Int. J. of Mathematical Sciences and Applications,
Vol. 1, No. 3, September 2011

www.journalshub.com

A simplified contour integration for the Probability integral.

Abstract

In this note, we present a self-contained and straightforward contour integration to evaluate the probability integral. The method will not only be suitable for undergraduate students in a beginning complex analysis course who are interested in applying contour integration to evaluate the integral; but would provide a response to common criticisms of older articles that used contour integration to evaluate the probability integral.

Keywords: contour integration; probability integral

The value of the probability integral $\int_{-\infty}^{\infty} e^{-x^2}\, dx$ is well known and has numerous applications in mathematics and statistics. The integral also has well known but indirect methods of evaluation that include [**1-3, 6-7, 9-10,12**]. The method that is frequently applied in most undergraduate course materials in Probability and statistics, switches the integral to a double integral involving two variables, then transforms the associated double integral into one involving polar coordinates. Too often, the background justifying the steps in these conversions are omitted (see, for example,[**11, pp. 204-211**]). The background includes the Fubini theorem and the Jacobi's theorem [**12**]. A statement and discussion of these theorems are found in many real analysis texts (see, for example, [**5, pp. 296-297**] and [**4, pp. 346-348**] respectively).

Although there are very few published methods for evaluating the probability integral that use contour integration techniques, they lack details that would facilitate understanding of the derivations by students of introductory complex analysis courses (see [**1,3,9**]). In this note, we present a self-contained contour integral evaluation of the probability integral that was motivated by (1) recent criticism by Desbrow [**3**] of the articles [**1**] and [**9**], and (2) our desire to give a straightforward procedure suitable for undergraduate students in a beginning complex variable course, who are interested in using contour integration to evaluate the probability integral. The evaluation builds an integrand and a contour suggested in [**8, p. 304**].

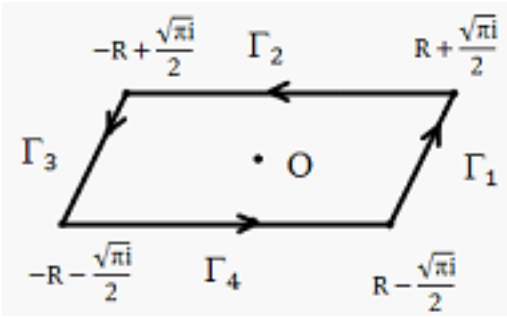

Figure 1

Consider

$$\lim_{R\to\infty} \oint_{\Gamma} \frac{e^{-z^2+\sqrt{\pi}iz}}{e^{2\sqrt{\pi}iz}-1}\, dz,$$

where Γ is the positively oriented closed contour in Figure 1, consisting of the line segments Γ_1 from $z = R - \frac{\sqrt{\pi i}}{2}$ to $z = R + \frac{\sqrt{\pi i}}{2}$, Γ_2 from $z = R + \frac{\sqrt{\pi i}}{2}$ to $z = -R + \frac{\sqrt{\pi i}}{2}$, Γ_3 from $z = -R + \frac{\sqrt{\pi i}}{2}$ to $z = -R - \frac{\sqrt{\pi i}}{2}$, and Γ_4 from $z = -R - \frac{\sqrt{\pi i}}{2}$ to $z = R - \frac{\sqrt{\pi i}}{2}$; and $\sqrt{\pi i} = \sqrt{\pi} e^{\frac{\pi i}{4}}$.

Observe that $z_0 = 0$ is the only simple pole of $f(z) = \frac{e^{-z^2+\sqrt{\pi}iz}}{e^{2\sqrt{\pi}iz}-1}$ in the interior of the closed contour Γ; so

$$Res(f(z),0) = \lim_{z\to 0} zf(z) = \frac{1}{2\sqrt{\pi}i};\ and\ \oint_\Gamma f(z)\,dz\ =\ 2\pi i Res(f(z),0)\ = \sqrt{\pi}i = \sqrt{\pi}\,e^{\frac{\pi i}{4}}\ = \frac{\sqrt{\pi}}{\sqrt{2}}(1+i);$$

and

$$\lim_{R\to\infty}\oint_\Gamma f(z)\,dz\ =\ \lim_{R\to\infty}\left(\int_{\Gamma_1} f(z)\ dz + \int_{\Gamma_2} f(z)\ dz + \int_{\Gamma_3} f(z)\ dz + \int_{\Gamma_4} f(z)\ dz\right) = \frac{\sqrt{\pi}}{\sqrt{2}}(1+i). \qquad (1)$$

