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Fluted Tubes and Waisted Circles

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Abstract

Motivated by the aerodynamics of a supersonic vehicle operating in a hydrogen-filled tube, a "waisted circle" is the cross-section of a fluted or grooved tube. We investigate a new class of such closed curves having a remarkably simple polar equation: It is a function only of cosine, two real parameters, and a rational coefficient of the angle. The N waists, where N is any positive integer, correspond to local minima in the curve's distance from the pole, and their locations are N-fold rotationally symmetric about the pole. We prove that the ratio of the two real parameters (with the rational coefficient constant) solely determines the shape of the waisted circle. Its area formula is a second-degree polynomial of the same two parameters, and with these fixed, area is constant and independent of the number of waists.

Keywords: Cassini oval, closed curve, rotational symmetry, tube vehicle

1. Introduction

Ongoing research is investigating the foundations of a supersonic transport vehicle operating within a hydrogen-filled tube [1]. The high sonic speed and low density of hydrogen allow a speed of Mach 2.8 – with respect to air outside the tube – but normalized energy consumption comparable to conventional subsonic aircraft [2]. Aside from the hydrogen atmosphere, increasing tube cross-sectional area, relative to the vehicle, reduces aerodynamic drag [3-5].

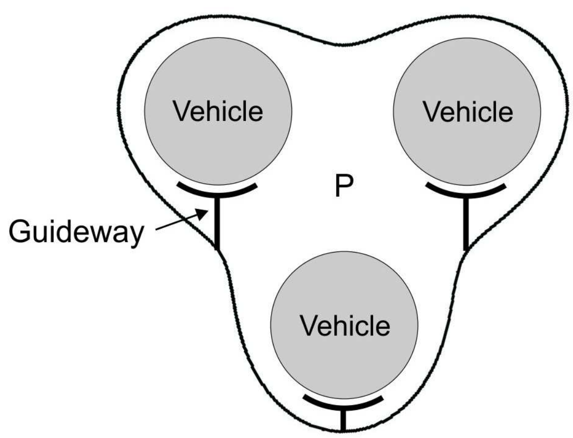

Figure 1. Schematic of conjoined tubes: Three circular-section tubes joined along their lengths form a fluted tube with a rotationally symmetric cross section having three waists. The central part of the tube (P) serves as a plenum for flows.

Because a practical tube-vehicle system must allow bidirectional and parallel transit, an efficient method of increasing cross-sectional area utilizes "conjoined tubes."

The idea of conjoined tubes is conceptually to join a bundle of two or more separate, circular-section tubes along their length to form a *fluted tube*, that is, a single tube with grooves or waists along its length. Figure 1 illustrates the idea for three conjoined tubes. A "waisted" cross-section offers the benefit of partially wrapping the tube around each vehicle and thereby (a) providing structural stiffness to the tube, (b) simplifying guideway placement, and (c) mitigating inference flow between passing vehicles. The shared central part of the fluted tube acts as a plenum for vehicular flows.

The subject of this note is a new curve, a "waisted circle," satisfying the necessary conditions for the cross section of a fluted tube: *N*-fold ($1 \le N < \infty$) rotationally symmetric, continuous, and closed. While there are indefinitely many curves satisfying these conditions, e.g., the well-known Cassini curve, the new curve is remarkable for its simplicity and mathematical tractability.

2. Results and Discussion

The waisted circle has the simple polar equation

$$r = f(t) = a + b\cos^2 ct,\ 0 \le t \le 2\pi \tag{1}$$

where *a* and *b* are positive real parameters and $|c|$ is an integral multiple of ½. *Waist* is defined as a point on the continuous curve where $f(t)$ assumes a local minimum value. Letting t_W denote the angle of a ray on which a waist occurs, we therefore require that the first two derivatives assume the values $f'(t_\mathrm{W}) = 0$ and $f''(t_\mathrm{W}) > 0$, where

$$f'(t) = -bc\sin 2ct,\ 0 \le t \le 2\pi \tag{2}$$

$$f''(t) = -2bc^2\cos 2ct,\ 0 \le t \le 2\pi \tag{3}$$

To give a closed, continuous curve on domain $T = [0,\ 2\pi]$, a solution t_W of (2) and (3) must satisfy three conditions: (i) the argument of equation (2) must satisfy $2ct_\mathrm{W} = n\pi$, where *n* is an integer, (ii) for a similar reason, parameter *c* must be an integral multiple of ½, and (iii) to correspond to a local minimum value, equation (3) requires the condition $\cos 2ct < 0$. A solution satisfying all three conditions is

$$t_i = \frac{n\pi}{2c},\quad i = 1, 2, 3, \ldots, 2|c|,\quad n = 1, 3, 5, \ldots, 4|c| - 1,\quad c \in C \tag{4}$$

