A simplified contour integration for the Probability integral.

Abstract

In this note, we present a self-contained and straightforward contour integration to evaluate the probability integral. The method will not only be suitable for undergraduate students in a beginning complex analysis course who are interested in applying contour integration to evaluate the integral; but would provide a response to common criticisms of older articles that used contour integration to evaluate the probability integral.

Keywords: contour integration; probability integral

The value of the probability integral \( \int_{-\infty}^{\infty} e^{-x^2} \, dx \) is well known and has numerous applications in mathematics and statistics. The integral also has well known but indirect methods of evaluation that include [1-3, 6-7, 9-10,12]. The method that is frequently applied in most undergraduate course materials in Probability and statistics, switches the integral to a double integral involving two variables, then transforms the associated double integral into one involving polar coordinates. Too often, the background justifying the steps in these conversions are omitted (see, for example,[11, pp. 204-211]). The background includes the Fubini theorem and the Jacobi’s theorem [12]. A statement and discussion of these theorems are found in many real analysis texts (see, for example, [5, pp. 296-297] and [4, pp. 346-348] respectively).

Although there are very few published methods for evaluating the probability integral that use contour integration techniques, they lack details that would facilitate understanding of the derivations by students of introductory complex analysis courses (see [1,3,9]). In this note, we present a self-contained contour integral evaluation of the probability integral that was motivated by (1) recent criticism by Desbrow [3] of the articles [1] and [9], and (2) our desire to give a straightforward procedure suitable for undergraduate students in a beginning complex variable course, who are interested in using contour integration to evaluate the probability integral. The evaluation builds an integrand and a contour suggested in [8, p. 304].

Consider

\[
\lim_{R \to \infty} \int_{\Gamma} \frac{e^{-z^2+\sqrt{\pi}iz}}{e^{2\sqrt{\pi}iz} - 1} \, dz,
\]

where \( \Gamma \) is the positively oriented closed contour in Figure 1, consisting of the line segments \( \Gamma_1 \) from \( z = R - \frac{\sqrt{\pi}i}{2} \) to \( z = R + \frac{\sqrt{\pi}i}{2} \), \( \Gamma_2 \) from \( z = R + \frac{\sqrt{\pi}i}{2} \) to \( z = -R + \frac{\sqrt{\pi}i}{2} \), \( \Gamma_3 \) from \( z = -R + \frac{\sqrt{\pi}i}{2} \) to \( z = -R - \frac{\sqrt{\pi}i}{2} \), and \( \Gamma_4 \) from \( z = -R - \frac{\sqrt{\pi}i}{2} \) to \( z = R - \frac{\sqrt{\pi}i}{2} \); and \( \sqrt{\pi}i = \sqrt{\pi}e^{\pi i/2} \).
Observe that \( z_0 = 0 \) is the only simple pole of \( f(z) = \frac{e^{-z^2 + \sqrt{\pi}iz}}{e^{\pi i z} - 1} \) in the interior of the closed contour \( \Gamma \); so

\[
Res(f(z), 0) = \lim_{z \to 0} zf(z) = \frac{1}{2\sqrt{\pi}i}; \quad \text{and} \quad \oint_{\Gamma} f(z) \, dz = 2\pi i Res(f(z), 0) = \sqrt{\pi}i = \sqrt{\pi} e^{\pi i} = \frac{\sqrt{\pi}}{\sqrt{2}}(1+i);
\]

and

\[
\lim_{R \to \infty} \oint_{\Gamma} f(z) \, dz = \lim_{R \to \infty} \left( \int_{\Gamma_1} f(z) \, dz + \int_{\Gamma_2} f(z) \, dz + \int_{\Gamma_3} f(z) \, dz + \int_{\Gamma_4} f(z) \, dz \right) = \frac{\sqrt{\pi}}{\sqrt{2}}(1+i). \quad (1)
\]

We will now show that

\[
\lim_{R \to \infty} \int_{\Gamma_1} f(z) \, dz = \lim_{R \to \infty} \int_{\Gamma_3} f(z) \, dz = 0, \quad \text{and} \quad \lim_{R \to \infty} \int_{\Gamma_2} f(z) \, dz + \lim_{R \to \infty} \int_{\Gamma_4} f(z) \, dz = \frac{1+i}{2\sqrt{2}} \int_{-\infty}^{\infty} e^{-x^2} \, dx.
\]

