

On general multiple Eulerian integrals involving the multivariable I-function , a general class of polynomials and the extension of Zeta-function

F.Y. AYANT¹

¹ Teacher in High School , France
 E-mail : fredericayant@gmail.com

ABSTRACT

The object of this paper is first, to evaluate a general multiple Eulerian integrals with general integrands involving the product of a multivariable I-function defined by Nambisan et al [4], a general Class of polynomials, the extension of Zeta-function and generalized hypergeometric function with general arguments. The second multiple Eulerian integral contain a general class of polynomials, the general polynomial set, the extension of Zeta-function and multivariable I-function defined by Nambisan et al [4] with general arguments. Our integral formulas are interesting and unified nature.

Keywords : Multivariable I-function, class of polynomial, general polynomials set, multiple Eulerian integral, extension of Zeta-function, multivariable H-function, Srivastava-Daoust function.

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1. Introduction

In this paper, we evaluate two multiple Eulerian integral involving the multivariable I-function , the extension of Zeta-function and multivariable class of polynomials with general arguments. The multivariable I-function is a extension of the multivariable H-function defined by Srivastava et al [12]. We will give the contracted form.

The I-function is defined and represented in the following manner.

$$I(z_1, \dots, z_r) = I_{p,q;p_1,q_1;\dots;p_r,q_r}^{0,n;m_1,n_1;\dots;m_r,n_r} \left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{matrix} \middle| \begin{matrix} (a_j; \alpha_j^{(1)}, \dots, \alpha_j^{(r)}; A_j)_{1,p} : \\ \\ \\ \\ (b_j; \beta_j^{(1)}, \dots, \beta_j^{(r)}; B_j)_{1,q} : \end{matrix} \right. \\ \left. \begin{matrix} (c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,p_1}; \dots; (c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{1,p_r} \\ \\ (d_j^{(1)}, \bar{\delta}_j^{(1)}; D_j^{(1)})_{1,q_1}; \dots; (d_j^{(r)}, \bar{\delta}_j^{(r)}; D_j^{(r)})_{1,q_r} \end{matrix} \right) \tag{1.1}$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \phi(s_1, \dots, s_r) \prod_{i=1}^r \theta_i(s_i) z_i^{s_i} ds_1 \dots ds_r \tag{1.2}$$

where $\phi(s_1, \dots, s_r), \theta_i(s_i), i = 1, \dots, r$ are given by :

$$\phi(s_1, \dots, s_r) = \frac{\prod_{j=1}^n \Gamma^{A_j} \left(1 - a_j + \sum_{i=1}^r \alpha_j^{(i)} s_j \right)}{\prod_{j=n+1}^p \Gamma^{A_j} \left(a_j - \sum_{i=1}^r \alpha_j^{(i)} s_j \right) \prod_{j=1}^q \Gamma^{B_j} \left(1 - b_j + \sum_{i=1}^r \beta_j^{(i)} s_j \right)} \tag{1.3}$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{n_i} \Gamma^{C_j^{(i)}} (1 - c_j^{(i)} + \gamma_j^{(i)} s_i) \prod_{j=1}^{m_i} \Gamma^{D_j^{(i)}} (d_j^{(i)} - \bar{\delta}_j^{(i)} s_i)}{\prod_{j=n_i+1}^{p_i} \Gamma^{C_j^{(i)}} (c_j^{(i)} - \gamma_j^{(i)} s_i) \prod_{j=m_i+1}^{q_i} \Gamma^{D_j^{(i)}} (1 - d_j^{(i)} - \bar{\delta}_j^{(i)} s_i)} \quad (1.4)$$

where $i = 1, \dots, r$. Also $z_i \neq 0$ for $i = 1, \dots, r$

The parameters $m_j, n_j, p_j, q_j (j = 1, \dots, r), n, p, q$ are non negative integers (for more details, see Nambisan [6])

$\alpha_j^{(i)} (j = 1, \dots, p; i = 1, \dots, r), \beta_j^{(i)} (j = 1, \dots, q; i = 1, \dots, r), \gamma_j^{(i)} (j = 1, \dots, p; i = 1, \dots, r)$ and $\bar{\delta}_j^{(i)}$

$(j = 1, \dots, q; i = 1, \dots, r)$ are assumed to be positive quantities for standardisation purpose.

$a_j (j = 1, \dots, p), b_j (j = 1, \dots, q), c_j^{(i)} (j = 1, \dots, p; i = 1, \dots, r), d_j^{(i)} (j = 1, \dots, q; i = 1, \dots, r)$ are complex numbers.

The exponents $A_j (j = 1, \dots, p), B_j (j = 1, \dots, q), C_j^{(i)} (j = 1, \dots, p; i = 1, \dots, r), D_j^{(i)} (j = 1, \dots, q; i = 1, \dots, r)$

of various gamma function involved in (2.2) and (2.3) may take non integer values.

The contour L_i in the complex s_i -plane is of Mellin Barnes type which runs from $c - i\infty$ to $c + i\infty$ (c real) with indentation, if necessary, in such a manner that all singularities of $\Gamma^{D_j^{(i)}} (d_j^{(i)} - \bar{\delta}_j^{(i)} s_i), j = 1, \dots, m_i$ lie to the right and $\Gamma^{C_j^{(i)}} (1 - c_j^{(i)} - \gamma_j^{(i)} s_i), j = 1, \dots, n_i$ are to the left of L_i .

Following the result of Braaksma [2] the I-function of r variables is analytic if :

$$U_i = \sum_{j=1}^p A_j \alpha_j^{(i)} - \sum_{j=1}^q B_j \beta_j^{(i)} + \sum_{j=1}^{p_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=1}^{q_i} D_j^{(i)} \bar{\delta}_j^{(i)}, i = 1, \dots, r \quad (1.5)$$

The integral (2.1) converges absolutely if

$|\arg(z_k)| < \frac{1}{2} \Delta_k \pi, k = 1, \dots, r$ where

$$\Delta_k = - \sum_{j=n+1}^p A_j \alpha_j^{(k)} - \sum_{j=1}^q B_j \beta_j^{(k)} + \sum_{j=1}^{m_k} D_j^{(k)} \delta_j^{(k)} - \sum_{j=m_k+1}^{q_k} D_j^{(k)} \bar{\delta}_j^{(k)} + \sum_{j=1}^{n_k} C_j^{(k)} \gamma_j^{(k)} - \sum_{j=n_k+1}^{p_k} C_j^{(k)} \gamma_j^{(k)} > 0 \quad (1.6)$$

Srivastava and Garg [8] introduced and defined a general class of multivariable polynomials as follows

$$S_L^{h_1, \dots, h_u} [z_1, \dots, z_u] = \sum_{R_1, \dots, R_u=0}^{h_1 R_1 + \dots + h_u R_u \leq L} (-L)_{h_1 R_1 + \dots + h_u R_u} B(E; R_1, \dots, R_u) \frac{z_1^{R_1} \dots z_u^{R_u}}{R_1! \dots R_u!} \quad (1.7)$$

The coefficients $B(E; R_1, \dots, R_u)$ are arbitrary constants, real or complex.