We will now show that

$$\lim_{R\to\infty}\int_{\Gamma_1} f(z)\,dz = \lim_{R\to\infty}\int_{\Gamma_3} f(z)\,dz = 0,\ and\ \lim_{R\to\infty}\int_{\Gamma_2} f(z)\,dz + \lim_{R\to\infty}\int_{\Gamma_4} f(z)\,dz = \frac{1+i}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-x^2}\,dx.$$

Using $z = z_1 + t(z_2 - z_1)$ with $t \in (-\infty,\infty)$, as a representation of the equation of a line through any points z_1 and z_2 in the complex plane, we have:

$$\begin{aligned}
\Gamma_1:\ \ z &= R - \tfrac{\sqrt{\pi}}{2\sqrt{2}} - \tfrac{\sqrt{\pi}}{2\sqrt{2}}i + t\left(\tfrac{\sqrt{\pi}}{\sqrt{2}} + \tfrac{\sqrt{\pi}}{\sqrt{2}}i\right) = R - \tfrac{\sqrt{\pi}}{2}e^{\frac{\pi i}{4}} + t\sqrt{\pi}e^{\frac{\pi i}{4}}\ for\ 0 \le t \le 1,\\
\Gamma_2:\ \ z &= \left(\tfrac{\sqrt{\pi}}{2\sqrt{2}} + t\right) + \tfrac{\sqrt{\pi}}{2\sqrt{2}}i\ for\ t = R\ to\ t = -R,\\
\Gamma_3:\ \ z &= -R + \tfrac{\sqrt{\pi}}{2\sqrt{2}} + \tfrac{\sqrt{\pi}}{2\sqrt{2}}i + s\left(-\tfrac{\sqrt{\pi}}{\sqrt{2}} - \tfrac{\sqrt{\pi}}{\sqrt{2}}i\right) = -R + \tfrac{\sqrt{\pi}}{2}e^{\frac{\pi i}{4}} - s\sqrt{\pi}e^{\frac{\pi i}{4}}\ for\ 0 \le s \le 1,\ and\\
\Gamma_4:\ \ z &= \left(\tfrac{-\sqrt{\pi}}{2\sqrt{2}} + t\right) - \tfrac{\sqrt{\pi}}{2\sqrt{2}}i\ for\ t = -R\ to\ t = R.
\end{aligned}$$

On Γ_1,

$$\begin{aligned}
\sqrt{\pi}iz &= \tfrac{\sqrt{\pi}}{\sqrt{2}}\left[R + i\left(R - \tfrac{\sqrt{\pi}}{\sqrt{2}} + 2t\tfrac{\sqrt{\pi}}{\sqrt{2}}\right)\right] = \tfrac{\sqrt{\pi}}{\sqrt{2}}R + i\tfrac{\sqrt{\pi}}{\sqrt{2}}R - i\tfrac{\pi}{2} + it\pi,\\
\mathrm{z}^2 &= R^2 - \tfrac{\sqrt{\pi}}{\sqrt{2}}R + \sqrt{2\pi}Rt - i\tfrac{\sqrt{\pi}}{\sqrt{2}}R + i\tfrac{\pi}{4} - i\pi t + i\sqrt{2\pi}Rt + i\pi t^2,\\
\text{-}\mathrm{z}^2 + \sqrt{\pi}iz &= -R^2 + \sqrt{2\pi}R - \sqrt{2\pi}Rt + i(-\tfrac{3\pi}{4} + 2\pi t + \sqrt{2\pi}Rt - \pi t^2),\\
\left|e^{-z^2+\sqrt{\pi}iz}\right| &= e^{-R^2+\sqrt{2\pi}R-\sqrt{2\pi}Rt} = \tfrac{e^{\sqrt{2\pi}R}}{e^{R^2}e^{\sqrt{2\pi}Rt}} \le \tfrac{e^{\sqrt{2\pi}R}}{e^{R^2}}\ for\ 0 \le t \le 1, \qquad (2)\\
\left|e^{2\sqrt{\pi}iz} - 1\right| &= \left|e^{\sqrt{2\pi}R}e^{i\left(\sqrt{2\pi}R+\pi-2t\pi\right)} - 1\right| \ge \left||e^{\sqrt{2\pi}R}||e^{i\left(\sqrt{2\pi}R+\pi-2t\pi\right)}| - 1\right| = \left|e^{\sqrt{2\pi}R} - 1\right|,\\
and\ \tfrac{1}{\left|e^{2\sqrt{\pi}iz}-1\right|} &\le \tfrac{1}{\left|e^{\sqrt{2\pi}R}-1\right|}. \qquad (3)
\end{aligned}$$