Using \( z = z_1 + t(z_2 - z_1) \) with \( t \in (-\infty, \infty) \), as a representation of the equation of a line through any points \( z_1 \) and \( z_2 \) in the complex plane, we have:

\[
\Gamma_1: \quad z = R - \frac{\sqrt{\pi}}{2\sqrt{2}} - \frac{\sqrt{\pi}}{2\sqrt{2}} i + t \left( \frac{\sqrt{\pi}}{\sqrt{2}} + \frac{\sqrt{\pi}}{\sqrt{2}} i \right) = R - \frac{\sqrt{\pi}}{2} e^{\pi i} + t\sqrt{\pi} e^{\pi i} \quad \text{for} \quad 0 \leq t \leq 1,
\]

\[
\Gamma_2: \quad z = \left( \frac{\sqrt{\pi}}{2\sqrt{2}} + t \right) + \frac{\sqrt{\pi}}{2\sqrt{2}} i \quad \text{for} \quad t = R \quad \text{to} \quad t = -R,
\]

\[
\Gamma_3: \quad z = -R + \frac{\sqrt{\pi}}{2\sqrt{2}} + \frac{\sqrt{\pi}}{2\sqrt{2}} i + s \left( -\frac{\sqrt{\pi}}{\sqrt{2}} - \frac{\sqrt{\pi}}{\sqrt{2}} i \right) = -R + \frac{\sqrt{\pi}}{2} e^{\pi i} - s\sqrt{\pi} e^{\pi i} \quad \text{for} \quad 0 \leq s \leq 1, \quad \text{and}
\]

\[
\Gamma_4: \quad z = \left( -\frac{\sqrt{\pi}}{2\sqrt{2}} + t \right) - \frac{\sqrt{\pi}}{2\sqrt{2}} i \quad \text{for} \quad t = -R \quad \text{to} \quad t = R.
\]

On \( \Gamma_1 \),

\[
\sqrt{\pi}iz = \frac{\sqrt{\pi}}{\sqrt{2}} \left[ R + i \left( R - \frac{\sqrt{\pi}}{\sqrt{2}} + 2t\sqrt{\frac{\pi}{2}} \right) \right] = \frac{\sqrt{\pi}}{\sqrt{2}} R + i\frac{\sqrt{\pi}}{\sqrt{2}} R - i \frac{\pi}{2} + it\pi,
\]

\[
z^2 = R^2 - \frac{\sqrt{\pi}}{2} R + \sqrt{2\pi} R t - i\frac{\sqrt{\pi}}{\sqrt{2}} R + \frac{\pi}{4} - i\pi t + i\sqrt{2\pi} R t + it^2,
\]

\[
-\frac{\sqrt{\pi}}{2}iz = -R^2 + \sqrt{2\pi} R - \sqrt{2\pi} R t + i(-\frac{3\pi}{4} + 2\pi t + \sqrt{2\pi} R t - t^2),
\]

\[
\left| e^{-z^2 + \sqrt{\pi}iz} \right| = e^{-R^2 + \sqrt{2\pi} R - \sqrt{2\pi} R t} \leq \frac{e^{\sqrt{2\pi} R}}{e^{\sqrt{2\pi} R t}} \text{ for } 0 \leq t \leq 1,
\]

\[
\left| e^{2\sqrt{\pi}iz} - 1 \right| = \left| e^{\sqrt{2\pi} R} e^{i(\sqrt{2\pi} R + \pi - 2t\pi)} \right| - 1 \geq \left| e^{\sqrt{2\pi} R} \left| e^i(\sqrt{2\pi} R + \pi - 2t\pi) \right| - 1 \right| = \left| e^{\sqrt{2\pi} R} - 1 \right|,
\]

and

\[
\frac{1}{\left| e^{2\sqrt{\pi}iz} - 1 \right|} \leq \frac{1}{\left| e^{2\sqrt{2\pi} R} - 1 \right|}.
\]

So by combining (2) and (3) we obtain

\[
\left| \int_{\Gamma_1} e^{z^2 + \sqrt{\pi}iz} \, dz \right| \leq \int_{\Gamma_1} e^{R^2} \left| e^{\sqrt{2\pi} R} \right| \left| dz \right| = \int_0^1 \frac{\sqrt{\pi}}{R} \, dt = \frac{\sqrt{\pi}}{R} \left| 1 - e^{-\sqrt{2\pi} R} \right| \to 0 \text{ as } R \to \infty,
\]

and

\[
\lim_{R \to \infty} \int_{\Gamma_1} \frac{e^{-z^2 + \sqrt{\pi}iz}}{e^{2\sqrt{\pi}iz} - 1} \, dz = 0. \quad (4)
\]

