Int. Jr. of Mathematical Sciences & Applications
Vol. 7, No. 2, (July-December, 2017)
Copyright: Mind Reader Publications
ISSN No: 2230-9888
www.ijmsa.yolasite.com

Applications of special functions in boundary value problems I

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ABSTRACT
In this paper, we shall employ, multivariable Aleph-function, class of multivariable polynomials [8], Srivastava-Daoust function [9] and multivariable I-function defined by Prathima et al [5] in two boundary value problems. First we evaluate an integral involving the product of multivariable Aleph-function, the class of multivariable polynomials, the Srivastava-Daoust function and the multivariable I-function, and then make its applications to solve following two boundary problems :
1. a boundary value problem on heat conduction in a finite bar and to establish an expansion formula involving the product of the above multivariable Aleph-function, the class of multivariable polynomials, the Srivastava-Daoust function and the multivariable I-function.
2. another boundary value problem on electrostatics potential in spherical region.
We shall see two particular cases concerning the I-function of two variables defined by Rathie et al [6] and the multivariable H-function defined by Srivastava et al [10,11].

Keywords: Boundary value problems, multivariable I-function, Lauricella function of several variables, class of multivariable polynomials, multivariable Aleph-function, I-function of two variables, multivariable H-function.

2010 Mathematics Subject Classification. 33C45, 33C60, 26D20

1.Introduction

Recently, Chandel and Dwivedi [2] have discussed a problem on heat conduction and the problem of electrostatics potential in spherical region involving the product of the multivariable H-function defined by Srivastava et al [10,11] and a class of multivariable polynomials defined by Srivastava [8]. Here in the present paper, we discuss the same problems by employing the product of multivariable Aleph-function, the class of multivariable polynomials [8], the Srivastava-Daoust function [9] and the multivariable I-function [5] concerning two boundary problems.

The generalized polynomials defined by Srivastava [8], is given in the following manner :

$$S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}[y_1,\cdots,y_u] = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_u)_{M_uK_u}}{K_u!}A[N_1,K_1;\cdots;N_u,K_u]y_1^{K_1}\cdots y_u^{K_u} \quad (1.1)$$

Where $M_1,\cdots,M_u$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_u,K_u]$ are arbitrary constants, real or complex.

The Srivastava-Daoust function is defined by (see [9]):

$$F_{\bar{C}:D^{(1)};\cdots;D^{(v)}}^{\bar{A}:B^{(1)};\cdots;B^{(v)}}\left(\begin{array}{c|c} z_1 & [(a);\theta',\cdots,\theta^{(v)}]:[(b');\phi'];\cdots;[(b^{(v)});\phi^{(v)}] \\ \cdot & \cdot \\ \cdot & \\ z_v & [(c);\psi',\cdots,\psi^{(v)}]:[(d');\delta'];\cdots;[(d^{(v)});\delta^{(v)}] \end{array}\right)$$

$$= \sum_{m_1,\cdots,m_v=0}^{\infty} A'\frac{z_1^{m_1}\cdots z_v^{m_v}}{m_1!\cdots m_v!} \quad (1.2)$$

where

$$A' = \frac{\prod_{j=1}^{\bar{A}}(a_j)_{m_1\theta'_j+\cdots+m_v\theta_j^{(v)}}\prod_{j=1}^{B^{(1)}}(b'_j)_{m_1\phi'_j}\cdots\prod_{j=1}^{B^{(v)}}(b_j^{(v)})_{m_v\phi_j^{(v)}}}{\prod_{j=1}^{\bar{C}}(c_j)_{m_1\psi'_j+\cdots+m_v\psi_j^{(v)}}\prod_{j=1}^{D^{(1)}}(d'_j)_{m_1\delta'_j}\cdots\prod_{j=1}^{D^{(v)}}(d_j^{(v)})_{m_v\delta_j^{(v)}}} \quad \textbf{(1.3)}$$

The series given by (1.2) converges absolutely if

$$1+\sum_{j=1}^{\bar{C}}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{\bar{A}}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0;i=1,\cdots,v \tag{1.4}$$

