Int Jr. of Mathematics Sciences & Applications
Vol. 2, No. 2, May 2012

ISSN No: 2230-9888

OPERATION PRINCIPLE OF INDEFINITE INTEGRALS

ZHOU Qin

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF JINAN,
JINAN -250022, SHANDONG PROVINCE, CHINA

Abstract

Some of the operations among indefinite integrals can be regarded as operations of sets. There is no precise definition or reasonable interpretation to this kind of operations. It may be due to coincidence that the missing of the definition did not affect the correctness of relevant calculation. Thus, the theoretical and logic problems caused by it have long been ignored. Taking "+" operation between indefinite integrals as an example, this paper constructs the quotient group $(C^2(I)/R_{\forall(x)};+^2)$ ***and tries to prove that the binary operation "***$+'$***"is the "+" operation in the indefinite integral*** $\int f(x)dx+\int g(x)dx$***.***

Keywords: calculus, indefinite integral, congruence relation, quotient group

1. INTRODUCTION

In calculus, the following are some basic equations [1]:

$$\int cf(x)dx = c\int f(x)dx \qquad (1)$$

$$\int[f(x)+g(x)]dx=\int f(x)dx+\int g(x)dx \qquad (2)$$

$$\int[f(x)-g(x)]dx = \int f(x)dx-\int g(x)dx \qquad (3)$$

According to the definition of indefinite integral [2]: any antiderivative of $f(x)$ on the interval I is called indefinite integral of $f(x)$ on I, expressed in the form $\int f(x)dx$. We can understand the indefinite integral $\int f(x)dx$ as sets or family of functions. Then, it is not hard to note that there is an obvious loophole in the equation. For example, the "+" at both ends of Equation (2) stand for different operations: the left "+" is the operation among sets (or function families), while the right one is the ordinary additive operation of functions. Using the same sign to express two operations of different elements, confusion is inevitable. Such a problem as follows can be found in Article [3],

$$\int e^x \sin xdx = \int \sin xde^x = e^x \sin x - \int e^x \cos xdx$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x dx \qquad (4)$$

Then, $2\int e^x \sin x dx = e^x \sin x - e^x \cos x + c_1 \qquad (5)$

$$\int e^x \sin xdx = \frac{1}{2} e^x (\sin x - \cos x) + c \qquad (6)$$

Thus, how to explain coefficient 2 of $\int e^x \sin xdx$ in equation (5)? According to the definition of indefinite integral, $\int e^x \sin xdx$ is a function set. The union of two same sets should be the same set, so coefficient 2 hear is illogical.

Here we should point out a mistake. The "+" in the equation is not defined as the union of sets (albeit quite a few treatises on sets define "+" as the union of sets). Thus, the starting point of this article is not accurate. Actually, the key to this problem is to find a reasonable definition of "+".

Then how to define the operation $\int f(x)dx + \int g(x)dx$? Here we will apply some algebraic theories to this question so as to make an accurate interpretation of the above problem in teaching and learning indefinite integral.

2. DEFINITION OF INDEFINITE INTEGRAL

2.1. Quotient group [4] [5]

In the study of the algebraic structure of a set, an "Equivalence Relation" is often used to decompose sets into subsets as well as "Congruence Relation" with respect to operations in sets when operations are needed in sets. With these concepts, we obtain subset (equivalence class) and quotient set. For group (a set with operation), they refer to subgroup and quotient group.

Definition 2.1: If there is a property R which makes any two elements a, b in the set A either "have the property R" or "do not have the property R", it must be one or the other, which means "R" gives a relation to A. If a, b has property R, a and b are said to have relations, denoted as aRb; and if a, b does not have property R, a is said not related to b, denoted as $a\not R b$.

For example, when R is used to represent the relationship "$\le$" in the set of real numbers, we have $3R6, 3R3, 6\not R3$. And the relationship "=" in the set of real numbers can also be generalized as the equivalence relation of general sets.

Definition 2.2: If a relation R in set A meets the following conditions:

(1) Reflexivity: $aRa, \quad a \in A$;

(2) Symmetry: $aRb \Rightarrow bRa, \quad a, b \in A$;

(3) Transitivity: $aRb, bRc \Rightarrow aRc, \quad a, b, c \in A$,

Then, the relation R is known as an equivalence relation of A. The set composed by all elements equivalent to a is called the equivalence class represented by a, denoted by $[a]_R$. The set $\{[a]_R | a \in A\}$, composed by all the equivalence classes (only take one if repeated) in A, is called quotient set of A for R, denoted as A/R.