Similarly on \( \Gamma_3 \),
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\[
\sqrt{\pi} i z = \sqrt{\frac{\pi}{2}} \left[ -R + i \left( -R + \sqrt{\frac{\pi}{2}} - 2s \sqrt{\frac{\pi}{2}} \right) \right] = -\sqrt{\frac{\pi}{2}} R - i \sqrt{\frac{\pi}{2}} R + i \frac{\pi}{2} - i s \pi,
\]

\[
z^2 = R^2 - \frac{\pi}{2} R + 2 \sqrt{\pi} R s - i \frac{\pi}{2} R + i \frac{\pi}{2} - i \pi s + i \sqrt{2} \pi R s + i \pi s^2,
\]

\[
-\sqrt{z^2 + \sqrt{\pi} i z} = -\sqrt{R^2 - \sqrt{\pi} R s + i \frac{\pi}{4} - \sqrt{\pi} R s - \pi s^2},
\]

\[
\left| e^{-z^2 + \sqrt{\pi} i z} \right| = e^{-R^2 - \sqrt{\pi} R s} \leq \frac{1}{e^{R^2}} \text{ for } 0 \leq s \leq 1,
\]

\[
\left| e^{2 \sqrt{\pi} i z} - 1 \right| = e^{-\sqrt{\pi} R} e^{i(\sqrt{\pi} R + \pi - 2s\pi)} - 1 \geq \left| e^{-\sqrt{\pi} R} \right| \left| e^{i(\sqrt{\pi} R + \pi - 2s\pi)} - 1 \right| = e^{-\sqrt{\pi} R} - 1,
\]

Combining (5) and (6), we have

\[
\left| \int_{\Gamma_3} e^{z^2 + \sqrt{\pi} i z} \frac{dz}{e^{2 \sqrt{\pi} i z} - 1} \right| \leq \int_{\Gamma_3} e^{-z^2 + \sqrt{\pi} i z} \frac{|dz|}{e^{2 \sqrt{\pi} i z} - 1} = \int_0^1 e^{\sqrt{\pi} ds} \frac{\sqrt{\pi}}{e^{-\sqrt{\pi} R} - 1} = \frac{\sqrt{\pi}}{e^{-\sqrt{\pi} R} - 1} \to 0 \text{ as } R \to \infty,
\]

and

\[
\lim_{R \to \infty} \int_{\Gamma_3} e^{z^2 + \sqrt{\pi} i z} \frac{dz}{e^{2 \sqrt{\pi} i z} - 1} = 0.
\]

On \(\Gamma_2\),

\[
\sqrt{\pi} i z = \sqrt{\frac{\pi}{2}} t + it \sqrt{\frac{\pi}{2}} + i \frac{\pi}{2}, \quad -z^2 = -\left( \frac{\sqrt{\pi}}{2\sqrt{2}} t + \frac{\sqrt{\pi}}{2\sqrt{2}} \right)^2 = -\frac{\sqrt{\pi}}{2\sqrt{2}} t - t^2 - \frac{\pi}{4} i - \frac{\sqrt{\pi}}{2\sqrt{2}} ti,
\]

and

\[
-\sqrt{z^2 + \sqrt{\pi} i z} = -t^2 + \frac{\pi}{4} i.
\]

Also, on \(\Gamma_4\),

\[
\sqrt{\pi} i z = \sqrt{\frac{\pi}{2}} t + it \sqrt{\frac{\pi}{2}} - i \frac{\pi}{2}, \quad -z^2 = -\left( -\frac{\sqrt{\pi}}{2\sqrt{2}} t + \frac{\sqrt{\pi}}{2\sqrt{2}} \right)^2 = \frac{\sqrt{\pi}}{2\sqrt{2}} t - t^2 - \frac{\pi}{4} i + \sqrt{\pi} ti,
\]

and

\[
-\sqrt{z^2 + \sqrt{\pi} i z} = 2 \sqrt{\frac{\pi}{2}} t - t^2 + it \sqrt{\frac{\pi}{2}} - 3 \frac{\pi}{4} i.
\]