So by combining (2) and (3) we obtain

$$\left|\int_{\Gamma_1}\frac{e^{-z^2+\sqrt{\pi}iz}}{e^{2\sqrt{\pi}iz}-1}\,dz\right| \le \int_{\Gamma_1}\frac{e^{\sqrt{2\pi}R}|dz|}{e^{R^2}\left|e^{\sqrt{2\pi}R}-1\right|} = \int_0^1\frac{\sqrt{\pi}dt}{e^{R^2}\left|1-e^{-\sqrt{2\pi}R}\right|} = \frac{\sqrt{\pi}}{e^{R^2}\left|1-e^{-\sqrt{2\pi}R}\right|}\ \to\ 0\ as\ R\to\infty,$$

and

$$\lim_{R\to\infty}\int_{\Gamma_1}\frac{e^{-z^2+\sqrt{\pi}iz}}{e^{2\sqrt{\pi}iz}-1}\,dz\ =\ 0. \qquad (4)$$

Similarly on Γ_3,

$$\begin{aligned}
\sqrt{\pi}iz &= \frac{\sqrt{\pi}}{\sqrt{2}}\left[-R+i\left(-R+\frac{\sqrt{\pi}}{\sqrt{2}}-2s\frac{\sqrt{\pi}}{\sqrt{2}}\right)\right] = -\frac{\sqrt{\pi}}{\sqrt{2}}R - i\frac{\sqrt{\pi}}{\sqrt{2}}R + i\frac{\pi}{2} - is\pi,\\
\text{z}^2 &= R^2 - \frac{\sqrt{\pi}}{\sqrt{2}}R + \sqrt{2\pi}Rs - i\frac{\sqrt{\pi}}{\sqrt{2}}R + i\frac{\pi}{4} - i\pi s + i\sqrt{2\pi}Rs + i\pi s^2,\\
\text{-z}^2 + \sqrt{\pi}iz &= -R^2 - \sqrt{2\pi}Rs + i(\tfrac{\pi}{4} - \sqrt{2\pi}Rs - \pi s^2),\\
\left|e^{-z^2+\sqrt{\pi}iz}\right| &= e^{-R^2-\sqrt{2\pi}Rs} = \frac{1}{e^{R^2}e^{\sqrt{2\pi}Rs}} \le \frac{1}{e^{R^2}}\ for\ 0 \le s \le 1, \qquad (5)\\
\left|e^{2\sqrt{\pi}iz}-1\right| &= \left|e^{-\sqrt{2\pi}R}e^{i\left(-\sqrt{2\pi}R+\pi-2s\pi\right)}-1\right| \ge \left|\left|e^{-\sqrt{2\pi}R}\right|\left|e^{i\left(-\sqrt{2\pi}R+\pi-2s\pi\right)}\right|-1\right| = \left|e^{-\sqrt{2\pi}R}-1\right|,\\
and\ \frac{1}{\left|e^{2\sqrt{\pi}iz}-1\right|} &\le \frac{1}{\left|e^{-\sqrt{2\pi}R}-1\right|}. \qquad (6)
\end{aligned}$$

Combining (5) and (6), we have

$$\left|\int_{\Gamma_3}\frac{e^{-z^2+\sqrt{\pi}iz}}{e^{2\sqrt{\pi}iz}-1}\,dz\right| \le \int_{\Gamma_3}\frac{|dz|}{e^{R^2}\left|e^{-\sqrt{2\pi}R}-1\right|} = \int_0^1\frac{\sqrt{\pi}ds}{e^{R^2}\left|e^{-\sqrt{2\pi}R}-1\right|} = \frac{\sqrt{\pi}}{e^{R^2}\left|e^{-\sqrt{2\pi}R}-1\right|} \to 0\ as\ R\to\infty,$$

and

$$\lim_{R\to\infty}\int_{\Gamma_3}\frac{e^{-z^2+\sqrt{\pi}iz}}{e^{2\sqrt{\pi}iz}-1}\,dz = 0. \qquad (7)$$