The Aleph-function of several variables is an extension of the multivariable I-function recently study by C.K. Sharma and Ahmad [7] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We define : $\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}:m'_1,n_1,\cdots,m'_r,n'_r}_{p_i,q_i,\tau_i;R:p_{i(1)},q_{i(1)},\tau_{i(1)};R^{(1)};\cdots;p_{i(r)},q_{i(r)};\tau_{i(r)};R^{(r)}}\left(\begin{array}{c|c} \mathrm{z}_1 & \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & \end{array}\right.$

$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}]$ $,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:$

.... $,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m'+1,q_i}]:$

$$\left.\begin{array}{l}[(\mathrm{c'}_j^{(1)});\gamma_j^{(1)})_{1,n'_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n'_1+1,p_i^{(1)}}];\cdots;\ [(\mathrm{c'}_j^{(r)});\gamma_j'^{(r)})_{1,n'_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n'_r+1,p_i^{(r)}}]\\ [(\mathrm{d'}_j^{(1)});\delta_j'^{(1)})_{1,m'_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m'_1+1,q_i^{(1)}}];\cdots;[(\mathrm{d'}_j^{(r)});\delta_j'^{(r)})_{1,m'_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m'_r+1,q_i^{(r)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)y_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.5}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.6}$$

and $$\theta_k(s_k)=\frac{\prod_{j=1}^{m'_k}\Gamma(d_j'^{(k)}-\delta_j'^{(k)}s_k)\prod_{j=1}^{n'_k}\Gamma(1-c_j'^{(k)}+\gamma_j'^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m'_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n'_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.7}$$

Suppose , as usual , that the parameters

$a_j,j=1,\cdots,p;b_j,j=1,\cdots,q;$

$c_j'^{(k)},j=1,\cdots,n_k;c_{ji^{(k)}}^{(k)},j=n'_k+1,\cdots,p_{i^{(k)}}$

$d_j'^{(k)},j=1,\cdots,m_k;d_{ji^{(k)}}^{(k)},j=m'_k+1,\cdots,q_{i^{(k)}}$

with $k=1\cdots,r,i=1,\cdots,R$, $i^{(k)}=1,\cdots,R^{(k)}$

are complex numbers , and the $\alpha's,\beta's,\gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} + \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n'_k} \gamma_j'^{(k)} + \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{m'_k} \delta_j'^{(k)}$$

$$-\tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.8}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n'_k} \gamma_j'^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m'_k} \delta_j'^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1, \cdots, r, i = 1, \cdots, R \ , i^{(k)} = 1, \cdots, R^{(k)} \tag{1.9}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}) , max(|z_1|, \cdots, |z_r|) \rightarrow 0$$

$$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}) , min(|z_1|, \cdots, |z_r|) \rightarrow \infty$$

where $k = 1, \cdots, r : \alpha_k = min[Re(d_j'^{(k)}/\delta_j'^{(k)})], j = 1, \cdots, m'_k$ and

$$\beta_k = max[Re((c_j'^{(k)} - 1)/\gamma_j'^{(k)})], j = 1, \cdots, n'_k$$

The serie representation of Aleph-function of several variables is given by

$$\aleph(y_1, \cdots, y_r) = \sum_{G_1, \cdots, G_r=0}^{\infty} \sum_{g_1=0}^{m'_1} \cdots \sum_{g_r=0}^{m'_r} \frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1} G_1! \cdots \delta_{g_r} G_r!} \psi(\eta_{G_1,g_1}, \cdots, \eta_{G_r,g_r})$$

$$\theta_1(\eta_{G_1,g_1}) \cdots \theta_r(\eta_{G_r,g_r}) y_1^{\eta_{G_1,g_1}} \cdots y_r^{\eta_{G_r,g_r}} \tag{1.10}$$