We will note

$$B_u = \frac{(-L)_{h_1 R_1 + \dots + h_u R_u} B(E; R_1, \dots, R_u)}{R_1! \dots R_u!} \quad (1.8)$$

2. Extension of the Hurwitz-Lerch Zeta function

The extension of the Hurwitz-Lerch Zeta function $\phi(z, s, a)$ is introduced by Srivastava et al ([12], eq.(6.2), page 503) as follows :

$$\phi_{(\lambda_1, \dots, \lambda_p, \sigma_1, \dots, \sigma_q)}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z; \mathfrak{s}, a) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (\lambda_j)_{n\rho_j}}{(a+n)^{\mathfrak{s}} \prod_{j=1}^q (\mu_j)_{n\sigma_j}} \times \frac{z^n}{n!} \quad (2.1)$$

with : $p, q \in \mathbb{N}_0, \lambda_j \in \mathbb{C} (j = 1, \dots, p), a, \mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^* (j = 1, \dots, q), \rho_j, \sigma_k \in \mathbb{R}^+$

$(j = 1, \dots, p; k = 1, \dots, q)$

where $\Delta > -1$ when $\mathfrak{s}, z \in \mathbb{C}; \Delta = -1$ and $\mathfrak{s} \in \mathbb{C}$, when $|z| < \nabla^*$, $\Delta = -1$ and $Re(\chi) > \frac{1}{2}$ when $|z| = \nabla^*$

$$\nabla^* = \prod_{j=1}^p \rho_j^{\rho_j} \prod_{j=1}^q \sigma_j^{\sigma_j}; \Delta = \sum_{j=1}^q \sigma_j - \sum_{j=1}^p \rho_j; \chi = \mathfrak{s} + \sum_{j=1}^q \mu_j - \sum_{j=1}^p \lambda_j + \frac{p-q}{2}$$

We will note these conditions the conditions (f) and $A_n = \frac{\prod_{j=1}^p (\lambda_j)_{n\rho_j}}{(a+n)^{\mathfrak{s}} \prod_{j=1}^q (\mu_j)_{n\sigma_j}}; \bar{A}_n = \frac{\prod_{j=1}^p (\bar{\lambda}_j)_{n\rho_j}}{(a+n)^{\mathfrak{s}} \prod_{j=1}^q (\mu_j)_{n\sigma_j}}$

3. Sequence of function

Agarwal and Chaubey [1], Salim [6] and several others have studied a general sequence of functions. In the present document we shall study the following useful series formula for a general sequence of functions.

$$R_n^{\alpha, \beta} [x; E, F, g, h; p, q; \gamma; \delta; e^{-sx^r}] = \sum_{w, v, u, t, e, k_1, k_2} \psi(w, v, u, t, e, k_1, k_2) x^R \quad (3.1)$$

$$\text{where } \sum_{w, v, u, t, e, k_1, k_2} = \sum_{w=0}^{\infty} \sum_{v=0}^n \sum_{u=0}^v \sum_{t=0}^n \sum_{c=0}^t \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \quad (3.2)$$

and the infinite series on the right side (2.1) is absolutely convergent, $R = ln + qv + pt + rw + k_1r + k_2q$ (3.3)

$$\text{and } \psi(w, v, u, t, e, k_1, k_2) = \frac{(-)^{t+w+k_2} (-v)_u (-t)_e (\alpha)_t l^n s^{w+k_1} F^{\gamma n-t}}{w!v!u!t!e!K_n k_1!k_2!} \frac{s^{w+k_1} F^{\gamma n-t}}{(1-\alpha-t)_e} (\alpha - \gamma n)_e$$

$$(-\beta - \delta n)_v g^{v+k_2} h^{\delta n-v-k_2} (v - \delta n)_{k_2} E^t \left(\frac{pe + rw + \lambda + qn}{l} \right)_n \quad (3.4)$$

where K_n is a sequence of constants. This function will note $R_n^{\alpha, \beta} [x]$

By suitably specializing the parameters involving in (2.1), a general sequence of function reduced to generalized polynomials set studied by Raizada [5], a class of polynomials introduced by Fujiwara [3] and several others authors.

4. Integral representation of generalized hypergeometric function

The following generalized hypergeometric function in terms of multiple contour integrals is also required [10, page 39 eq .30]

$$\frac{\prod_{j=1}^P \Gamma(A_j)}{\prod_{j=1}^Q \Gamma(B_j)} {}_P F_Q [(A_P); (B_Q); -(x_1 + \dots + x_r)]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \frac{\prod_{j=1}^P \Gamma(A_j + s_1 + \dots + s_r)}{\prod_{j=1}^Q \Gamma(B_j + s_1 + \dots + s_r)} \Gamma(-s_1) \dots \Gamma(-s_r) x_1^{s_1} \dots x_r^{s_r} ds_1 \dots ds_r \quad (4.1)$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j + s_1 + \dots + s_r)$ are separated from those of $\Gamma(-s_j), j = 1, \dots, r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j = 1, \dots, r$

The equivalent form of Eulerian beta integral is :

$$\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} dt = (b-a)^{\alpha+\beta} B(\alpha, \beta) (Re(\alpha) > 0, Re(\beta) > 0, a \neq b) \quad (4.2)$$

5. First integral

We note :

$$A = (a_j; \alpha_j^{(1)}, \dots, \alpha_j^{(r)}; A_j)_{1,p}; B = (b_j; \beta_j^{(1)}, \dots, \beta_j^{(r)}; B_j)_{1,q} \quad (5.1)$$

$$C = (c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,p_1}; \dots; (c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{1,p_r}; (1, 0; 1); \dots; (1, 0; 1); (1, 0; 1); \dots; (1, 0; 1) \quad (5.2)$$

$$D = (d_j^{(1)}, \bar{\delta}_j^{(1)}; D_j^{(1)})_{1,q_1}; \dots; (d_j^{(r)}, \bar{\delta}_j^{(r)}; D_j^{(r)})_{1,q_r}; (0, 1; 1); \dots; (0, 1; 1); (0, 1; 1); \dots; (0, 1; 1) \quad (5.3)$$

$$X = m_1, n_1; \dots; m_r, n_r; 1, 0; \dots; 1, 0; 1, 0; \dots; 1, 0 \quad (5.4)$$

$$Y = p_1, q_1; \dots; p_r, q_r; 0, 1; \dots; 0, 1; 0, 1; \dots; 0, 1 \quad (5.5)$$

$$A^* = [1 + \sigma'_i - n'c'_i - \sum_{k=1}^u R_k \xi_i^{(1,k)}; \rho_i^{(1,1)}, \dots, \rho_i^{(1,r)}, \tau_i^{(1,1)}, \dots, \tau_i^{(1,l)}, 1, 0, \dots, 0; 1]_{1,s}, \dots,$$

$$[1 + \sigma_i^{(T)} - n'c_i^{(T)} - \sum_{k=1}^u R_k \xi_i^{(T,k)}; \rho_i^{(T,1)}, \dots, \rho_i^{(T,r)}, \tau_i^{(T,1)}, \dots, \tau_i^{(T,l)}, 0, \dots, 0, 1; 1]_{1,s},$$

$$[1 - A_j; 0, \dots, 0, 1, \dots, 1, 0, \dots, 0; 1]_{1,P},$$

$$[1 - \alpha_i - n'a_i - \sum_{k=1}^u R_k \alpha_i^{(k)}; \delta'_i, \dots, \delta_i^{(r)}, \mu'_i, \dots, \mu_i^{(l)}, 1, \dots, 1, 0, \dots, 0; 1]_{1,s},$$

W-items (T-W)-items

$$[1 - \beta_i - n'b_i - \sum_{k=1}^u R_k \beta_i^{(k)}; \eta'_i, \dots, \eta_i^{(r)}, \theta'_i, \dots, \theta_i^{(l)}, 0, \dots, 0, 1, \dots, 1; 1]_{1,s} \quad (5.6)$$

