lambda_j^{(1)})_{1,l}; \cdots ; (\sigma_j^{(r)}; \lambda_j^{(r)})_{1,l}, (a_k^{(r)}; \alpha_k^{(r)})_{1,p^{(r)}}, (1 + \mu\sigma_j^{(r)}; \mu\lambda_j^{(r)})_{1,l} \tag{3.140}$$

$$B_{16} = (b_k^{(1)}; \beta_k^{(1)})_{1,q^{(1)}}, (1 + \sigma_j^{(1)}; \lambda_j^{(1)})_{1,l}; \cdots ; (b_k^{(r)}; \beta_k^{(r)})_{1,q^{(r)}}, (1 + \sigma_j^{(r)}; \lambda_j^{(r)})_{1,l} \tag{3.141}$$

$$X'_{16} = m^{(1)}, n^{(1)}; \cdots ; m^{(r)}, n^{(r)} \tag{3.142}$$

$$Y'_{16} = p^{(1)} + l, q^{(1)}; \cdots ; p^{(r)} + l, q^{(r)} \tag{3.143}$$

$$A'_{16} = (a_k^{(1)}; \alpha_k^{(1)})_{1,p^{(1)}}, (\mu_j\sigma_j^{(1)}; \mu_j\lambda_j^{(1)})_{1,l}; \cdots ; (a_k^{(r)}; \alpha_k^{(r)})_{1,p^{(r)}}, (\mu_j\sigma_j^{(r)}; \mu_j\lambda_j^{(r)})_{1,l} \tag{3.144}$$

$$B'_{14} = (b_k^{(1)}; \beta_k^{(1)})_{1,q^{(1)}}; \cdots ; (b_k^{(r)}; \beta_k^{(r)})_{1,q^{(r)}} \tag{3.145}$$

We obtain the following formula

$$I_{U:p_r,q_r;Y_{16}}^{V;0,n_r;X_{16}}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}_{15} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}:\mathrm{B}_{15} \\ \mathrm{z}_r & \end{array}\right) = \prod_{m=1}^{l}\frac{1}{(\mu_m)^r}\, I_{U:p_r,q_r;Y'_{16}}^{V;0,n_r;X'_{16}}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{A}'_{16} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}:\mathrm{B}'_{16} \\ \mathrm{z}_r & \end{array}\right) \tag{3.146}$$

for $\mu_m > 0, m = 1, \cdots, l$.

To prove the theorems 2 to 16, we use the similar method that (3.9).

Remark :

We obtain the same relations with the multivariable H-function [6,7], the H-function of two variables [5].

4. Conclusion

In this paper, we have obtained sixteen reductions involving the multivariable I-function. This function is quite general in nature. Therefore, on specializing the parameters of I-function of several variables, we may obtain various reductions concerning the special functions of one and several variables.

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