Where $\psi(., \cdots, .), \theta_i(.), i = 1, \cdots, r$ are given respectively in (1.2), (1.3) and

$$\eta_{G_1,g_1} = \frac{d_{g_1}'^{(1)} + G_1}{\delta_{g_1}'^{(1)}}, \cdots, \ \eta_{G_r,g_r} = \frac{d_{g_r}'^{(r)} + G_r}{\delta_{g_r}'^{(r)}}$$

which is valid under the conditions $\delta_{g_i}^{(i)}[d'^{\,i}_j + p_i] \neq \delta'^{(i)}_j[d'^{(i)}_{g_i} + G_i]$ (1.11)

for $j \neq m'_i, m'_i = 1, \cdots \eta_{G_i,g_i}; p'_i, n'_i = 0, 1, 2, \cdots, ; y_i \neq 0, i = 1, \cdots, r$ (1.12)

Let

$$G(\eta_{G_1,g_1}, \cdots, \eta_{G_r,g_r}) = \phi(\eta_{G_1,g_1}, \cdots, \eta_{G_r,g_r})\theta_1(\eta_{G_1,g_1}) \cdots \theta_r(\eta_{G_r,g_r}) \tag{1.13}$$

where $\phi(\eta_{G_1,g_1}, \cdots, \eta_{G_r,g_r}), \theta_1(\eta_{G_1,g_1}), \cdots, \theta_r(\eta_{G_r,g_r})$ are given respectively in (1.6) and (1.7)

The multivariable I-function defined by Prathima et al [5] is an extension of the multivariable H-function defined by Srivastava et al [10,11]. It is defined in term of multiple Mellin-Barnes type integral :

$$\bar{I}(z_1, \cdots, z_s) = I^{0,N:M_1,N_1;\cdots;M_s,N_s}_{P,Q:P_1,Q_1;\cdots;P_s,Q_s}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(s)}; A_j)_{1,P} : \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_s & (\mathrm{b}_j; \beta_j^{(1)}, \cdots, \beta_j^{(s)}; B_j)_{1,Q} : \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{c}_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,N_1}, (c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{N_1+1,P_1}; \cdots; (c_j^{(s)}, \gamma_j^{(s)}; C_j^{(s)})_{1,N_s}, (c_j^{(s)}, \gamma_j^{(s)}; C_j^{(s)})_{N_s+1,P_s} \\ \\ (\mathrm{d}_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{1,M_1}, (d_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{M_1+1,Q_1}; \cdots; (d_j^{(s)}, \delta_j^{(s)}; D_j^{(s)})_{1,M_s}, (d_j^{(s)}, \delta_j^{(s)}; D_j^{(s)})_{M_s+1,Q_s} \end{array}\right) \tag{1.14}$$

$$= \frac{1}{(2\pi\omega)^s}\int_{L_1} \cdots \int_{L_s} \phi(t_1, \cdots, t_s)\prod_{i=1}^{s}\theta_i(t_i) z_i^{t_i} \mathrm{d}t_1 \cdots \mathrm{d}t_s \tag{1.15}$$

where $\phi(t_1, \cdots, t_s), \theta_i(t_i), i = 1, \cdots, s$ are given by :

$$\phi(t_1, \cdots, t_s) = \frac{\prod_{j=1}^{N}\Gamma^{A_j}\left(1 - a_j + \sum_{i=1}^{s}\alpha_j^{(i)}t_j\right)}{\prod_{j=N+1}^{P}\Gamma^{A_j}\left(a_j - \sum_{i=1}^{s}\alpha_j^{(i)}t_j\right)\prod_{j=1}^{Q}\Gamma^{B_j}\left(1 - b_j + \sum_{i=1}^{s}\beta_j^{(i)}t_j\right)} \tag{1.16}$$

$$\phi_i(t_i) = \frac{\prod_{j=1}^{N_i}\Gamma^{C_j^{(i)}}\left(1 - c_j^{(i)} + \gamma_j^{(i)}t_i\right)\prod_{j=1}^{M_i}\Gamma^{D_j^{(i)}}\left(d_j^{(i)} - \delta_j^{(i)}t_i\right)}{\prod_{j=N_i+1}^{P_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)} - \gamma_j^{(i)}t_i\right)\prod_{j=M_i+1}^{Q_i}\Gamma^{D_j^{(i)}}\left(1 - d_j^{(i)} + \delta_j^{(i)}t_i\right)} \tag{1.17}$$

For more details, see Prathima et al [5].

