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Exact solutions for nonlinear partial differential equations by using the extended tanh- method

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Abstract

The tanh method is a powerful solution method; various extension forms of the tanh method have been developed with a computerized symbolic computation and is used for constructing the exact travelling wave solutions, of coupled nonlinear equations arising in physics. The obtained solutions include solitons, kinks and plane periodic solutions. First a power series in tanh was used as an ansatz to obtain analytical solutions of traveling wave type of certain nonlinear evolution equations .The main properties of the method will be explained and then applied to particular and well-chosen examples in further works to establish more entirely new solutions for other kinds of nonlinear evolution equations arising in physics

Keywords: Extended tanh method, nonlinear physical models, Solitons, kinks solution,plane periodic solutions

1 Introduction

Nonlinear evolution equations have a major role in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, solid state physics, chemical kinematics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed. A variety of powerful methods, such as inverse scattering method [1, 13], bilinear tranformation [7, 8, 12], the tanh-sech method [9, 11, 14], extended tanh method [4, 6], sine-cosine method [15–16, 19], F-expansion method [3], and homogeneous balance method [5] were used to investigate nonlinear dispersive and dissipative problems.The pioneer work Malfiet in [9, 10] introduced the powerful tanh method for a reliable treatment of the nonlinear wave equations. The useful tanh method is widely used by many work such as in [18–20, 21] and by the references there in.Later, the extended tanh method, developed by Wazwaz [20, 18], is a direct and effective algebraic method for handling nonlinear equations. Various extensions of the method were developed as well.

Our first interest in present work being in implementing the extended tanh method to stress its power in handling nonlinear equations so that one can apply it to models of various types of nonlinearity. The next interest is in the determination of exact traveling wave solutions for nonlinear partial differential equations. Searching for exact solutions of nonlinear problems has attracted a considerable amount of research work where computer symbolic systems facilitate the computational work.

2. EXTENDED TANH METHOD

We now describe the tanh method for a given partial differential equation. This Method was defined by Malfliet [21] and Fan and Hon [23]. Wazwaz summarized the main steps introduced for using this method as follows [20],

1- first considered a general form of nonlinear equation

$$H\left(u, u_t, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{xxy}, \ldots\right) = 0, \qquad \textbf{(1)}$$

2- To find the traveling wave solution of Eq. (1) he introduced the wave variable:

$$(2) \quad \xi = k\,x + \lambda\,y + c\,t,$$

So that

$$(3)\, u(t,x,y) = U(\xi),$$

which are of important physical significance, , and c are constants **, to** k, λ be determined later. Then system (1) reduces to a system of nonlinear ordinary differential equations

$$(4)\; H\left(U, cU', kU', \lambda U', k^2 U'', \ldots\right) = 0,$$

$$\frac{d}{d\xi} = \left(1 - Y^2\right)\frac{d}{dY},$$

$$\frac{d^2}{d\xi^2} = \left(1 - Y^2\right)\left[-2Y\frac{d}{dY} + \left(1 - Y^2\right)\frac{d^2}{dY^2}\right], \tag{6}$$

$$\frac{d^3}{d\xi^3} = \left(1 - Y^2\right)\left[\left(6Y^2 - 2\right)\frac{d}{dY} - 6Y\left(1 - Y^2\right)\frac{d^2}{dY^2} + \left(1 - Y^2\right)^2\frac{d^3}{dY^3}\right],$$

and the remaining derivatives may be derived similarly.

5- Introduce the ansatz and then solution of $U(\xi)$ is in the form of Tanh Method:

$$U(\xi) = \sum_{p=0}^{m} \alpha_p Y^p = \alpha_0 + \alpha_1 Y + \ldots + \alpha_m Y^m, \tag{7}$$

In order to construct more general, it is reasonable to introduce the following ansatz [6]

. Extended Tanh Method:

$$U(\xi) = \sum_{p=-m}^{m} \alpha_p Y^p = \alpha_{-m} Y^{-m} + \ldots + \alpha_0 + \ldots + \alpha_m Y^m \tag{8}$$

α_p where *m* is positive integer which is unknown to be later determined, is unknown constant

6- To determine the parameter *m* , we usually balance linear terms of the highest order in the resulting equation with the highest order nonlinear terms: With *m* determined as described, equating the coefficients of powers of *Y* in the resulting equation will give a system of algebraic equations involving the $\alpha_p, (p = -m, \ldots, 0, \ldots, m), k, \lambda$ and c. Having determined these parameters, taking into account that in most cases these parameters are positive, we find, on using (24), an analytic solution in closed form. The traveling wave solutions of many nonlinear ODEs and PDEs from soliton theory (and elsewhere) can be expressed as polynomials of hyperbolic or elliptic functions. For instance, the bell shaped sech- solutions and kink shaped tanh-solutions model wave phenomena in fluid dynamics, plasmas, elastic media, electrical circuits, optical fibers, chemical reactions, eactions, and bio-genetics.