W-items (T-W)-items

$$\begin{aligned}
B^* &= [1 + \sigma'_i - n'c'_i - \sum_{k=1}^u R_k \xi_i^{(1,k)}; \rho_i^{(1,1)}, \dots, \rho_i^{(1,r)}, \tau_i^{(1,1)}, \dots, \tau_i^{(1,l)}, 0, \dots, 0; 1]_{1,s}, \dots, \\
&[1 + \sigma_i^{(T)} - n'c_i^{(T)} - \sum_{k=1}^u R_k \xi_i^{(T,k)}; \rho_i^{(T,1)}, \dots, \rho_i^{(T,r)}, \tau_i^{(T,1)}, \dots, \tau_i^{(T,l)}, 0, \dots, 0; 1]_{1,s}, \\
&[1 - B_j; 0, \dots, 0, 1, \dots, 1, 0, \dots, 0; 1]_{1,Q}, \\
&[1 - \alpha_i - \beta_i - n'(a_i + b_i) - \sum_{k=1}^u R_k (\alpha_i^{(k)} + \beta_i^{(k)}); (\delta'_i + \eta'_i), \dots, (\delta_i^{(r)} + \eta_i^{(r)}) \\
&(\mu'_i + \theta'_i), \dots, (\mu_i^{(l)} + \theta_i^{(l)}), 1, \dots, 1; 1]_{1,s} \tag{5.7}
\end{aligned}$$

$$\begin{aligned}
&\int_{u_1}^{v_1} \cdots \int_{u_s}^{v_s} \prod_{i=1}^s \left[(x_i - u_i)^{\alpha_i - 1} (v_i - x_i)^{\beta_i - 1} \prod_{j=1}^T (u_i^{(j)} x_i + v_i^{(j)})^{\sigma_i^{(j)}} \right] \\
&\phi_{(\lambda_1, \dots, \lambda_p, \mu_1, \dots, \mu_q)}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)} \left[\begin{array}{c} \prod_{i=1}^s \left[\frac{(x_i - u_i)^{a_i} (v_i - x_i)^{b_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{c_i^{(j)}}} \right]; \mathfrak{s}, a \end{array} \right] \\
&S_L^{h_1, \dots, h_u} \left(\begin{array}{c} y_1 \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\alpha'_i} (v_i - x_i)^{\beta'_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\xi_i^{(j,1)}}} \right] \\ \vdots \\ y_u \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\alpha_i^{(u)}} (v_i - x_i)^{\beta_i^{(u)}}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\xi_i^{(j,u)}}} \right] \end{array} \right) I \left(\begin{array}{c} z_1 \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\delta'_i} (v_i - x_i)^{\eta'_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\rho_i^{(j,1)}}} \right] \\ \vdots \\ z_r \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\delta_i^{(r)}} (v_i - x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\rho_i^{(j,r)}}} \right] \end{array} \right)
\end{aligned}$$

$${}_pF_Q \left[(A_P); (B_Q); - \sum_{k=1}^l g_k \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\mu_i^{(k)}} (v_i - x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\tau_i^{(j,k)}}} \right] \right] dx_1 \cdots dx_s$$

$$= \frac{\prod_{j=1}^Q \Gamma(B_j)}{\prod_{j=1}^P \Gamma(A_j)} \prod_{i=1}^s \left[(v_i - u_i)^{\alpha_i + \beta_i - 1} \prod_{j=1}^W (u_i U_i^{(j)} + V_i^{(j)})^{\sigma_i^{(j)}} \prod_{j=W+1}^T (u_i U_i^{(j)} + V_i^{(j)})^{\sigma_i^{(j)}} \right]$$

$$\sum_{n'=0}^{\infty} \sum_{R_1, \dots, R_u=0}^{h_1 R_1 + \dots + h_u R_u \leq L} B_u A_{n'} \frac{z^{n'}}{n'!} y_1^{R_1} \dots y_u^{R_u} \prod_{i=1}^s \left[(v_i - u_i)^{n'(a_i + b_i) + \sum_{k=1}^u R_k (\alpha_i^{(k)} + \beta_i^{(k)})} \right]$$

$$\prod_{i=1}^s \left[\prod_{j=1}^W \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{-n' c_i^{(j)} - \sum_{k=1}^u R_k \xi_i^{(j,k)}} \prod_{j=W+1}^T \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{-n' c_i^{(j)} - \sum_{k=1}^u R_k \xi_i^{(j,k)}} \right]$$

$$I_{p+sT+P+2s; X}^{0, n+sT+P+2s; Y} \left(\begin{array}{c|c} z_1 w_1 & A ; A^* : C \\ \dots & \cdot \\ \dots & \cdot \\ z_r w_r & \cdot \\ g_1 W_1 & \cdot \\ \dots & \cdot \\ \dots & \cdot \\ g_l W_l & \cdot \\ G_1 & \cdot \\ \dots & \cdot \\ \dots & \cdot \\ G_T & B ; B^* : D \end{array} \right) \quad (5.8)$$

Where

$$w_m = \prod_{i=1}^s \left[(v_i - u_i)^{\delta_i^{(m)} + \eta_i^{(m)}} \prod_{j=1}^W \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{-\rho_i^{(j,m)}} \prod_{j=W+1}^T \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{-\rho_i^{(j,m)}} \right], m = 1, \dots, r \quad (5.9)$$

$$W_k = \prod_{i=1}^s \left[(v_i - u_i)^{\mu_i^{(k)} + \theta_i^{(k)}} \prod_{j=1}^W \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{-\tau_i^{(j,k)}} \prod_{j=W+1}^T \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{-\tau_i^{(j,k)}} \right], k = 1, \dots, l \quad (5.10)$$

$$G_j = \prod_{i=1}^s \left[\frac{(v_i - u_i) U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}} \right], j = 1, \dots, W \quad (5.11)$$

$$G_j = - \prod_{i=1}^s \left[\frac{(v_i - u_i) U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}} \right], j = W + 1, \dots, T \quad (5.12)$$

Provided that :

(A) $0 \leq W \leq T; u_i, v_i \in \mathbb{R}; i = 1, \dots, r \min\{a_i, b_i, c_i^{(j)}, \alpha_i^{(k)}, \beta_i^{(k)}, \xi_i^{(j,k)}\} > 0, i = 1, \dots, s; k = 1, \dots, u; j = 1, \dots, T$

(B) $\min\{\delta_i^{(t)}, \eta_i^{(t)}, \rho_i^{(j,t)}, \mu_i^{(k)}, \theta_i^{(k)}, \tau_i^{(j,k)}\} \geq 0; j = 1, \dots, T; i = 1, \dots, s; k = 1, \dots, l; t = 1, \dots, r$

(C) $\sigma_i^{(j)} \in \mathbb{R}, U_i^{(j)}, V_i^{(j)} \in \mathbb{C}, z_t, g_k \in \mathbb{C}; j = 1, \dots, s; t = 1, \dots, r; k = 1, \dots, l; t = 1, \dots, r$

$$(D) \max \left[\frac{(v_i - u_i)U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}} \right] < 1, i = 1, \dots, s; j = 1, \dots, W \text{ and}$$

$$\max \left[\frac{(v_i - u_i)U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}} \right] < 1, i = 1, \dots, s; j = W + 1, \dots, T$$

$$(E) U_i = \sum_{j=1}^p A_j \alpha_j^{(i)} - \sum_{j=1}^q B_j \beta_j^{(i)} + \sum_{j=1}^{p_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=1}^{q_i} D_j^{(i)} \bar{\delta}_j^{(i)} \leq 0, i = 1, \dots, r$$

$$(F) \Delta_k = - \sum_{j=n+1}^p A_j \alpha_j^{(k)} - \sum_{j=1}^q B_j \beta_j^{(k)} + \sum_{j=1}^{m_k} D_j^{(k)} \bar{\delta}_j^{(k)} - \sum_{j=m_k+1}^{q_k} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n_k} C_j^{(k)} \gamma_j^{(k)} - \sum_{j=n_k+1}^{p_k} C_j^{(k)} \gamma_j^{(k)} - \delta_i^{(k)} - \eta_i^{(k)} - \sum_{j=1}^T \rho_i^{(j,k)} > 0$$

$$(G) \left| \arg \left(z_i \prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{-\rho_i^{(j,k)}} \right) \right| < \frac{1}{2} \Delta_i \pi$$

$$(H) \operatorname{Re} \left[\alpha_i + n' a_i + \sum_{t=1}^r \delta_i^{(t)} \min_{1 \leq j \leq m_i} \frac{d_j^{(t)}}{\bar{\delta}_j^{(t)}} \right] > 0;$$

$$\operatorname{Re} \left[\beta_i + n' b_i + \sum_{t=1}^r \eta_i^{(t)} \min_{1 \leq j \leq m_i} \frac{d_j^{(t)}}{\bar{\delta}_j^{(t)}} \right] > 0; i = 1, \dots, s$$