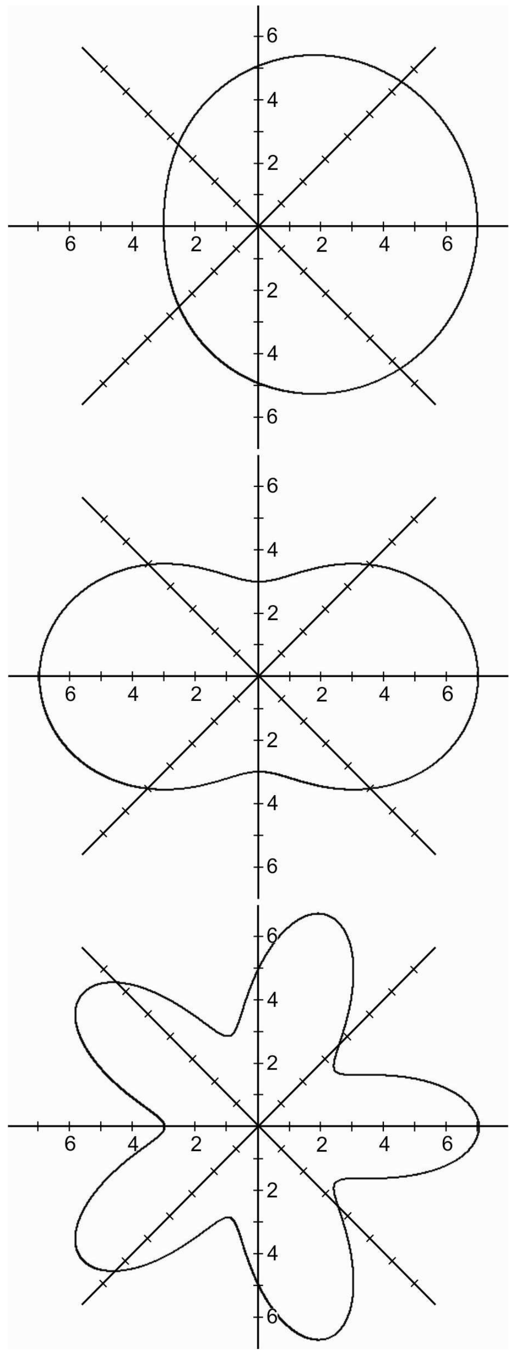

Figure 2. The waisted circle has *N* waists, where $1 \leq N < \infty$. For example, when coefficient $c = \frac{1}{2}$, 1, or $\frac{5}{2}$, the continuous curve has respectively one (top), two, or five waists. Each curve shown has parameter $a = 3$ and $b = 4$.

where index i corresponds to the ith waist and set

$$C = \{c : |c| = \tfrac{1}{2}, 1, \tfrac{3}{2}, ...\} \quad (5)$$

Because of its symmetry, the curve described by equations (1) and (5) is termed a "waisted circle." The curve has $N = 2|c|$ waists on domain T, where integer $1 \le N < \infty$, and has N-fold rotational symmetry. When $c \in C$ and $|c|$ is an integer, the closed curve has an even-number of waists; when $|c|$ is an integral multiple of ½ but is not an integer, the curve has an odd-number of waists. Figure 2 shows closed curves for $c = \frac{1}{2}, 1, \text{and} \frac{5}{2}$, which correspond respectively to one, two, and five waists. If $c \notin C$, the curve is not closed on T; when $0 < |c| < 1$ but $c \notin C$, the curve becomes a distorted open curve, or it becomes a spiral.

Waisted circle (1) is related to the double-egg curve [6, 7], which has the equation $r = \cos^2 t,\ 0 \le t \le 2\pi$, but the double egg has (two) waists in only a trivial sense: The waists bisect the curve into two separate, tangent ovals.

Define *waist width* as the common distance

$$w_i = f(t_i), \quad i = 1, 2, 3, ..., 2|c| \quad (6)$$

where t_i is given by (4). By substitution of (4) into (1), we have the waist width of the ith waist as

$$w_i = a + b\cos^2 \frac{n\pi}{2}, \quad i = 1, 2, 3, ..., 2|c|, \quad n = 1, 3, 5, ..., 4|c| - 1, \quad c \in C \quad (7)$$

Because the argument of cosine is an integral multiple of $\pi/2$, the second term of (7) is zero, and hence

$$w_i = a, \quad i = 1, 2, 3, ..., 2|c|, \quad c \in C \quad (8)$$

Thus, all N waists have the same width, namely, the value of parameter a, and for fixed b, the common width w_i decreases as $a/b \to 0$. For fixed a, since the absolute value of second derivative (3) is an increasing function of b, the curvature at a waist increases as $a/b \to 0$.