3. APPLICATIONS

In this section, we will demonstrate the proposed method on three nonlinear evolution equations of special interest in physics

Example 1. Let us first consider the system of nonlinear evolution equations which has the form

$$\begin{aligned} q_t &= a\left[q_{xxx} - 6\,qq_x + 3\,\partial_x^{-1} q_{yy} + 6\,pr_x - 6r\,p_x\right], \\ p_t &= b\left[-\,p_{xxx} - 3\,q\,p_x - 3\,p_{xy} + w\,p^2\,\partial_x^{-1} q_y\right], \\ r_t &= E\left[-\,r_{xxx} - 3qr_x - 3r_{xy} - hr^2\partial_x^{-1} q_y\right], \end{aligned} \tag{9}$$

To investigate Eq.(9), let us now apply the transformation $q_{y=S_x}$, so as to convert it into

$$\begin{aligned} q_y &= S_x, \\ q_t &= a\left[q_{xxx} - 6qq_x + 3S_y + 6\,pr_x - 6\,p_x r\right], \\ p_t &= b\left[-\,p_{xxx} - 3qp_x - 3\,p_{xy} + w\,p^2 S\right], \\ r_t &= E\left[-\,r_{xxx} - 3q\,r_x - 3r_{xy} - h\,r^2 S\right], \end{aligned} \tag{10}$$

Where a, b, E, w and h are positive constants, Following the above procedure ,By introducing a complex variation ξ defined as Eq. (2),

$$q(t,x,y) = q(\xi), \qquad \mathrm{S}(\mathrm{t},\mathrm{x},\mathrm{y}) = S(\xi), \qquad \mathrm{p}(\mathrm{t},\mathrm{x},\mathrm{y}) = p(\xi), \qquad \mathrm{r}(\mathrm{t},\mathrm{x},\mathrm{y}) = R(\xi). \tag{11}$$

We transform system (10) to system of ordinary differential equations in the form

$$\begin{aligned} &\lambda\, q' = kS' \\ &c\,q' = a\left[k^3 q''' - 6k\,q\,q' + 3\lambda\,S' + 6k\,pR' - 6kR\,p'\right], \\ &c\,p' = b\left[-\,k^3 p''' - 3k\,p'q - 3k\lambda\,p'' + w\,p^2 S\right], \\ &c\,R' = E\left[-\,k^3 R''' - 3k\,q\,R' - 3k\lambda\,R'' - hR^2 S\right] \end{aligned} \tag{12}$$

The extended tanh method admits the use of the finite expansion

$$q(\xi) = \sum_{p=-n}^{n} \alpha_p Y^P, \quad \mathrm{P}(\xi) = \sum_{P=-L}^{L} \beta_p Y^p, \quad \mathrm{S}(\xi) = \sum_{p=-m}^{m} \gamma_p Y^p, \quad R(\xi) = \sum_{p=-e}^{e} \phi_p Y^p \tag{13}$$

where

$$Y = \tanh(\xi).$$

Balancing q' term with S' in the first equation and q''' term with qq' in the second equation and p''' term with p^2S in the third equation and R''' term with R^2S in the four equation leads to the following ansatz: (we take m □2, n □2,L □1,e =1)

$$
\begin{aligned}
q(\xi) &= \alpha_{-2}Y^{-2} + \alpha_{-1}Y^{-1} + \alpha_0 + \alpha_1 Y + \alpha_2 Y^2, \\
S(\xi) &= \gamma_{-2}Y^{-2} + \gamma_{-1}Y^{-1} + \gamma_0 + \gamma_1 Y + \gamma_2 Y^2, \\
p(\xi) &= \beta_{-1}Y^{-1} + \beta_0 + \beta_1 Y, \\
R(\xi) &= \phi_{-1}Y^{-1} + \phi_0 + \phi_1 Y,
\end{aligned}
\tag{14}
$$

Substituting Eqs. (14) into Eqs. (12) and equating the coefficients of the powers Y then we get the following algebraic relations:

$$0 = \lambda\alpha_{-2} - k\gamma_{-2},$$

$$0 = \lambda\alpha_{-1} - k\gamma_{-1},$$

$$0 = \lambda\alpha_{0} - k\gamma_{0},$$

$$0 = \lambda\alpha_{1} - k\gamma_{1},$$

$$0 = \lambda\alpha_{2} - k\gamma_{2},$$

$$0 = 2c\alpha_{-2} + 16ak^3\alpha_{-2} + 6ak\alpha_{-1}^2 + 12ak\alpha_0\alpha_{-2} - 6a\lambda\gamma_{-2} - 6ak\beta_{-1}\phi_{-1} + 6ak\phi_{-1}\beta_{-1},$$

$$0 = c\alpha_{-1} + 2ak^3\alpha_{-1} - 6ak\alpha_{-2}\alpha_1 + 6ak\alpha_0\alpha_{-1} + 12ak\alpha_1\alpha_{-2} - 3a\lambda\gamma_{-1} - 6ak\beta_0\phi_{-1} + 6ak\phi_0\beta_{-1},$$

$$0 = -c\alpha_{-1} - 2ak^3\alpha_1 - 12ak\alpha_{-1}\alpha_2 - 6ak\alpha_0\alpha_1 + 6ak\alpha_{-1}\alpha_2 + 3a\lambda\gamma_1 + 6ak\beta_0\phi_1 - 6ak\phi_0\beta_1,$$