For example, in the n-order complex matrix set, "congruence" is an equivalence relation, so is "similarity". Thus, there can be several different equivalence relations in one set.

Definition 2.3: Given a binary operation "$\bullet$" in set A, if an equivalence relation R of A is maintained in this operation, that is,

$aRb, cRd \Rightarrow (a \bullet c)R(b \bullet d), \quad a, b, c, d \in A$

Then, R is known as a "congruence relation" with respect to operation "•" in A. At this point, the equivalence class $[a]_R$ represented by a is also called congruence class of a.

Theorem 2.1[6] if $(A;\forall)$ is a group, in which • is the binary operation on A, e is an identity element, and R is the congruence relation on A, then the quotient algebra $(A/R;\forall^{2})$ of algebraic system $(A;\bullet)$ with respect to R is a group, in which the binary operation $\forall^{2}$ in set A/R is defined as $[a]_R \forall^{2} [b]_R = [a\forall b]_R$.

Prove:

(1) $\forall^{2}$ satisfies the closure law, which is obvious.

(2) To $a,b,c \square A$, there is $([a]_R \forall^{2} [b]_R) \forall^{2} [c]_R = [a\forall b]_R \forall^{2} [c]_R = [(a\forall b)\forall c]_R$ $[a\forall (b\forall c)]_R = [a]_R \forall^{2} [b\forall c]_R = [a]_R \forall^{2} ([b]_R \forall^{2} [c]_R)$, so $\forall^{2}$ satisfies the associative law;

(3) To $a \square A$, $[e]_R \forall^{2} [a]_R = [e\forall a]_R = [a]_R = [a\forall e]_R = [a]_R \forall^{2} [e]_R$, $[e]_R$ is the identity element of $\forall^{2}$;

(4) $(A;\forall)$ is a group, to $a \square A$, there is inverse element $a^{-1} \square A$, which makes $a^{-1}\forall a = a\forall a^{-1} = e$, so

$[a]_R \forall^{2} [a^{-1}]_R = [a\forall a^{-1}]_R = [e]_R = [a^{-1}\forall a]_R = [a^{-1}]_R \forall^{2} [a]_R$. Therefore, there exists an inverse element of $[a]_R \square A/R$, that is, $[a]_R^{-1} = [a^{-1}]_R$.

In summary, $(A/R;\forall^{2})$ is group?

Definition 2.4: the group $(A/R;\forall^{2})$ is called the quotient group of $(A;\forall)$ with respect to congruence relation R, referred to as quotient group.

2.2. Indefinite integral and quotient group

If function set is designated as

$C^{2}(I) = \{F(x) | F(x)\ is\ differentiable\ at\ any\ point\ on\ I\}$, and property $R_{\forall(x)}$ has the same derived function on I, then $R_{\forall(x)}$ is an equivalence relation of $C^{2}(I)$. If a binary operation "+" in $C^{2}(I)$ is an ordinary addition of two functions, then $R_{\bullet(x)}$ is a congruence relation about "+" in $C^{2}(I)$. It is easy to verify that the algebraic system $(C^{2}(I);+)$ is a group, whose identity element is 0 function, the inverse element to $F(x) \square C^{2}(I)$ is $\updownarrow F(x)$ and its quotient group about congruence relation $R_{\forall(x)}$ is $(C^{2}(I)/R_{\forall(x)};+^{2})$.

If $F^{2}(x)=f(x)$, the congruence class represented by $F(x)$ is marked as $[F(x)]_{R_{\forall\ (x)}}$. And $+f(x)dx$ denotes all the antiderivatives of $f(x)$, then,

$$+f(x)dx=[F(x)]_{R_{\forall\ (x)}} \tag{7}$$

So far, we can see clearly that $+f(x)dx$ are the elements that construct the quotient set $C^{2}(I)/R_{\forall\ (x)}=\{[F(X)]_{R\forall\ (x)}\big|F(x)\,◘\,C^{2}(I)\}$. Of course, $+f(x)dx$ can also be regarded as the elements that construct the quotient group $(C^{2}(I)/R_{\forall\ (x)}\ ;+^{2})$.

3. ADDITIONAL PRINCIPLE OF INDEFINITE INTEGRAL

If $F(x),G(x)\,◘\,C^{2}(I)$ and $F^{2}(x)=f(x)$, $G^{2}(x)=g(x)$, then $+f(x)dx=[F(x)]_{R_{\forall\ (x)}}$, $+g(x)dx=[G(x)]_{R_{\forall\ (x)}}$.