(I) The conditions (f) are satisfied

(J) The series occurring on the right-hand side of (5.13) are absolutely and uniformly convergent

(K) $m_j, n_j, p_j, q_j (j = 1, \dots, r), n, p, q \in \mathbb{N}^*; \delta_j^{(i)} \in \mathbb{R}_+ (j = 1, \dots, q_i; i = 1, \dots, r)$

$\alpha_j^{(i)} \in \mathbb{R}_+ (j = 1, \dots, p; i = 1, \dots, r), \beta_j^{(i)} \in \mathbb{R}_+ (j = 1, \dots, q; i = 1, \dots, r), \gamma_j^{(i)} \in \mathbb{R}_+ (j = 1, \dots, p_i; i = 1, \dots, r)$

$a_j (j = 1, \dots, p), b_j (j = 1, \dots, q), c_j^{(i)} (j = 1, \dots, p_i, i = 1, \dots, r), d_j^{(i)} (j = 1, \dots, q_i, i = 1, \dots, r) \in \mathbb{C}$

The exponents $A_j (j = 1, \dots, p), B_j (j = 1, \dots, q), C_j^{(i)} (j = 1, \dots, p_i; i = 1, \dots, r), D_j^{(i)} (j = 1, \dots, q_i; i = 1, \dots, r)$

of various gamma function involved in (1.3) and (1.4) may take non integer values.

Proof

To establish the formula (5.3), we first use series representation (2.1) and (1.7) for $\phi_{(\lambda_1, \dots, \lambda_p, \sigma_1, \dots, \sigma_q)}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(.; s, a)$ and $S_L^{h_1, \dots, h_u} [.]$ respectively, we use contour integral representation with the help of (1.1) for the multivariable I-function occurring in its left-hand side and use the contour integral representation with the help of (4.1) for the generalized hypergeometric function ${}_pF_q(.).$

Changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process). Now we write :

$$\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} = \prod_{j=1}^W (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} \prod_{j=W+1}^T (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}}$$

$$\text{where } K_i^{(j)} = \eta_i^{(j)} - n' c_i^{(j)} - \sum_{k=1}^u R_k \xi_i^{(j,k)} - \sum_{t=1}^r \rho_i^{(j,t)} \zeta_t - \sum_{k=1}^l \tau_i^{(j,k)} \zeta_k \quad i = 1, \dots, s; j = 1, \dots, T \quad (5.8)$$

and express the factor occurring in R.H.S. Of (5.8) in terms of following Mellin-Barnes contour integral with the help of the result [9, page 18, eq.(2.6.4)]

$$\frac{1}{(2\pi\omega)^W} \int_{L'_1} \cdots \int_{L'_W} \prod_{j=1}^W \left[\Gamma(-\zeta'_j) \Gamma(-K_i^{(j)} + \zeta'_j) \prod_{j=1}^W (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} \right] = \prod_{j=1}^W \left[\frac{(U_i^{(j)} u_i + V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})} \right]$$

$$\prod_{j=1}^W \left[\frac{(U_i^{(j)}(x_i - u_i))}{(u_i U_i^{(j)} + V_i^{(j)})} \right]^{\zeta'_j} d\zeta'_1 \cdots d\zeta'_W \quad (5.9)$$

and

$$\frac{1}{(2\pi\omega)^{T-W}} \int_{L'_{W+1}} \cdots \int_{L'_T} \prod_{j=W+1}^T \left[\Gamma(-\zeta'_j) \Gamma(-K_i^{(j)} + \zeta'_j) \prod_{j=W+1}^T (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} \right] = \prod_{j=W+1}^T \left[\frac{(U_i^{(j)} v_i + V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})} \right]$$

$$\prod_{j=W+1}^T \left[-\frac{(U_i^{(j)}(v_i - x_i))}{(v_i U_i^{(j)} + V_i^{(j)})} \right]^{\zeta'_j} d\zeta'_{W+1} \cdots d\zeta'_T \quad (5.10)$$

We apply the Fubini's theorem for multiple integral. Finally evaluating the innermost \mathbf{x} -integral with the help of (4.2) and reinterpreting the multiple Mellin-Barnes contour integral in terms of multivariable I-function of $(r + l + T)$ -variables, we obtain the formula (5.8).

6. Second integral

We note :

$$A = (a_j; \alpha_j^{(1)}, \dots, \alpha_j^{(r)}; A_j)_{1,p}; B = (b_j; \beta_j^{(1)}, \dots, \beta_j^{(r)}; B_j)_{1,q} \quad (6.1)$$

$$C = (c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,p_1}; \dots; (c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{1,p_r}; (1, 0; 1); \dots; (1, 0; 1) \quad (6.2)$$

$$D = (d_j^{(1)}, \bar{\delta}_j^{(1)}; D_j^{(1)})_{1,q_1}; \dots; (d_j^{(r)}, \bar{\delta}_j^{(r)}; D_j^{(r)})_{1,q_r}; (0, 1; 1); \dots; (0, 1; 1) \quad (6.3)$$

$$X = m_1, n_1; \dots; m_r, n_r; 1, 0; \dots; 1, 0; Y = p_1, q_1; \dots; p_r, q_r; 0, 1; \dots; 0, 1 \quad (6.4)$$

$$A^* = [1 + \sigma'_i - \theta'_i R - n' c'_i - \sum_{k=1}^u R_k \xi_i^{(1,k)}; \rho_i^{(1,1)}, \dots, \rho_i^{(1,r)}, 1, 0, \dots, 0; 1]_{1,s}, \dots,$$

$$[1 + \sigma_i^{(T)} - \theta_i^{(T)} R - n' c_i^{(T)} - \sum_{k=1}^u R_k \xi_i^{(T,k)}; \rho_i^{(T,1)}, \dots, \rho_i^{(T,r)}, 0, \dots, 0, 1; 1]_{1,s},$$

$$[1 - \alpha_i - \zeta_i R - n' a_i - \sum_{k=1}^u R_k \alpha_i^{(k)}; \delta'_i, \dots, \delta_i^{(r)}, 1, \dots, 1, 0, \dots, 0; 1]_{1,s},$$

W-items (T-W)-items

$$[1 - \beta_i - \lambda_i R - n' b_i - \sum_{k=1}^u R_k \beta_i^{(k)}; \eta'_i, \dots, \eta_i^{(r)}, 0, \dots, 0, 1, \dots, 1]_{1,s} \quad (6.5)$$