$$0 = -2c\alpha_2 - 16ak^3\alpha_2 - 12ak\alpha_2\alpha_0 - 6ak\alpha_1^2 + 6a\lambda\gamma_2 + 6ak\beta_1\phi_1 - 6ak\phi_{1\beta_1},$$

$$0 = 6ak^3\alpha_1 - 12ak\alpha_1\alpha_2 - 6ak\alpha_2\alpha_1,$$

$$0 = 24ak^3\alpha_2 - 12ak\alpha_2^2,$$

$$0 = -24ak^3\alpha_{-2} + 12ak\alpha_{-2}^2,$$

$$0 = -6ak^3\alpha_{-1} + 6ak\alpha_{-2}\alpha_{-1} + 12ak\alpha_{-2}\alpha_{-1},$$

$$0 = -6ak\alpha_{-1}\alpha_{1+6ak\alpha_1}\alpha_{-1} + 12ak\alpha_2\alpha_{-2} + 6ak\beta_{-1}\phi_1 - 6ak\beta_1\phi_{-1} - 6ak\phi_{-1}\beta_{1+6ak\phi_1}\beta_{-1} - 12ak\alpha_{-2}\alpha_2$$

$$0 = -c\beta_1 - c\beta_{-1} + 2bk^3\beta_1 + 2bk^3\beta_{-1} + 3bk\beta_{-1}\alpha_2 - 3bk\beta_1\alpha_0 - 3bk\beta_{-1}\alpha_0 + 3bk\beta_1\alpha_{-2} + wb\beta_{-1}^2\gamma_2 + 2bw\beta_{-1}\beta_0\gamma_1 + 2bw\beta_{-1}\beta_1\gamma_0 + bw\beta_0^2\gamma_0 + 2bw\beta_0\beta_1\gamma_{-1} + bw\beta_1^2\gamma_{-2},$$

$$0 = c\beta_1 - 8bk^3\beta_1 - 3bk\beta_1\alpha_2 - 3bk\beta_{-1}\alpha_2 + 3bk\beta_1\alpha_0 + 2bw\beta_{-1}\beta_1\gamma_2 + bw\beta_0^2\gamma_2 + 2bw\beta_0\beta_1\gamma_1 + bw\beta_1^2\gamma_0,$$

$$0 = 6\,b\,k^3\beta_{-1} + 3\,b\,k\,\beta_{-1}\,\alpha_{-2} + w\,b\,\beta_{-1}^2\,\gamma_{-2},$$

$$0 = 6\,b\,k^3\beta_1 + 3\,b\,k\,\beta_1\,\alpha_2 + w\,\beta_1^2\,\gamma_2,$$

$$0 = 3\,k\,b\,\beta_{-1}\alpha_{-1} - 6\,b\,k\,\lambda\,\beta_{-1} + w\,b\,\beta_{-1}^2\,\gamma_{-1} + 2\,b\,w\,\beta_{-1}\,\beta_0\,\gamma_{-2},$$

$$0 = 3\,k\,b\,\beta_1\,\alpha_1 - 6\,b\,k\,\lambda\,\beta_1 + 2\,b\,w\,\beta_0\,\beta_1\,\gamma_2 + b\,w\,\beta_1^2\,\gamma_1,$$

$$0 = 3\,b\,k\,\beta_{-1}\,\alpha_1 - 3\,b\,k\,\beta_1\,\alpha_{-1} - 3b\,k\,\beta_{-1}\,\alpha_{-1} + 6\,k\,\lambda\,\beta_{-1} + w\,b\,\beta_{-1}^2\,\gamma_1$$
$$+ 2\,b\,w\,\gamma_0\,\beta_{-1}\,\beta_0 + 2b\,w\,\beta_{-1}\beta_1\,\gamma_{-1} + b\,w\,\beta_0^2\,\gamma_{-1} + 2\,b\,w\,\beta_0\,\beta_1\,\gamma_{-2},$$

$$0 = -3\,b\,k\,\beta_1\,\alpha_1 - 3\,b\,k\,\beta_{-1}\,\alpha_1 + 3\,k\,b\,\beta_1\,\alpha_{-1} + 6\,b\,k\,\lambda\,\beta_1 + 2\,b\,w\,\beta_{-1}\,\beta_0\,\gamma_2$$
$$+ 2\,b\,w\,\beta_{-1}\,\beta_1\,\gamma_1 + b\,w\,\beta_0^2\,\gamma_1 + 2\,b\,w\,\beta_0\,\beta_1\,\gamma_0 + b\,w\,\beta_1^2\,\gamma_{-1},$$

$$0 = c\,\phi_{-1} - 8\,E\,k^3\,\phi_{-1} - 3\,E\,k\,\phi_1\,\alpha_{-2} - 2\,E\,k\,\alpha_{-2}\,\phi_{-1} + 3\,E\,k\,\alpha_0\,\phi_{-1} - E\,h\,\phi_{-1}^2\,\gamma_0$$
$$- 2\,E\,h\,\phi_{-1}\,\phi_0\,\gamma_{-1} - 2\,E\,h\,\phi_{-1}\,\phi_1\,\gamma_{-2} - E\,h\,\phi_0^2\,\gamma_{-2},$$