For each waist, there necessarily corresponds a local maximum value for continuous, bounded function f, and the half-line containing the maximum value locates a geometrical "lobe." Using an argument analogous to that used to find the rays of the waists, the maxima of the N lobes occur at angle

$$t_j = \frac{m\pi}{2c}, \quad j = 1, 2, 3, ..., 2|c|, \quad m = 0, 2, 4, ..., 4|c| - 2, \quad c \in C \quad (9)$$

One maximum always occurs at angle $t_1 = 0$. Because $f(0) = a + b$, all maximum values lie on a circle *K*, centered at the pole, with radius $r = a + b$, which circumscribes the waisted circle, and the range of function (1) is

$$0 \le a \le f(t) \le a + b,\ \ 0 \le t \le 2\pi,\ \ b \ge 0 \tag{10}$$

With the number of waists *N* given by fixing coefficient *c*, the shape of waisted circle (1) is determined solely by the ratio a/b. This result is equivalent to the following:

Proposition. With *c* fixed, and as *a*, *b*, and *t* are varied, distance *r* is unchanged up to multiplication by real constant *k* (a scale factor) if and only if *a*/*b* is constant.

Proof. Suppose distance *r* is multiplied by constant *k* so that $r' = kr = k(a + b\cos^2 ct)$ $= ka + kb\cos^2 ct = a' + b'\cos^2 ct$. Hence, the ratio of constants is $a'/b' = ka/kb = a/b$, and the ratio is the same constant at each *t*. Suppose the ratio *a*/*b* is constant as *a*, *b*, and *t* are varied so that $a/b = ka/kb = a'/b'$. Distance $r' = a' + b'\cos^2 ct = ka + kb\cos^2 ct = k(a + b\cos^2 ct) = kr$ and distance *r* is unchanged up to multiplication by constant *k*.

The area formula for waisted circle (1) has an antiderivative in terms of elementary functions, and area A is a simple polynomial function of *a* and *b*:

$$A = \int_0^{2\pi} \tfrac{1}{2} f(t)^2\, dt = \pi\left(a^2 + ab + \tfrac{3}{8} b^2\right) \tag{11}$$

Thus, area is independent of coefficient *c*, and for given *a* and *b*, the area is constant and independent of the number of waists *N*.

Arc length, also relatively simple, has the formula

$$L = \int_0^{2\pi} \sqrt{(a + b\cos^2 ct)^2 + (bc\sin 2ct)^2}\, dt \tag{12}$$

If *a* and *b* are finite so that the radius of circumscribing circle *K* is finite, then as $c \to \infty$, equation (12) shows that arc length $L \to \infty$. Hence, a waisted circle with unbounded arc length is contained within a finite-radius circle. Integral (12) evidently has no antiderivative in terms of a finite combination of elementary functions, but the numerical integral converges rapidly for reasonable values of the parameters.

The two-parameter Cassini oval, $r = \varphi(t) = \alpha\sqrt{\cos 2t + \sqrt{(\beta/\alpha)^4 - \sin^2 2t}}$, where α and β are the parameters, is an example of a closed curve having two nontrivial waists [8]. It has numerous applications. Some applications, such as modeling bistatic radar systems [9], derive from its constant product of focal chords. Others – for example, modeling the geometry of the red blood cell [10], the shape of coalescing urban centers [11], and nuclear fission [12] – derive only from its being waisted. The Cassini curve can have more than two waists [13], with multi-static radar being an application [14]. In comparison to the simple relation of equation (8), the Cassini oval has waist width $\alpha\sqrt{(\beta/\alpha)^2-1}$. Although requiring an argument more involved than the proof above, the shape of the Cassini oval is also determined by the ratio of its parameters α and β [8]. In contrast to formula (11), the Cassini oval's area function, an elliptic integral, has no antiderivative expressible as a finite combination of elementary functions.

3. Conclusions

The new class of closed curves may have applications similar to the Cassini curve. It shares the characteristics of being waisted, having a shape determined solely by the ratio of two parameters, having *N*-fold rotational symmetry, and having any integral number *N* of waists. The new curve, however, is mathematically more tractable than the Cassini oval. It may serve as an approximation to the Cassini oval or be the basis of entirely new applications based on its waists, rotational symmetry, or constant-area characteristic.

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