Following the result of Braaksma [1] the I-function of r variables is analytic if :

$$U_i = \sum_{j=1}^{P} A_j\alpha_j^{(i)} - \sum_{j=1}^{Q} B_j\beta_j^{(i)} + \sum_{j=1}^{P_i} C_j^{(i)}\gamma_j^{(i)} - \sum_{j=1}^{Q_i} D_j^{(i)}\delta_j^{(i)} \leqslant 0, i = 1, \cdots, s \tag{1.18}$$

The integral (2.1) converges absolutely if where$|arg(z_k)| < \frac{1}{2}\Delta_k \pi, k = 1, \cdots, s$

$$\Delta_k = -\sum_{j=N+1}^{P} A_j \alpha_j^{(k)} - \sum_{j=1}^{Q} B_j \beta_j^{(k)} + \sum_{j=1}^{M_k} D_j^{(k)} \delta_j^{(k)} - \sum_{j=M_k+1}^{Q_k} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{N_k} C_j^{(k)} \gamma_j^{(k)} - \sum_{j=N_k+1}^{P_k} C_j^{(k)} \gamma_j^{(k)} > 0 \quad (1.19)$$

For convenience, we will use the following notations in this paper.

$$A = (a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(s)}; A_j)_{1,P} : \ (c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,P_1}; \cdots; (c_j^{(s)}, \gamma_j^{(s)}; C_j^{(s)})_{1,P_s} \quad (1.20)$$

$$B = (b_j; \beta_j^{(1)}, \cdots, \beta_j^{(s)}; B_j)_{1,Q} : (d_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{1,Q_1}; \cdots; (d_j^{(s)}, \delta_j^{(s)}; D_j^{(s)})_{1,Q_s} \quad (1.21)$$

$$X = M_1, N_1; \cdots; M_s, N_s; Y = P_1, Q_1; \cdots; P_s, Q_s \quad (1.22)$$

$$B' = \frac{(-N_1)_{M_1 K_1}}{K_1!} \cdots \frac{(-N_u)_{M_u K_u}}{K_u!} A[N_1, K_1; \cdots; N_u, K_u] \quad (1.23)$$

$$F = F_{\bar{C}:D^{(1)};\cdots;D^{(v)}}^{\bar{A}:B^{(1)};\cdots;B^{(v)}} \quad (1.24)$$

$$C = \frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1} G_1! \cdots \delta_{g_r} G_r!} G(\eta_{G_1,g_1}, \cdots, \eta_{G_r,g_r}) \quad (1.25)$$

2. Main integral

Lemma (Erdelyi [4, page 276(6)]

$$\int_{-1}^{1} (1-x)^{\rho-1}(1+x)^{\sigma-1} P_m(x)\mathrm{d}x = 2^{\rho+\sigma-1} {}_3F_2(-n, n+1, \rho; 1, 1+\sigma : 1) = \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k \Gamma(\rho+k)}{k!(1)_k \Gamma(\rho+\sigma+k)} \quad (2.1)$$

where $Re(\rho) > 0, Re(\sigma) > 0$

In this section, we evaluate the integral

Theorem

$$\int_{-1}^{1} (1-x)^{\rho-1}(1+x)^{\sigma-1} P_{\mathbf{m}}(x) S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u} \begin{pmatrix} \mathrm{y}_1(1-x)^{f_1}(1+x)^{w_1} \\ \cdot \ \cdot \ \cdot \\ \cdot \ \cdot \ \cdot \\ \mathrm{y}_u(1-x)^{f_u}(1+x)^{w_u} \end{pmatrix} F \begin{pmatrix} \mathrm{t}_1(1-x)^{a_1}(1+x)^{b_1} \\ \cdot \ \cdot \ \cdot \\ \cdot \ \cdot \ \cdot \\ \mathrm{t}_v(1-x)^{a_v}(1+x)^{b_v} \end{pmatrix}$$