$$0 = -c\,\phi_1 - c\,\phi_{-1} + 2E\,k^3\phi_1 + 2\,E\,k^3\phi_{-1} + 3\,k\,E\,\alpha_{-2}\,\phi_1 - 3\,E\,k\,\alpha_0\,\phi_1 - 3E\,k\,\alpha_0\,\phi_{-1}$$
$$+ 3\,E\,k\,\alpha_2\,\phi_{-1} - E\,h\,\phi_{-1}^2\,\gamma_2 - 2E\,h\,\phi_{-1}\,\phi_0\,\gamma_1 - 2\,E\,h\,\phi_{-1}\,\phi_1\,\gamma_0 - E\,h\,\phi_0^2\,\gamma_0$$
$$- 2E\,h\,\phi_0\,\phi_1\,\gamma_{-1} - E\,h\,\phi_1^2\,\gamma_{-2}$$

$$0 = c\,\phi_1 - 8\,E\,k^3\,\phi_1 + 3\,E\,k\,\alpha_0\,\phi_1 - 3\,E\,k\,\alpha_2\,\phi_1 - 3\,E\,k\,\alpha_2\,\phi_{-1} - 2\,E\,h\,\phi_{-1}\,\phi_1\,\gamma_2$$
$$- E\,h\,\phi_0^2\,\gamma_2 - 2\,E\,h\,\phi_0\,\phi_1\,\gamma_1 - E\,h\,\phi_1^2\,\gamma_0$$

$$0 = 6\,E\,k^3\phi_{-1} + 3\,E\,k\,\alpha_{-2}\,\phi_{-1} - E\,h\,\phi_{-1}^2\,\gamma_{-2},$$

$$0 = 6\,E\,k^3\phi_1 + 3\,E\,k\,\phi_1\,\alpha_2 - E\,h\,\phi_1^2\,\gamma_2,$$

$$0 = 3\,E\,k\,\alpha_{-1}\,\phi_{-1} - 6\,E\,k\,\lambda\,\phi_{-1} - E\,h\,\phi_{-1}^2\,\gamma_{-1} - 2E\,h\,\phi_{-1}\,\phi_0\,\gamma_{-2},$$

$$0 = 3E\,k\,\alpha_1\,\phi_1 - 6E\,k\,\lambda\,\phi_1 - 2E\,h\,\phi_0\,\phi_1\,\gamma_2 - E\,h\,\phi_1^2\,\gamma_1,$$

$$0 = 3E\,k\,\alpha_{-1}\,\phi_1 - 3Ek\,\alpha_1\phi_1 - 3E\,k\alpha_1\phi_{-1} + 6E\,k\lambda\,\phi_1 + 6E\,k\lambda\,\phi_{-1} - 6E\,k\lambda\,\phi_{-1}$$
$$- 2Eh\,\phi_{-1}\phi_0\gamma_2 - 2Eh\,\phi_{-1}\phi_1\gamma_1 - Eh\,\phi_0^2\gamma_1 - 2Eh\,\phi_0\,\phi_1\,\gamma_0 - Eh\,\phi_1^2\gamma_{-1},$$

$$0 = -3Ek\,\alpha_{-1}\phi_1 - 3Ek\,\alpha_{-1}\phi_{-1} + 3Ek\,\alpha_1\phi_{-1} + 6E\,k\lambda\,\phi_{-1} - Eh\,\phi_{-1}^2\gamma_1 - 2Eh\,\phi_{-1}\phi_0\gamma_0$$
$$- 2E\,h\phi_{-1}\phi_1\,\gamma_{-1} - Eh\,\phi_0^2\,\gamma_{-1} - 2Eh\,\phi_0\phi_1\,\gamma_{-2}, \tag{15}$$

Solving the system of equations (15) we obtain that

$$\alpha_1 = 0, \quad \gamma_1 = 0, \quad \alpha_{-1} = 0, \quad \gamma_{-1} = 0,$$
$$\alpha_2 = 2k^2, \quad \alpha_{-2} = 2k^2, \quad \gamma_2 = 2k\lambda, \quad \gamma_{-2} = 2k\lambda, \tag{16}$$
$$\alpha_0 = -4k^2, \quad \gamma_0 = -4k\lambda, \quad \phi_0 = \frac{-3}{2h}, \quad \phi_1 = \frac{6k^2}{h\lambda},$$
$$\phi_{-1} = \frac{6k^2}{h\lambda}, \quad \beta_0 = \frac{3}{2w}, \quad \beta_{-1} = \frac{-6k^2}{\lambda w}, \quad \beta_1 = \frac{-6k^2}{\lambda w},$$
$$c = 32bk^3 + \frac{3}{4}\frac{\lambda^2}{k}b, \quad \lambda = \pm\sqrt{\frac{64[2b-a]}{3[4a-b]}}\,k^2, \quad E = b$$

are arbitrary constants . we are resulting the solution of Eq. k, a, b, w, h and E where (9) in the form

$$q = 2k^2\left[-2 + cath^2\xi + \tanh^2\xi\right], \qquad S = 2k\lambda\left[-2 + \text{cath}^2\xi + \tanh^2\xi\right],$$
$$p = -\frac{6k^2}{\lambda w}\left[cath\,\xi + \tanh\xi\right] + \frac{3}{2w}, \qquad R = \frac{6k^2}{\lambda h}\left[cath\,\xi + \tanh\xi\right] - \frac{3}{2h},$$

where (17) $$\xi = kx \pm \sqrt{\frac{64[2b-a]}{3[4a-b]}}\,k^2 y + \left[32bk^3 + \frac{3\lambda^2 b}{4k}\right]t,$$