On Γ_2,

$$\sqrt{\pi}iz = \frac{\sqrt{\pi}}{\sqrt{2}}t + it\frac{\sqrt{\pi}}{\sqrt{2}} + i\frac{\pi}{2}, \quad \text{-z}^2 = -\left[\left(\frac{\sqrt{\pi}}{2\sqrt{2}}+t\right)+\frac{\sqrt{\pi}}{2\sqrt{2}}i\right]^2 = -\frac{\sqrt{\pi}}{\sqrt{2}}t - t^2 - \frac{\pi}{4}i - \frac{\sqrt{\pi}}{\sqrt{2}}ti,$$

and $\text{-z}^2 + \sqrt{\pi}iz = -t^2 + \frac{\pi}{4}i.$

Also, on Γ_4

$$\sqrt{\pi}iz = \frac{\sqrt{\pi}}{\sqrt{2}}t + it\frac{\sqrt{\pi}}{\sqrt{2}} - i\frac{\pi}{2}, \quad \text{-z}^2 = -\left[-\frac{\sqrt{\pi}}{2\sqrt{2}}+t-\frac{\sqrt{\pi}}{2\sqrt{2}}i\right]^2 = \frac{\sqrt{\pi}}{\sqrt{2}}t - t^2 - \frac{\pi}{4}i + \frac{\sqrt{\pi}}{\sqrt{2}}ti,$$

and $\text{-z}^2 + \sqrt{\pi}iz = 2\frac{\sqrt{\pi}}{\sqrt{2}}t - t^2 + it\frac{2\sqrt{\pi}}{\sqrt{2}} - \frac{3\pi}{4}i.$

Hence,

$$\begin{aligned}
\int_{\Gamma_2} f(z)\,dz + \int_{\Gamma_4} f(z)\,dz &= \int_R^{-R}\frac{e^{(-t^2+\frac{\pi}{4}i)}}{e^{(2\frac{\sqrt{\pi}}{\sqrt{2}}t+it\frac{2\sqrt{\pi}}{\sqrt{2}}+i\pi)}-1}\,dt + \int_{-R}^{R}\frac{e^{(2\frac{\sqrt{\pi}}{\sqrt{2}}t-t^2+it\frac{2\sqrt{\pi}}{\sqrt{2}}-\frac{3\pi}{4}i)}}{e^{(2\frac{\sqrt{\pi}}{\sqrt{2}}t+it\frac{2\sqrt{\pi}}{\sqrt{2}}-i\pi)}-1}\,dt.\\
&= \int_R^{-R}\frac{e^{(-t^2+\frac{\pi}{4}i)}}{-e^{(2\frac{\sqrt{\pi}}{\sqrt{2}}t+it\frac{2\sqrt{\pi}}{\sqrt{2}})}-1}\,dt + \int_{-R}^{R}\frac{e^{(2\frac{\sqrt{\pi}}{\sqrt{2}}t-t^2+it\frac{2\sqrt{\pi}}{\sqrt{2}}-\frac{3\pi}{4}i)}}{-e^{(2\frac{\sqrt{\pi}}{\sqrt{2}}t+it\frac{2\sqrt{\pi}}{\sqrt{2}})}-1}\,dt.\\
&= \int_{-R}^{R}\frac{e^{(-t^2+\frac{\pi}{4}i)}}{e^{(2\frac{\sqrt{\pi}}{\sqrt{2}}t+it\frac{2\sqrt{\pi}}{\sqrt{2}})}+1}\,dt - \int_{-R}^{R}\frac{e^{(2\frac{\sqrt{\pi}}{\sqrt{2}}t-t^2+it\frac{2\sqrt{\pi}}{\sqrt{2}}-\frac{3\pi}{4}i)}}{e^{(2\frac{\sqrt{\pi}}{\sqrt{2}}t+it\frac{2\sqrt{\pi}}{\sqrt{2}})}+1}\,dt.\\
&= \int_{-R}^{R}\frac{e^{(-t^2-\frac{3\pi}{4}i)}[e^{\pi i}-e^{(2\frac{\sqrt{\pi}}{\sqrt{2}}t+it\frac{2\sqrt{\pi}}{\sqrt{2}})}]}{e^{(2\frac{\sqrt{\pi}}{\sqrt{2}}t+it\frac{2\sqrt{\pi}}{\sqrt{2}})}+1}\,dt\\
&= e^{(-\frac{3\pi}{4}i)}\int_{-R}^{R}\frac{e^{-t^2}[-1-e^{(2\frac{\sqrt{\pi}}{\sqrt{2}}t+it\frac{2\sqrt{\pi}}{\sqrt{2}})}]}{e^{(2\frac{\sqrt{\pi}}{\sqrt{2}}t+it\frac{2\sqrt{\pi}}{\sqrt{2}})}+1}\,dt
\end{aligned}$$