W-items (T-W)-items

$$B^* = [1 + \sigma'_i - \theta'_i R - n' c'_i - \sum_{k=1}^u R_k \xi_i^{(1,k)}; \rho_i^{(1,1)}, \dots, \rho_i^{(1,r)}, 0, \dots, 0]_{1,s}, \dots,$$

$$[1 + \sigma_i^{(T)} - \theta_i^{(T)} R - n' c_i^{(T)} - \sum_{k=1}^u R_k \xi_i^{(T,k)}; \rho_i^{(T,1)}, \dots, \rho_i^{(T,r)}, 0, \dots, 0]_{1,s},$$

$$[1 - \alpha_i - \beta_i (\zeta_i + \lambda_i) R - n' (a_i + b_i) - \sum_{k=1}^u R_k (\alpha_i^{(k)} + \beta_i^{(k)}); (\delta'_i + \eta'_i), \dots, (\delta_i^{(r)} + \eta_i^{(r)})$$

$$(\mu'_i + \theta'_i), \dots, (\mu_i^{(l)} + \theta_i^{(l)}), 1, \dots, 1]_{1,s} \quad (6.6)$$

We have the following multiple Eulerian integral

$$\int_{u_1}^{v_1} \cdots \int_{u_s}^{v_s} \prod_{i=1}^s \left[(x_i - u_i)^{\alpha_i - 1} (v_i - x_i)^{\beta_i - 1} \prod_{j=1}^T (u_i^{(j)} x_i + v_i^{(j)})^{\sigma_i^{(j)}} \right]$$

$$\phi_{(\lambda_1, \dots, \lambda_p, \sigma_1, \dots, \sigma_q)}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)} \left[z \prod_{i=1}^s \left[\frac{(x_i - u_i)^{a_i} (v_i - x_i)^{b_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{c_i^{(j)}}} \right]; \mathfrak{s}, a \right]$$

$$S_L^{h_1, \dots, h_u} \left(\begin{array}{c} y_1 \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\alpha'_i} (v_i - x_i)^{\beta'_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\xi_i^{(j,1)}}} \right] \\ \vdots \\ y_u \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\alpha_i^{(u)}} (v_i - x_i)^{\beta_i^{(u)}}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\xi_i^{(j,u)}}} \right] \end{array} \right) I \left(\begin{array}{c} z_1 \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\delta'_i} (v_i - x_i)^{\eta'_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\rho_i^{(j,1)}}} \right] \\ \vdots \\ z_r \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\delta_i^{(r)}} (v_i - x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\rho_i^{(j,r)}}} \right] \end{array} \right)$$

$$R_n^{\alpha, \beta} \left[Z \prod_{j=1}^s \left[\frac{(x_i - u_i)^{\zeta_i} (v_i - x_i)^{\lambda_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\theta_i^{(j)}}} \right] \right] dx_1 \cdots dx_r$$

$$\begin{aligned}
&= \prod_{i=1}^s \left[(v_i - u_i)^{\alpha_i + \beta_i - 1} \prod_{j=1}^W (u_i U_i^{(j)} + V_i^{(j)})^{\rho_i^{(j)}} \prod_{j=W+1}^T (u_i U_i^{(j)} + V_i^{(j)})^{\rho_i^{(j)}} \right] \sum_{n'=0}^{\infty} \sum_{R_1, \dots, R_u=0}^{h_1 R_1 + \dots + h_u R_u \leq L} \\
&\sum_{w, v, u, t, e, k_1, k_2} B_u A_{n'} \frac{z^{n'}}{n'!} y_1^{R_1} \dots y_u^{R_u} \psi'(w, v, u, t, e, k_1, k_2) Z^R \prod_{i=1}^s \left[(v_i - u_i)^{n'(a_i + b_i) + \sum_{k=1}^u R_k (\alpha_i^{(k)} + \beta_i^{(k)})} \right] \\
&\prod_{i=1}^s \left[\prod_{j=1}^W (u_i U_i^{(j)} + V_i^{(j)})^{-n' c_i^{(j)} - \sum_{k=1}^u R_k \xi_i^{(j,k)}} \prod_{j=W+1}^T (u_i U_i^{(j)} + V_i^{(j)})^{-n' c_i^{(j)} - \sum_{k=1}^u R_k \xi_i^{(j,k)}} \right] \\
&\psi'(w, v, u, t, e, k_1, k_2) I_{p+sT+2s; q+sT+s; Y}^{0, n+sT+2s; X} \left(\begin{array}{c|c} z_1 w_1 & \mathbf{A}; \mathbf{A}^* : C \\ \dots & \cdot \\ \dots & \cdot \\ z_r w_r & \cdot \\ \mathbf{G}_1 & \cdot \\ \dots & \cdot \\ \dots & \cdot \\ \mathbf{G}_T & \mathbf{B}; \mathbf{B}^* : D \end{array} \right) \tag{6.7}
\end{aligned}$$

where

$$\psi'(w, v, u, t, e, k_1, k_2) = \frac{\psi(w, v, u, t, e, k_1, k_2,) \prod_{i=1}^s (v_i - u_i)^{(\zeta_i + \lambda_i)R}}{\prod_{i=1}^s \left[\prod_{j=1}^W (u_i U_i^{(j)} + V_i^{(j)})^{\theta_i^{(j)} R} \prod_{j=W+1}^T (u_i U_i^{(j)} + V_i^{(j)})^{\theta_i^{(j)} R} \right]} \tag{6.8}$$

$$w_l = \prod_{i=1}^s \left[(v_i - u_i)^{\delta_i^{(l)} + \eta_i^{(l)}} \prod_{j=1}^W (u_i U_i^{(j)} + V_i^{(j)})^{-\rho_i^{(j,l)}} \prod_{j=W+1}^T (u_i U_i^{(j)} + V_i^{(j)})^{\rho_i^{(j,l)}} \right], l = 1, \dots, r \tag{6.9}$$

$$G_j = \prod_{i=1}^s \left[\frac{(v_i - u_i) U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}} \right], j = 1, \dots, W \tag{6.10}$$

$$G_j = - \prod_{i=1}^s \left[\frac{(v_i - u_i) U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}} \right], j = W + 1, \dots, T \tag{6.11}$$

Provided that :

(A) $\min\{a_i, b_i, c_i^{(j)}, \alpha_i^{(k)}, \beta_i^{(k)}, \xi_i^{(j,k)}\} > 0, i = 1, \dots, s; k = 1, \dots, u; j = 1, \dots, T; 0 \leq W \leq T; u_i, v_i \in \mathbb{R}$

(B) $\min\{\zeta_i, \lambda_i, \theta_i^{(j)}, \delta_i^{(l)}, \eta_i^{(l)}, \rho_i^{(j,l)}\} \geq 0; j = 1, \dots, T; i = 1, \dots, s; l = 1, \dots, r$

(C) $\operatorname{Re}(\alpha_i, \beta_i, v_i^{(j)}) > 0 (i = 1, \dots, s; j = 1, \dots, T); |\tau| < 1$

$$(D) \max \left[\frac{(v_i - u_i)U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}} \right] < 1, i = 1, \dots, s; j = 1, \dots, W \text{ and}$$

$$\max \left[\frac{(v_i - u_i)U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}} \right] < 1, i = 1, \dots, s; j = W + 1, \dots, T$$

$$(E) U_i = \sum_{j=1}^p A_j \alpha_j^{(i)} - \sum_{j=1}^q B_j \beta_j^{(i)} + \sum_{j=1}^{p_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=1}^{q_i} D_j^{(i)} \bar{\delta}_j^{(i)} \leq 0, i = 1, \dots, r$$

$$(F) \Delta_k = - \sum_{j=n_k+1}^p A_j \alpha_j^{(k)} - \sum_{j=1}^q B_j \beta_j^{(k)} + \sum_{j=1}^{m_k} D_j^{(k)} \delta_j^{(k)} - \sum_{j=m_k+1}^{q_k} D_j^{(k)} \bar{\delta}_j^{(k)} + \sum_{j=1}^{n_k} C_j^{(k)} \gamma_j^{(k)} - \sum_{j=n_k+1}^{p_k} C_j^{(k)} \gamma_j^{(k)}$$

$$-\delta_i^{(k)} - \eta_i^{(k)} - \sum_{j=1}^T \rho_i^{(j,k)} > 0$$

$$(G) \left| \arg \left(z_i \prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{-\rho_i^{(j,k)}} \right) \right| < \frac{1}{2} \Delta_i \pi$$