Example 2. We study the following class of nonlinear systems of partial differential equations:

$$u_t + v_x + u\,u_x + u_{xx} = 0,$$
$$v_t + (uv)_x + u_{xxx} = 0, \tag{18}$$

To solve the system of Eq.(18) by means of the modified extended tanh-function method ,we used

$$\xi = kx + ct, \qquad u(t,x) = U(\xi), \qquad v(t,x) = V(\xi) \tag{19}$$

System of Eq.(18) into the ODE Carries the

$$cU' + kV' + kUU' + k^2U'' = 0,$$
$$cV' + k(UV)' + k^3U''' = 0. \tag{20}$$

once, we obtain ξ Integrating (20)with respect to

$$cU + kV + k\left(\frac{1}{2}U^2\right) + k^2U' = c_1,$$
$$cV + k(UV) + k^3U'' = c_2. \tag{21}$$

From the first equation of (21),we have that $c_1, c_2 = 0,$ where

$$V = \frac{-c}{k}U - \frac{1}{2}U^2 - kU' \tag{22}$$

Substituting Eq.(22) into the second equation of (21) ,we get

$$-\frac{c^2}{k}U - \frac{3}{2}cU^2 - \frac{k}{2}U^3 - ckU' - k^2UU' + k^3U'' = 0, \tag{23}$$

Introduciong a new independent variable $Y = \tanh(\xi)$.

The extended tanh method admits the use of the finite expansion

$$U(\xi) = \sum_{p=-m}^{m} \alpha_p Y^p, \tag{24}$$

Balancing the order of with the order of U'' in Eq (23) leads to the U^3 following ansatz: (we take m □1).

$$U(\xi) = \alpha_{-1}Y^{-1} + \alpha_0 + \alpha_1 Y \tag{25}$$

Substituting Eq. (25) into Eq. (23), and equating the coefficients of the powers Y then we obtain a system of algebraic equations in the form

$$0 = -\frac{3}{2}c\alpha_1^2 - \frac{3}{2}k\alpha_0\alpha_1^2 + ck\alpha_1 + k^2\alpha_0\alpha_1,$$

$$0 = -\frac{k}{2}\alpha_1^3 + k^2\alpha_1^2 + 2k^3\alpha_1,$$

$$0 = -\frac{c^2}{k}\alpha_0 - 3c\alpha_{-1}\alpha_1 - \frac{3}{2}c\alpha_0^2 - 3k\alpha_{-1}\alpha_0\alpha_1 - \frac{k}{2}\alpha_0^3 - 2ck\alpha_1 - k^2\alpha_0\alpha_1 - k^2\alpha_{-1}\alpha_0,$$

$$0 = -\frac{c^2}{k}\alpha_1 - 3c\alpha_0\alpha_1 - \frac{3}{2}k\alpha_{-1}\alpha_1^2 - \frac{3}{2}k\alpha_0^2\alpha_1 - k^2\alpha_1^2 - 2k^3\alpha_1,$$

$$0 = -\frac{c^2}{k}\alpha_{-1} - 3c\alpha_{-1}\alpha_0 - \frac{3}{2}k\alpha_{-1}^2\alpha_1 - \frac{3}{2}k\alpha_{-1}\alpha_0^2 - k^2\alpha_{-1}^2 - 2k^3\alpha_{-1},$$

$$0 = -\frac{3}{2}c\alpha_{-1}^2 - \frac{3}{2}k\alpha_{-1}^2\alpha_0 + ck\alpha_{-1} + k^2\alpha_0\alpha_{-1},$$

$$0 = -\frac{k}{2}\alpha_{-1}^3 + k^2\alpha_{-1}^2 + 2k^3\alpha_{-1}. \tag{26}$$

by solving the system of Eq (26). we obtained the six cases of solutions

i) *The first case:*

$$\alpha_0 = -\frac{c}{k}, \qquad \alpha_1 = \alpha_{-1} = k\left[1+\sqrt{5}\right], \qquad c = \pm k^2\left[22+8\sqrt{5}\right]^{\frac{1}{2}}, \tag{27}$$