$$= -e^{(-\frac{3\pi}{4}i)} \int_{-R}^{R} \frac{e^{-t^2}[1 + e^{(2\frac{\sqrt{\pi}}{\sqrt{2}}t + it\frac{2\sqrt{\pi}}{\sqrt{2}})}]}{e^{(2\frac{\sqrt{\pi}}{\sqrt{2}}t + it\frac{2\sqrt{\pi}}{\sqrt{2}})} + 1}\, dt = \frac{1+i}{\sqrt{2}} \int_{-R}^{R} e^{-t^2}\, dt;$$

so that

$$\lim_{R\to\infty} \left[\int_{\Gamma_2} f(z)\, dz \; + \; \int_{\Gamma_4} f(z)\, dz \right] = \frac{1+i}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-t^2}\, dt. \tag{8}$$

Combining (1), (4), (7) and (8) we obtain

$$\lim_{R\to\infty} \left(\int_{\Gamma_1} f(z)\, dz + \int_{\Gamma_2} f(z)\, dz + \int_{\Gamma_3} f(z)\, dz + \int_{\Gamma_4} f(z)\, dz \right) = \frac{1+i}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-t^2}\, dt \; = \frac{\sqrt{\pi}}{\sqrt{2}}(1+i),$$

$$from\ which \int_{-\infty}^{\infty} e^{-t^2}\, dt \; = \sqrt{\pi}.$$

References

[1] J. H. Cadwell, Three Integrals, *The Mathematical Gazette* **31** (1947) 239-240

[2] D. Desbrow, Evaluating the probability integral, *The Mathematical Gazette* **74** (1990) 169-170.

[3] D. Desbrow, On Evaluating $\int_{-\infty}^{\infty} e^{ax(x-2b)}\, dx$ by Contour Integration Round a Parallelogram, *The American Mathematical Monthly* **105** (1998) 726-731.

[4] S.A. Fulling, M.N. Sinyakov, and S.V. Tischchenko, *Linearity and the mathematics of several variables*, World Scientific Pub Co Inc, New Jersey, 2000.

[5] A.I. Khuri, *Advanced Calculus with Applications in Statistics*, John Wiley & Sons Inc., New York, 1993

[6] P. Levrie and W. Daems, Evaluating the Probability Integral Using Wallis's Product Formula π, *The American Mathematical Monthly* **116** (2009) 538-541.

[7] N. Lord, An elementary single-variable proof of $\int_{-\infty}^{\infty} e^{-x^2/2}\, dx \; = \sqrt{2\pi}$, *The Mathematical Gazette* **87** (2003) 308-311.

[8] J.E. Marsden and M.J. Hoffman, *Basic complex analysis*, W.H. Freeman and Company, New York, 1999.

[9] L. Mirsky, The Probability Integral, *The Mathematical Gazette* **33** (1949) 279

[10] C. P. Nicholas and R. C. Yates, The probability integral, *The American Mathematical Monthly* **57** (1950) 412-413.

[11] S. Ross, *A first course in probability*, Prentice-Hall, New Jersey, 1998

[12] R.M. Young, On Evaluating the Probability Integral: A Simple One-Variable, *The Mathematical Gazette* **89**(2005) 252-254.

Author Details:

Philip Wagala Gwanyama
Department of Mathematics
Northeastern Illinois University
5500 N. St. Louis Avenue, Chicago, IL 60625
E-Mail: P-Gwanyama@neiu.edu