(H) The conditions (f) are satisfied

$$(I) m_j, n_j, p_j, q_j (j = 1, \dots, r), n, p, q \in \mathbb{N}^*; \bar{\delta}_j^{(i)} \in \mathbb{R}_+ (j = 1, \dots, q_i; i = 1, \dots, r)$$

$$\alpha_j^{(i)} \in \mathbb{R}_+ (j = 1, \dots, p; i = 1, \dots, r), \beta_j^{(i)} \in \mathbb{R}_+ (j = 1, \dots, q; i = 1, \dots, r), \gamma_j^{(i)} \in \mathbb{R}_+ (j = 1, \dots, p_i; i = 1, \dots, r)$$

$$a_j (j = 1, \dots, p), b_j (j = 1, \dots, q), c_j^{(i)} (j = 1, \dots, p_i, i = 1, \dots, r), d_j^{(i)} (j = 1, \dots, q_i, i = 1, \dots, r) \in \mathbb{C}$$

The exponents $A_j (j = 1, \dots, p), B_j (j = 1, \dots, q), C_j^{(i)} (j = 1, \dots, p_i; i = 1, \dots, r), D_j^{(i)} (j = 1, \dots, q_i; i = 1, \dots, r)$

of various gamma function involved in (1.3) and (1.4) may take non integer values.

(J) The series occurring on the right-hand side of (6.7) is absolutely and uniformly convergent

$$(K) \operatorname{Re} \left[\alpha_i + n' a_i + R \zeta_i + \sum_{t=1}^r \delta_i^{(t)} \min_{1 \leq j \leq m_i} \frac{d_j^{(t)}}{\bar{\delta}_j^{(t)}} \right] > 0;$$

$$\operatorname{Re} \left[\beta_i + n' b_i + R \lambda_i + \sum_{t=1}^r \eta_i^{(t)} \min_{1 \leq j \leq m_i} \frac{d_j^{(t)}}{\bar{\delta}_j^{(t)}} \right] > 0; i = 1, \dots, s$$

Proof

To establish the formula (6.8), we first use series representation (2.1), (1.7) and (3.1) for $\phi_{(\lambda_1, \dots, \lambda_p, \sigma_1, \dots, \sigma_q)}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)} (.; \mathfrak{s}, a)$, $\mathcal{S}_L^{h_1, \dots, h_u} [.]$ and $R_n^{\alpha, \beta} [.]$ respectively and the contour integral representation with the help of (1.1) for the multivariable I-function occurring in its left-hand side. Changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process). Now, we write:

$$\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} = \prod_{j=1}^W (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} \prod_{j=W+1}^T (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} \quad (6.12)$$

$$\text{where } K_i^{(j)} = \eta_i^{(j)} - n'c_i^{(j)} - R\theta_i^{(j)} - \sum_{k=1}^u R_k \xi_i^{(j,k)} - \sum_{t=1}^r \rho_i^{(j,t)} \zeta_t; i = 1, \dots, s; j = 1, \dots, T \quad (6.13)$$

and express the factors occurring in R.H.S. Of (6.8) in terms of following Mellin-Barnes contour integral , we obtain :

$$\prod_{j=1}^W (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} = \prod_{j=1}^W \left[\frac{(U_i^{(j)} u_i + V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})} \right] \frac{1}{(2\pi\omega)^W} \int_{L'_1} \cdots \int_{L'_W} \prod_{j=1}^W \left[\Gamma(-\zeta'_j) \Gamma(-K_i^{(j)} + \zeta'_j) \right. \\ \left. \prod_{j=1}^W \left[\frac{(U_i^{(j)}(x_i - u_i))}{(u_i U_i^{(j)} + V_i^{(j)})} \right]^{\zeta'_j} \right] d\zeta'_1 \cdots d\zeta'_W \quad (6.14)$$

and

$$\prod_{j=W+1}^T (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} = \prod_{j=W+1}^T \left[\frac{(U_i^{(j)} v_i + V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})} \right] \frac{1}{(2\pi\omega)^{T-W}} \int_{L'_{W+1}} \cdots \int_{L'_T} \prod_{j=W+1}^T \left[\Gamma(-\zeta'_j) \Gamma(-K_i^{(j)} + \zeta'_j) \right. \\ \left. \prod_{j=W+1}^T \left[\frac{(U_i^{(j)}(x_i - v_i))}{(v_i U_i^{(j)} + V_i^{(j)})} \right]^{\zeta'_j} \right] d\zeta'_{W+1} \cdots d\zeta'_T \quad (6.15)$$

We apply the Fubini's theorem for multiple integral. Finally evaluating the innermost \mathbf{x} -integral with the help of (4.2) and reinterpreting the multiple Mellin-Barnes contour integral in terms of I-function of $(r + T)$ -variables, we obtain the formula (6.8).

7. Multivariable H-function

If $A_j = B_j = C_j^{(i)} = D_j^{(i)} = 1$, the multivariable I-function defined by Nambisan et al [6] reduces to the multivariable H-funcction defined by Srivastava et al [11] and we have the two following formulas.

Formula 1

$$\int_{u_1}^{v_1} \cdots \int_{u_s}^{v_s} \prod_{i=1}^s \left[(x_i - u_i)^{\alpha_i - 1} (v_i - x_i)^{\beta_i - 1} \prod_{j=1}^T (u_i^{(j)} x_i + v_i^{(j)})^{\sigma_i^{(j)}} \right] \\ \phi_{(\lambda_1, \dots, \lambda_p, \sigma_1, \dots, \sigma_q)}^{(\rho_1, \dots, \rho_p, \mu_1, \dots, \mu_q)} \left[z \prod_{i=1}^s \left[\frac{(x_i - u_i)^{a_i} (v_i - x_i)^{b_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{c_i^{(j)}}} \right]; \mathfrak{s}, a \right] \\ S_L^{h_1, \dots, h_u} \left(\begin{matrix} y_1 \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\alpha'_i} (v_i - x_i)^{\beta'_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\xi_i^{(j,1)}}} \right] \\ \vdots \\ y_u \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\alpha_i^{(u)}} (v_i - x_i)^{\beta_i^{(u)}}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\xi_i^{(j,u)}}} \right] \end{matrix} \right) H \left(\begin{matrix} z_1 \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\delta'_i} (v_i - x_i)^{\eta'_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\rho_i^{(j,1)}}} \right] \\ \vdots \\ z_r \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\delta_i^{(r)}} (v_i - x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\rho_i^{(j,r)}}} \right] \end{matrix} \right)$$

$$\begin{aligned}
& {}_P F_Q \left[(A_P); (B_Q); - \sum_{k=1}^l g_k \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\mu_i^{(k)}} (v_i - x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\tau_i^{(j,k)}}} \right] \right] dx_1 \cdots dx_s \\
&= \frac{\prod_{j=1}^Q \Gamma(B_j)}{\prod_{j=1}^P \Gamma(A_j)} \prod_{i=1}^s \left[(v_i - u_i)^{\alpha_i + \beta_i - 1} \prod_{j=1}^W (u_i U_i^{(j)} + V_i^{(j)})^{\sigma_i^{(j)}} \prod_{j=W+1}^T (u_i U_i^{(j)} + V_i^{(j)})^{\sigma_i^{(j)}} \right] \\
& \sum_{n'=0}^{\infty} \sum_{R_1, \dots, R_u=0}^{h_1 R_1 + \dots + h_u R_u \leq L} B_u A_{n'} \frac{z^{n'}}{n'!} y_1^{R_1} \cdots y_u^{R_u} \prod_{i=1}^s \left[(v_i - u_i)^{n'(a_i + b_i) + \sum_{k=1}^u R_k (\alpha_i^{(k)} + \beta_i^{(k)})} \right] \\
& \prod_{i=1}^s \left[\prod_{j=1}^W (u_i U_i^{(j)} + V_i^{(j)})^{-n' c_i^{(j)} - \sum_{k=1}^u R_k \xi_i^{(j,k)}} \prod_{j=W+1}^T (u_i U_i^{(j)} + V_i^{(j)})^{-n' c_i^{(j)} - \sum_{k=1}^u R_k \xi_i^{(j,k)}} \right]
\end{aligned}$$