In view of this we obtain the solution of system Eq(18), in the form

$$u(t,x)=-\frac{c}{k}+k\left[1+\sqrt{5}\right]\left[\coth(kx+ct)+\tanh(kx+ct)\right],$$
$$v(t,x)=-\frac{c}{k}\left[-\frac{c}{k}+k\left(1+\sqrt{5}\right)\left[cath(kx+ct)+\tanh(kx+ct)\right]\right]$$
$$-\frac{1}{2}\left[\frac{-c}{k}+k\left(1+\sqrt{5}\right)\left[cath(kx+ct)+\tanh(kx+ct)\right]\right]^2 \tag{28}$$
$$-\mathrm{k}\left[\mathrm{k}\left(1+\sqrt{5}\right)\right]\left[-\csc h^2(kx+ct)+\sec h^2(kx+ct)\right],$$

ii) ***The second case:***

$$\alpha_0=-\frac{c}{k},\qquad \alpha_1=\alpha_{-1}=k\left[1-\sqrt{5}\right],\qquad \mathrm{c}=\pm\mathrm{k}^2\left[18-8\sqrt{5}\right]^{\frac{1}{2}} \tag{29}$$

In view of this we obtain the solution of system Eq(18), in the form

$$u(t,x)=-\frac{c}{k}+k\left[1-\sqrt{5}\right]\left[cath(kx+ct)+\tanh(kx+ct)\right],$$
$$v(t,x)=-\frac{c}{k}\left[-\frac{c}{k}+k\left(1-\sqrt{5}\right)\left[cath(kx+ct)+\tanh(kx+ct)\right]\right]$$
$$-\frac{1}{2}\left[\frac{-c}{k}+k\left(1-\sqrt{5}\right)\left[cath(kx+ct)+\tanh(kx+ct)\right]\right]^2 \tag{30}$$
$$-\mathrm{k}\left[\mathrm{k}\left(1-\sqrt{5}\right)\right]\left[-\csc h^2(kx+ct)+\sec h^2(kx+ct)\right],$$

iii) ***The third case:***

$$\alpha_0=-\frac{c}{k},\qquad \alpha_{-1}=0,\qquad \alpha_1=k\left[1+\sqrt{5}\right],\qquad \mathrm{c}=\pm\mathrm{k}^2\left[6+2\sqrt{5}\right]^{\frac{1}{2}}, \tag{31}$$

In view of this we obtain the solution of system Eq(18), in the form

$$u(t,x)=-\frac{c}{k}+k\left[1+\sqrt{5}\right]\left[\tanh(kx+ct)\right],$$
$$v(t,x)=-\frac{c}{k}\left[-\frac{c}{k}+k\left(1+\sqrt{5}\right)\left[\tanh(kx+ct)\right]\right]-\frac{1}{2}\left[\frac{-c}{k}+k\left(1+\sqrt{5}\right)\left[\tanh(kx+ct)\right]\right]^2$$
$$-\mathrm{k}\left[\mathrm{k}\left(1+\sqrt{5}\right)\right]\left[\sec h^2(kx+ct)\right], \tag{32}$$

iv) ***The fourth case :***

$$\alpha_0 = -\frac{c}{k}, \qquad \alpha_1 = 0, \qquad \alpha_{-1} = k\left[1+\sqrt{5}\right], \qquad \mathrm{c} = \pm \mathrm{k}^2\left[6+2\sqrt{5}\right]^{\frac{1}{2}}, \tag{33}$$

In view of this we obtain the solution of system Eq(18), in the form

$$u(t,x) = -\frac{c}{k} + k\left[1+\sqrt{5}\right]\left[\coth(kx+ct)\right],$$

$$v(t,x) = -\frac{c}{k}\left[-\frac{c}{k} + k\left(1+\sqrt{5}\right)\left[cath(kx+ct)\right]\right] - \frac{1}{2}\left[\frac{-c}{k} + k\left(1+\sqrt{5}\right)\left[cath(kx+ct)\right]\right]^2$$
$$- \mathrm{k}\left[\mathrm{k}\left(1+\sqrt{5}\right)\right]\left[-\csc h^2(kx+ct)\right], \tag{34}$$

v) ***The fifth case:***

$$(35) \quad \alpha_0 = -\frac{c}{k}, \qquad \alpha_1 = 0 \qquad \alpha_{-1} = k\left[1-\sqrt{5}\right], \qquad \mathrm{c} = \pm \mathrm{k}^2\left[6-2\sqrt{5}\right]^{\frac{1}{2}}$$

In view of this we obtain the solution of system Eq(18), in the form

$$u(t,x) = -\frac{c}{k} + k\left[1-\sqrt{5}\right]\left[cath(kx+ct)\right],$$

$$v(t,x) = -\frac{c}{k}\left[-\frac{c}{k} + k\left(1-\sqrt{5}\right)\left[cath(kx+ct)\right]\right] - \frac{1}{2}\left[\frac{-c}{k} + k\left(1-\sqrt{5}\right)\left[cath(kx+ct)\right]\right]^2$$
$$- \mathrm{k}\left[\mathrm{k}\left(1-\sqrt{5}\right)\right]\left[-\csc h^2(kx+ct)\right], \tag{36}$$

vi) ***The sixth case:***

$$\alpha_0 = -\frac{c}{k}, \qquad \alpha_{-1} = 0 \qquad \alpha_1 = k\left[1-\sqrt{5}\right], \qquad \mathrm{c} = \pm \mathrm{k}^2\left[6-2\sqrt{5}\right]^{\frac{1}{2}} \tag{37}$$