Hence,

\[
\int_{\Gamma_2} f(z) \, dz + \int_{\Gamma_4} f(z) \, dz = \int_{-R}^R \frac{e^{(-t^2 + \frac{\pi}{4} i)}}{e^{(2 \sqrt{\frac{\pi}{2}} t + it + \sqrt{\frac{\pi}{2}} + i\pi) + 1}} \, dt + \int_{-R}^R \frac{e^{(2 \sqrt{\frac{\pi}{2}} t - t^2 + it + 2 \sqrt{\frac{\pi}{2}} + 3 \frac{\pi}{4} i)}}{e^{(2 \sqrt{\frac{\pi}{2}} t - t^2 + it + 2 \sqrt{\frac{\pi}{2}} + 3 \frac{\pi}{4} i) - 1}} \, dt.
\]

\[
= \int_{-R}^R \frac{e^{(-t^2 + \frac{\pi}{4} i)}}{e^{(2 \sqrt{\frac{\pi}{2}} t + it + 2 \sqrt{\frac{\pi}{2}} + i\pi)}} \, dt + \int_{-R}^R \frac{e^{(2 \sqrt{\frac{\pi}{2}} t - t^2 + it + 2 \sqrt{\frac{\pi}{2}} + 3 \frac{\pi}{4} i)}}{e^{(2 \sqrt{\frac{\pi}{2}} t - t^2 + it + 2 \sqrt{\frac{\pi}{2}} + 3 \frac{\pi}{4} i)}} \, dt.
\]

\[
= \int_{-R}^R \frac{e^{(-t^2 + \frac{\pi}{4} i)}}{e^{(2 \sqrt{\frac{\pi}{2}} t + it + 2 \sqrt{\frac{\pi}{2}})}} \, dt - \int_{-R}^R \frac{e^{(2 \sqrt{\frac{\pi}{2}} t - t^2 + it + 2 \sqrt{\frac{\pi}{2}} + 3 \frac{\pi}{4} i)}}{e^{(2 \sqrt{\frac{\pi}{2}} t + it + 2 \sqrt{\frac{\pi}{2}} + 1)}} \, dt.
\]

\[
= \int_{-R}^R \frac{e^{(-t^2 + \frac{\pi}{4} i)}}{e^{(2 \sqrt{\frac{\pi}{2}} t + it + 2 \sqrt{\frac{\pi}{2}} + 1) + 1}} \, dt - \int_{-R}^R \frac{e^{(2 \sqrt{\frac{\pi}{2}} t - t^2 + it + 2 \sqrt{\frac{\pi}{2}} + 3 \frac{\pi}{4} i)}}{e^{(2 \sqrt{\frac{\pi}{2}} t - t^2 + it + 2 \sqrt{\frac{\pi}{2}}) + 1}} \, dt.
\]

\[
= \int_{-R}^R e^{(-t^2 + \frac{\pi}{4} i)} \left[ e^{it} - e^{(2 \sqrt{\frac{\pi}{2}} t + it + 2 \sqrt{\frac{\pi}{2}})} \right] \, dt - \int_{-R}^R e^{(2 \sqrt{\frac{\pi}{2}} t - t^2 + it + 2 \sqrt{\frac{\pi}{2}} + 3 \frac{\pi}{4} i)} \, dt.
\]

\[
= e^{(-\frac{3 \pi}{4} i)} \int_{-R}^R e^{-t^2} - 1 - e^{(2 \sqrt{\frac{\pi}{2}} t + it + 2 \sqrt{\frac{\pi}{2}})} \, dt.
\]
\begin{equation}
-e^{(-3\pi/4)i} \int_{-R}^{R} e^{-t^2} \frac{[1 + e^{(2\sqrt{\pi}t + it)(2\sqrt{\pi})}]}{e^{(2\sqrt{\pi}t + it)(2\sqrt{\pi})}} dt = \frac{1 + i}{\sqrt{2}} \int_{-R}^{R} e^{-t^2} dt;
\end{equation}

so that

\[ \lim_{R \to \infty} \left[ \int_{\Gamma_2} f(z) \, dz + \int_{\Gamma_4} f(z) \, dz \right] = \frac{1 + i}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-t^2} \, dt. \]  

Combining (1), (4), (7) and (8) we obtain

\[ \lim_{R \to \infty} \left( \int_{\Gamma_1} f(z) \, dz + \int_{\Gamma_2} f(z) \, dz + \int_{\Gamma_3} f(z) \, dz + \int_{\Gamma_4} f(z) \, dz \right) = \frac{1 + i}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-t^2} \, dt = \frac{\sqrt{\pi}}{\sqrt{2}} (1 + i), \]

from which \[ \int_{-\infty}^{\infty} e^{-t^2} \, dt = \sqrt{\pi}. \]

References


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