$$H_{p+sT+P+2s; q+sT+Q+s; Y}^{0, n+sT+P+2s; X} \left(\begin{array}{c|c} z_1 w_1 & A ; A^* : C \\ \cdots & \cdot \\ \cdots & \cdot \\ z_r w_r & \cdot \\ g_1 W_1 & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ g_l W_l & \cdot \\ G_1 & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ G_T & B ; B^* : D \end{array} \right) \tag{7.1}$$

under the same notations and conditions that (5.8) with $A_j = B_j = C_j^{(i)} = D_j^{(i)} = 1$

Formula 2

$$\int_{u_1}^{v_1} \cdots \int_{u_s}^{v_s} \prod_{i=1}^s \left[(x_i - u_i)^{\alpha_i - 1} (v_i - x_i)^{\beta_i - 1} \prod_{j=1}^T (u_i^{(j)} x_i + v_i^{(j)})^{\sigma_i^{(j)}} \right]$$

$$\phi_{(\bar{\lambda}_1, \dots, \bar{\lambda}_p, \mu_1, \dots, \mu_q)}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)} \left[z \prod_{i=1}^s \left[\frac{(x_i - u_i)^{a_i} (v_i - x_i)^{b_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{c_i^{(j)}}} \right] ; \mathfrak{s}, a \right]$$

$$S_L^{h_1, \dots, h_u} \left(\begin{array}{c} y_1 \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\alpha'_i} (v_i - x_i)^{\beta'_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\xi_i^{(j,1)}}} \right] \\ \vdots \\ y_u \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\alpha_i^{(u)}} (v_i - x_i)^{\beta_i^{(u)}}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\xi_i^{(j,u)}}} \right] \end{array} \right) H \left(\begin{array}{c} z_1 \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\delta'_i} (v_i - x_i)^{\eta'_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\rho_i^{(j,1)}}} \right] \\ \vdots \\ z_r \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\delta_i^{(r)}} (v_i - x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\rho_i^{(j,r)}}} \right] \end{array} \right)$$

$$R_n^{\alpha, \beta} \left[Z \prod_{j=1}^s \left[\frac{(x_i - u_i)^{\zeta_i} (v_i - x_i)^{\lambda_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\theta_i^{(j)}}} \right] \right] dx_1 \cdots dx_r$$

$$= \prod_{i=1}^s \left[(v_i - u_i)^{\alpha_i + \beta_i - 1} \prod_{j=1}^W (u_i U_i^{(j)} + V_i^{(j)})^{\rho_i^{(j)}} \prod_{j=W+1}^T (u_i U_i^{(j)} + V_i^{(j)})^{\rho_i^{(j)}} \right] \sum_{n'=0}^{\infty} \sum_{R_1, \dots, R_u=0}^{h_1 R_1 + \dots + h_u R_u \leq L}$$

$$\sum_{w, v, u, t, e, k_1, k_2} B_u \bar{A}_{n'} \frac{z^{n'}}{n'!} y_1^{R_1} \cdots y_u^{R_u} \psi'(w, v, u, t, e, k_1, k_2) Z^R \prod_{i=1}^s \left[(v_i - u_i)^{n'(a_i + b_i) + \sum_{k=1}^u R_k (\alpha_i^{(k)} + \beta_i^{(k)})} \right]$$

$$\prod_{i=1}^s \left[\prod_{j=1}^W (u_i U_i^{(j)} + V_i^{(j)})^{-n' c_i^{(j)} - \sum_{k=1}^u R_k \xi_i^{(j,k)}} \prod_{j=W+1}^T (u_i U_i^{(j)} + V_i^{(j)})^{-n' c_i^{(j)} - \sum_{k=1}^u R_k \xi_i^{(j,k)}} \right]$$

$$\psi'(w, v, u, t, e, k_1, k_2) H_{p+sT+2s; q+sT+s; Y}^{0, n+sT+2s; X} \left(\begin{array}{c|c} z_1 w_1 & A ; A^* : C \\ \dots & \cdot \\ \dots & \cdot \\ z_r w_r & \cdot \\ \dots & \cdot \\ G_1 & \cdot \\ \dots & \cdot \\ \dots & \cdot \\ G_T & B ; B^* : D \end{array} \right) \quad (7.2)$$

under the same notations and conditions that (6.7) with $A_j = B_j = C_j^{(i)} = D_j^{(i)} = 1$

8. Srivastava-daoust function

$$\text{If } B(L; R_1, \dots, R_u) = \frac{\prod_{j=1}^{\bar{A}} (a_j)_{R_1 \theta'_j + \dots + R_u \theta_j^{(u)}} \prod_{j=1}^{B'} (b'_j)_{R_1 \phi'_j} \cdots \prod_{j=1}^{B^{(u)}} (b_j^{(u)})_{R_u \phi_j^{(u)}}}{\prod_{j=1}^{\bar{C}} (c_j)_{m_1 \psi'_j + \dots + m_u \psi_j^{(u)}} \prod_{j=1}^{D'} (d'_j)_{R_1 \delta'_j} \cdots \prod_{j=1}^{D^{(u)}} (d_j^{(u)})_{R_u \delta_j^{(u)}}} \quad (8.1)$$

then the general class of multivariable polynomial $S_L^{h_1, \dots, h_u} [z_1, \dots, z_u]$ reduces to generalized Lauricella function defined by Srivastava et al [7].

$$F_{\bar{C}:D'; \dots; D^{(u)}}^{1+\bar{A}:B'; \dots; B^{(u)}} \left(\begin{array}{c} y_1 \\ \dots \\ y_u \end{array} \middle| \begin{array}{l} [(-L); R_1, \dots, R_u] [(a); \theta', \dots, \theta^{(u)}] : [(b'); \phi']; \dots; [(b^{(u)}); \phi^{(u)}] \\ [(c); \psi', \dots, \psi^{(u)}] : [(d'); \delta']; \dots; [(d^{(u)}); \delta^{(u)}] \end{array} \right) \quad (8.2)$$

and we have the two following formulas

Formula 1

$$\int_{u_1}^{v_1} \cdots \int_{u_s}^{v_s} \prod_{i=1}^s \left[(x_i - u_i)^{\alpha_i - 1} (v_i - x_i)^{\beta_i - 1} \prod_{j=1}^T (u_i^{(j)} x_i + v_i^{(j)})^{\sigma_i^{(j)}} \right]$$

$$\phi_{(\bar{\lambda}_1, \dots, \bar{\lambda}_p, \mu_1, \dots, \mu_q)}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)} \left[a, b, c; z \prod_{i=1}^s \left[\frac{(x_i - u_i)^{a_i} (v_i - x_i)^{b_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{c_i^{(j)}}} \right] \right]$$