In view of this we obtain the solution of system Eq(18), in the form

$$u(t,x) = -\frac{c}{k} + k\left[1-\sqrt{5}\right]\left[\tanh(kx+ct)\right],$$

$$v(t,x) = -\frac{c}{k}\left[-\frac{c}{k} + k\left(1-\sqrt{5}\right)\left[\tanh(kx+ct)\right]\right] - \frac{1}{2}\left[\frac{-c}{k} + k\left(1-\sqrt{5}\right)\left[\tanh(kx+ct)\right]\right]^2$$
$$- \mathrm{k}\left[\mathrm{k}\left(1-\sqrt{5}\right)\right]\left[\sec h^2(kx+ct)\right], \tag{38}$$

Example 3. A new coupled modified KdV-type equation

$$\begin{aligned} u_t &= u_{xxx} - 12u\,u_x + \lambda v_x, \\ v_t &= b\,u_{xxx} + v_{xxx} - 12v\,u_x - 12uv_x \end{aligned} \tag{39}$$

Using the wave variable

$$\xi = kx + ct, \quad \mathrm{u}(\mathrm{t},\mathrm{x}) = U(\xi), \qquad v(\mathrm{t},\mathrm{x}) = V(\xi). \tag{40}$$

carries Eq. (39) into the ODE

$$\begin{aligned} &-cU' + k^3U''' - 12kUU' + \lambda kV' = 0, \\ &-cV' + bk^3U''' + k^3V''' - 12k(UV)' = 0, \end{aligned} \tag{41}$$

By integrating this Eq.(41)and by considering the constant of integration to be zero , we obtain a simplified ODE

$$\begin{aligned} &-cU + k^3U'' - 6kU^2 + \lambda kV = 0, \\ &-cV + bk^3U'' + k^3V'' - 12k(UV) = 0, \end{aligned} \tag{42}$$

The extended tanh method admits the use of the finite expansion

$$U(\xi) = \sum_{p=-m}^{m} \alpha_p Y^p, \qquad \mathrm{V}(\xi) = \sum_{p=n}^{n} \beta_p Y^p, \tag{43}$$

where

$$Y = \tanh(\xi).$$

Balancing U'' term with U^2 , in the first equation and U'' term with UV in the second equation of (42), leads to the following ansatz: (we take m □2, n □2).

$$\begin{aligned} U(\xi) &= \alpha_{-2}Y^{-2} + \alpha_{-1}Y^{-1} + \alpha_0 + \alpha_1 Y + \alpha_2 Y^2, \\ V(\xi) &= \beta_{-2}Y^{-2} + \beta_{-1}Y^{-1} + \beta_0 + \beta_1 Y + \beta_2 Y^2, \end{aligned} \tag{44}$$

Substituting Eq. (44) into Eq. (42), and proceeding as before we obtain the solution in the form

$$\begin{aligned} &\alpha_{-2} = k^2, \qquad \alpha_2 = k^2, \qquad \alpha_0 = -\frac{c}{12k} - \frac{2k^2}{3} + \frac{\lambda b}{12}, \qquad \alpha_1 = \alpha_{-1} = 0, \\ &\beta_{-2} = bk^2, \qquad \beta_2 = bk^2, \qquad \beta_0 = -\frac{2bk^2}{3} - \frac{\lambda b^2}{12}, \qquad \beta_1 = \beta_{-1} = 0, \\ &k = \pm\frac{\lambda b}{16}, \qquad \mathrm{c} = \pm\sqrt{24\mathrm{k}\left(\frac{32}{3}k^5 + \frac{2}{3}k^3\lambda b + \frac{1}{24}k\lambda^2 b^2 + \frac{2}{3}b\lambda k^4 + \frac{1}{12}k\lambda^2 b^2\right)}, \end{aligned} \tag{45}$$

Substituting Eq. (45) into Eq. (44), we kinks exact solutions for Eqs. (39) of the form

$$u(t,x)=k^2\left[cath^2(kx+ct)+\tanh^2(kx+ct)\right]-\frac{c}{12k}-\frac{2}{3}k^3+\frac{1}{12}\lambda\, b,$$
$$v(t,x)=bk^2\left[cath^2(kx+ct)+\tanh^2(kx+ct)\right]-\frac{2}{3}bk^2-\frac{1}{12}\lambda b^2, \tag{46}$$

Example 4. we consider the system of nonlinear partial differential equation in the Form

$$v_{xxt}+\alpha\, vv_t+\beta\, u_t v_x-v_t-v_x=0,$$
$$v=u_{x.} \tag{47}$$

α and β Where are constants. Using the wave variable

$$\xi=kx+ct,\quad \mathrm{u(t,x)}=U(\xi),\qquad v(\mathrm{t,x})=V(\xi). \tag{48}$$