Thus, in $(C^{2}(I)/R_{\forall\ (x)}\ ;+^{2})$, there is

$+f(x)dx+^{2}+g(x)dx=[F(x)+G(x)]_{R_{\forall\ (x)}}=[F(x)]_{R_{\forall\ (x)}}+^{2}[G(x)]_{R_{\forall\ (x)}}=+(f(x)+g(x))dx$, each of which has $\Pi(x)=F(x)+C$ to $\Pi(x)\,◘[F(x)]_{R\forall\ (x)}$.

It can be denoted as $[F(x)]_{R\forall\ (x)}=F(x)+C$.

Thus, $+f(x)dx+^{2}+g(x)dx=+(f(x)+g(x))dx=F(x)+G(x)+C$. (8)

It is obvious that the operator "+" on the left and right ends has different meaning: the former refers to the operation among equivalence classes, but the latter refers to the ordinary addition among functions. It is mere that the indefinite integral group $(C^{2}(I)/R_{\forall\ (x)}\ ;+^{2})$ is composed by the operations derived by congruence relation $R_{\forall\ (x)}$ of quotient set $C^{2}(I)/R_{\forall(x)}$, and operations in $(C^{2}(I)/R_{\forall\ (x)}\ ;+^{2})$ are those between equivalence classes and belong to the operations of representative element in equivalence class. This is the reason why the correct results can be obtained in the analysis and calculation of indefinite integral even though the expression and definition of "$+^{2}$" and "+" are not differentiated.

4. SUBTRACTION AND MULTIPLICATION OF INDEFINITE INTEGRAL

The ordinary subtraction operation "-" of two functions and the operation of multiplication functions "$k\forall\ f(x)$" (referred to as "$\forall$") in $C^{2}(I)$, $R_{\forall(x)}$ is the congruence relation about "-" and "$\forall$" in $C^{2}(I)$. Here, the algebraic system $(C^{2}(I)\,;\,\updownarrow)$、$(C^{2}(I)\,;\forall\,)$ are groups, whose identity element is 0 function，the inverse element to $F(x) \square C^2(I)$ is $\updownarrow F(x)$，and its quotient group about congruence relation $R_{\forall(x)}$ is $(C^2(I)/R_{\forall(x)};\updownarrow^2)$ and $(C^2(I)/R_{\forall(x)};\forall^2)$. In accordance with the above method, the operation principle of $+f(x)dx\updownarrow^2+g(x)dx=+(f(x)\updownarrow g(x))dx=(F(x)\updownarrow G(x))+C$ and $k\forall+f(x)dx=+kf(x)dx=kF(x)+C$ can also be explained.

To the ordinary multiplication operation “$\times$” of two functions in $C^2(I)$, $R_{\forall(x)}$ is not the congruence relation about “$\times$” in $C^2(I)$, so there is no operation similar to $+f(x)dx\cdot^2+g(x)dx=+(f(x)\cdot g(x))dx$.

5. CONCLUSION

In reality, in calculus courses for Mathematics majors, indefinite integral can be defined by congruence class $[F(x)]_{R_{\forall(x)}}$ [7]. But the content related to set theory should be added to the textbooks. As for non-mathematics majors, the original definition can be adopted in their calculus courses. However, it is necessary to make the principles outlined in this article as an explanatory note for teachers’ reference.

REFERENCES

[1] H. Anton, I. Bivens, and S.Davis, Calculus, John Wiley & Sons, Inc., New York, (2005)

[2] Tongji University., Advanced Mathematics. Higher Education Press, Beijing (2002)

[3]Pan, Dingkun, Some Common Flaws in Advanced Mathematics Textbooks. Xi’an Jiaotong University Press, Xi’an, (2006)

[4] Xu, Zongben, From College Mathematics to Modern Mathematics. Science Press, Beijing (2007)

[5] Ni, Ziwei, Cai, Jingqiu, Discrete Mathematics. Science Press, Beijing, (2001)

[6] Zhang, Zhouping, Jiao, Zhanya & Hu, Yupu, Congruence Relation and Quotient Group. Journal of Shanxi University of Science & Technology, l22,6, (2004)128-130.

[7] Zhou, Qin, A More Precise Statement of the Definition of Indefinite Integral, College Mathematics, 26,4, (2010) 176-177.