$$F_{\bar{C}:D'; \dots; D^{(u)}}^{1+\bar{A}:B'; \dots; B^{(u)}} \left(\begin{array}{c} y_1 \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\alpha'_i} (v_i - x_i)^{\beta'_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\xi_i^{(j,1)}}} \right] \\ \vdots \\ y_u \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\alpha_i^{(u)}} (v_i - x_i)^{\beta_i^{(u)}}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\xi_i^{(j,u)}}} \right] \end{array} \middle| \begin{array}{l} [(-L); R_1, \dots, R_u] [(a); \theta', \dots, \theta^{(u)}] : [(b'); \phi']; \dots; [(b^{(u)}); \phi^{(u)}] \\ [(c); \psi', \dots, \psi^{(u)}] : [(d'); \delta']; \dots; [(d^{(u)}); \delta^{(u)}] \end{array} \right)$$

$$I \begin{pmatrix} z_1 \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\delta'_i} (v_i - x_i)^{\eta'_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\rho_i^{(j,1)}}} \right] \\ \vdots \\ z_r \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\delta_i^{(r)}} (v_i - x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\rho_i^{(j,r)}}} \right] \end{pmatrix}$$

$${}_P F_Q \left[(A_P); (B_Q); - \sum_{k=1}^l g_k \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\mu_i^{(k)}} (v_i - x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\tau_i^{(j,k)}}} \right] \right] dx_1 \cdots dx_s$$

$$= \frac{\prod_{j=1}^Q \Gamma(B_j)}{\prod_{j=1}^P \Gamma(A_j)} \prod_{i=1}^s \left[(v_i - u_i)^{\alpha_i + \beta_i - 1} \prod_{j=1}^W (u_i U_i^{(j)} + V_i^{(j)})^{\sigma_i^{(j)}} \prod_{j=W+1}^T (u_i U_i^{(j)} + V_i^{(j)})^{\sigma_i^{(j)}} \right]$$

$$\sum_{n'=0}^{\infty} \sum_{R_1, \dots, R_u=0}^{h_1 R_1 + \dots + h_u R_u \leq L} B_u A_{n'} \frac{z^{n'}}{n'!} y_1^{R_1} \cdots y_u^{R_u} \prod_{i=1}^s \left[(v_i - u_i)^{n'(a_i + b_i) + \sum_{k=1}^u R_k (\alpha_i^{(k)} + \beta_i^{(k)})} \right]$$

$$\prod_{i=1}^s \left[\prod_{j=1}^W (u_i U_i^{(j)} + V_i^{(j)})^{-n' c_i^{(j)} - \sum_{k=1}^u R_k \xi_i^{(j,k)}} \prod_{j=W+1}^T (u_i U_i^{(j)} + V_i^{(j)})^{-n' c_i^{(j)} - \sum_{k=1}^u R_k \xi_i^{(j,k)}} \right]$$

$$I_{p+sT+P+2s; X}^{0, n+sT+P+2s; Y} \left(\begin{array}{c|c} z_1 w_1 & A ; A^* : C \\ \cdots & \cdot \\ \cdots & \cdot \\ z_r w_r & \cdot \\ g_1 W_1 & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ g_l W_l & \cdot \\ G_1 & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ G_T & B ; B^* : D \end{array} \right) \quad (8.3)$$

under the same notations and conditions that (5.8)

and $B'_u = \frac{(-L)_{h_1 R_1 + \dots + h_u R_u} B(E; R_1, \dots, R_u)}{R_1! \cdots R_u!}$; $B(L; R_1, \dots, R_u)$ is defined by (8.1)

Formula 2

$$\int_{u_1}^{v_1} \cdots \int_{u_s}^{v_s} \prod_{i=1}^s \left[(x_i - u_i)^{\alpha_i - 1} (v_i - x_i)^{\beta_i - 1} \prod_{j=1}^T (u_i^{(j)} x_i + v_i^{(j)})^{\sigma_i^{(j)}} \right]$$

$$\phi_{(\lambda_1, \dots, \lambda_p, \sigma_1, \dots, \sigma_q)}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)} \left[a, b, c; z \prod_{i=1}^s \left[\frac{(x_i - u_i)^{a_i} (v_i - x_i)^{b_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{c_i^{(j)}}} \right] \right]$$

$$F_{\bar{C}:D'; \dots; D^{(u)}}^{1+\bar{A}:B'; \dots; B^{(u)}} \left(\begin{array}{c} y_1 \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\alpha'_i} (v_i - x_i)^{\beta'_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\xi_i^{(j,1)}}} \right] \\ \vdots \\ y_u \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\alpha_i^{(u)}} (v_i - x_i)^{\beta_i^{(u)}}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\xi_i^{(j,u)}}} \right] \end{array} \right)$$

$$\left[(-L); R_1, \dots, R_u \right] [(a); \theta', \dots, \theta^{(u)}] : [(b'); \phi']; \dots; [(b^{(u)}); \phi^{(u)}] \\ [(c); \psi', \dots, \psi^{(u)}] : [(d'); \delta']; \dots; [(d^{(u)}); \delta^{(u)}]$$

$$I \left(\begin{array}{c} z_1 \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\delta'_i} (v_i - x_i)^{\eta'_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\rho_i^{(j,1)}}} \right] \\ \vdots \\ z_r \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\delta_i^{(r)}} (v_i - x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\rho_i^{(j,r)}}} \right] \end{array} \right)$$

$$R_n^{\alpha, \beta} \left[Z \prod_{j=1}^s \left[\frac{(x_i - u_i)^{\zeta_i} (v_i - x_i)^{\lambda_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\theta_i^{(j)}}} \right] \right] dx_1 \cdots dx_r$$

$$\begin{aligned}
&= \prod_{i=1}^s \left[(v_i - u_i)^{\alpha_i + \beta_i - 1} \prod_{j=1}^W (u_i U_i^{(j)} + V_i^{(j)})^{\rho_i^{(j)}} \prod_{j=W+1}^T (u_i U_i^{(j)} + V_i^{(j)})^{\rho_i^{(j)}} \right] \sum_{n'=0}^{\infty} \sum_{R_1, \dots, R_u=0}^{h_1 R_1 + \dots + h_u R_u \leq L} \\
&\sum_{w, v, u, t, e, k_1, k_2} B_u \bar{A}_{n'} \frac{z^{n'}}{n'!} y_1^{R_1} \dots y_u^{R_u} \psi'(w, v, u, t, e, k_1, k_2) Z^R \prod_{i=1}^s \left[(v_i - u_i)^{n'(a_i + b_i) + \sum_{k=1}^u R_k (\alpha_i^{(k)} + \beta_i^{(k)})} \right] \\
&\prod_{i=1}^s \left[\prod_{j=1}^W (u_i U_i^{(j)} + V_i^{(j)})^{-n' c_i^{(j)} - \sum_{k=1}^u R_k \xi_i^{(j, k)}} \prod_{j=W+1}^T (u_i U_i^{(j)} + V_i^{(j)})^{-n' c_i^{(j)} - \sum_{k=1}^u R_k \xi_i^{(j, k)}} \right] \\
&\psi'(w, v, u, t, e, k_1, k_2) I_{p+n+sT+2s; X}^{0, n+sT+2s; X} \left(\begin{array}{c|c} z_1 w_1 & A ; A^* : C \\ \dots & \cdot \\ \dots & \cdot \\ z_r w_r & \cdot \\ \dots & \cdot \\ G_1 & \cdot \\ \dots & \cdot \\ \dots & \cdot \\ G_T & B ; B^* : D \end{array} \right) \tag{8.4}
\end{aligned}$$

under the same notations and conditions that (6.7)

and $B'_u = \frac{(-L)_{h_1 R_1 + \dots + h_u R_u} B(E; R_1, \dots, R_u)}{R_1! \dots R_u!}$; $B(L; R_1, \dots, R_u)$ is defined by (8.1)

6. Conclusion

In this paper we have evaluated two generalized multiple Eulerian integrals involving the multivariable I-functions defined by Nambisan et al [6] with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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