Eqs. (47) become ordinary differential equations in the form of

$$k^2cV'''+\alpha\, cVV'+\beta\, c\, kU'V'-cV'-kV'=0,$$
$$V=kU' \tag{49}$$

Substituting second equation of (49) into the first equation of (49) ,we get

$$k^2c\,V''+\frac{1}{2}c\left[\alpha+\beta\right]V^2-\left[c+k\right]V=0, \tag{50}$$

obtained after integrating the ODE once and setting the constant of integration equal in (50) gives

V^2 with V'' to zero. Balancing

2m=m+2

So that m =2

The extended tanh method (8) admits the use of the finite expansion

Substituting Eq. (51) (51)
$$V(\xi)=\sum_{p=-m}^{m}\gamma_p Y^p=\gamma_{-2}Y^{-2}+\gamma_{-1}Y^{-1}+\gamma_0+\gamma_1 Y+\gamma_2 Y^2,$$
$$Y=\tanh\xi$$

into Eq. (50), and proceeding as before we obtain the solution in the form

$$\gamma_{-2}=\gamma_2=-\frac{12k^2}{[\alpha+\beta]},\qquad \gamma_0=\frac{8k^2}{[\alpha+\beta]}+\frac{[c+k]}{c[\alpha+\beta]},\qquad \gamma_1=\gamma_{-1}=0,$$
$$c=\frac{k}{16k^2-1}. \tag{52}$$

Substituting Eq. (52) into Eq. (51), we arrived to

$$V(\xi)=\frac{-12k^2}{[\alpha+\beta]}\left[cath^2\xi+\tanh^2\xi\right]+\frac{8k^2}{[\alpha+\beta]}+\frac{[c+k]}{c\,[\alpha+\beta]}, \tag{53}$$

and substituting in second equation of(49),yields ξ Integrating (53) with respect to

$$U(\xi)=-\frac{12k}{[\alpha+\beta]}\left[-cath\xi-\tanh\xi+2\xi\right]+\left[\frac{8k}{[\alpha+\beta]}+\frac{[c+k]}{c\,k\,[\alpha+\beta]}\right]\xi+c_1, \qquad (54)$$

constant ,We obtain the solution of system (47),in the form c_1 where

$$v(t,x)=\frac{-12k^2}{[\alpha+\beta]}\left[cath^2(kx+ct)+\tanh^2(kx+ct)\right]+\frac{8k^2}{[\alpha+\beta]}+\frac{[c+k]}{c\,[\alpha+\beta]},$$

$$u(t,x)=\frac{-12k}{\alpha+\beta}\left[-cath\,(kx+ct)-\tanh\,(kx+ct)+2\,(kx+ct)\right] \qquad (55)$$

$$+\left[\frac{8k}{\alpha+\beta}+\frac{c+k}{ck(\alpha+\beta)}\right]\,(kx+ct)+c_1.$$

Example 5. Consider the (2+1)dimensional nonlinear partial differential equation in the form

$$w_{xt}+w_{xxxy}+a\,w_x w_{xy}+b\,w_{xx}w_y+Lw_{yy}=0. \qquad (56)$$

We seek the following traveling wave solutions :

$$w(t,x,y)=W(\xi), \qquad \xi=\mathrm{k\,x}+\lambda\,\mathrm{y}+\mathrm{ct}. \qquad (57)$$

Substituting (57) into Eq.(56), then the Eq.(56) reduce to nonlinear ordinary Differential equation

$$\left(ck+L\lambda^2\right)W''+k^3\lambda\,W''''+\left(a\,k^2\lambda+b\lambda\,k^2\right)W'W''=0, \qquad (58)$$

By integrating this Eq.(58)and by considering the constant of integration to Be zero, we obtain a simplified ODE

$$\left(ck+L\lambda^2\right)W'+k^3\lambda\,W'''+\frac{1}{2}\left(a\,k^2\lambda+b\lambda\,k^2\right)(W')^2=0, \qquad (59)$$

leads to the following ansatz $(W')^2$ term with W''' Balancing

$$W(\xi)=\sum_{p=-1}^{1}\alpha_p Y^P=\alpha_{-1}Y^{-1}+\alpha_0+\alpha_1 Y \qquad (60)$$

$$\mathrm{Y}=\tanh(\xi),$$

Substituting Eq.(60) into Eq. (59), and proceeding as before we obtain the solution in the form

$$\alpha_1=\alpha_{-1}=\frac{12k}{a+b}, \qquad \alpha_0=\alpha_0, \qquad \mathrm{c}=\frac{-L\lambda^2-16k^3\lambda}{k}. \qquad (61)$$

Then the Eq.(60) became

$$W(\xi)=\alpha_0+\frac{12k}{a+b}\left[cath\xi+\tanh\xi\right]. \qquad (62)$$

We get $\xi \to \infty \left(\mathrm{Y} \to 1\right)$, If we require that the solution vanishes for

$$\alpha_0 = \frac{-24k}{a+b}, \qquad (63)$$

Then the solution of Eq.(56)in the original variables,

$$(64)\ w(t,x.y) = \frac{-24k}{a+b} + \frac{12k}{a+b}\left[cath\left(k\,x + \lambda\,y + c\,t\right) + \tanh\left(k\,x + \lambda\,y + ct\right)\right].$$

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