$$\aleph \begin{pmatrix} \mathrm{z}_1(1-x)^{c_1}(1+x)^{d_1} \\ \cdot \ \cdot \ \cdot \\ \cdot \ \cdot \ \cdot \\ \mathrm{z}_r(1-x)^{c_r}(1+x)^{d_r} \end{pmatrix} I \begin{pmatrix} \mathrm{Z}_1(1-x)^{h_1}(1+x)^{k_1} \\ \cdot \ \cdot \ \cdot \\ \cdot \ \cdot \ \cdot \\ \mathrm{Z}_s(1-x)^{h_s}(1+x)^{k_s} \end{pmatrix} \mathrm{d}x = 2^{\rho+\sigma-1} \sum_{\mathbf{p}=0}^{\mathbf{m}} \frac{(-\mathbf{m})_{\mathbf{p}}(\mathbf{m}+1)_{\mathbf{p}} 2^{\mathbf{p}}}{\mathbf{p}!\Gamma(\mathbf{p})}$$

$$\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m'_1}\cdots\sum_{g_r=0}^{m'_r}\sum_{m_1,\cdots,m_v=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]} A'B'C\,\frac{t_1^{m_1}\cdots t_v^{m_v}}{m_1!\cdots m_v!}\,y_1^{K_1}\cdots y_u^{K_u}\,z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}$$

$$2^{\sum_{i=1}^u K_i(f_i+w_i)+\sum_{i=1}^v(a_i+b_i)m_i+\sum_{i=1}^r \eta_{G_i,g_i}(c_i+d_i)}\, I_{P+2,Q+1;Y}^{0,N+2;X}\left(\begin{array}{c} \mathrm{Z}_1 2^{h_1+k_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{Z}_s 2^{h_s+k_s}\end{array}\right|$$

$$(1\text{-}\rho-\mathbf{p}-\textstyle\sum_{i=1}^u f_iK_i-\sum_{i=1}^v a_im_i-\sum_{i=1}^r c_i\eta_{G_i,g_i}:h_1,\cdots,h_s;1),$$
$$\cdot$$
$$\cdot$$
$$\cdots$$

$$\left.\begin{array}{c}(1\text{-}\sigma-\sum_{i=1}^u w_iK_i-\sum_{i=1}^v b_im_i-\sum_{i=1}^r d_i\eta_{G_i,g_i}:k_1,\cdots,k_s;1),A\\ \cdot\\ \cdot\\ (1\text{-}\sigma-\rho-\mathbf{p}-\sum_{i=1}^u(f_i+w_i)K_i-\sum_{i=1}^v(a_i+b_i)m_i-\sum_{i=1}^r(c_i+d_i)\eta_{G_i,g_i}:h_1+k_1,\cdots,h_s+k_s;1),B\end{array}\right) \quad (2.2)$$

Provided that

$min\{f_i,w_i,a_j,b_j,c_{k'},d_{k'},h_l,k_l\}>0$ for $i=1,\cdots,v;j=1,\cdots,u;k'=1,\cdots,r;l=1,\cdots,s$

$$1+\sum_{j=1}^{\bar{C}}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{\bar{A}}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0;i=1,\cdots,v\ ;\quad Re(\rho)>0,Re(\sigma)>0$$

$$Re\big[\rho+\sum_{u=1}^{u}K_if_i+\sum_{i=1}^{v}m_ia_i+\sum_{i=1}^{r}c_i\eta_{G_i,g_i}+\sum_{i=1}^{s}h_i\min_{1\leqslant j\leqslant M_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}\big]>0 \text{ and}$$

$$Re\big[\sigma+\sum_{i=1}^{u}K_iw_i+\sum_{i=1}^{v}m_ib_i+\sum_{i=1}^{r}d_i\eta_{G_i,g_i}+\sum_{i=1}^{s}k_i\min_{1\leqslant j\leqslant M_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}\big]>0$$

$\big|arg(Z_k(1-x)^{h_i}(1+x)^{k_i})\big|<\frac{1}{2}\Delta_k\pi$ where

$$\Delta_k=-\sum_{j=N+1}^{P}A_j\alpha_j^{(k)}-\sum_{j=1}^{Q}B_j\beta_j^{(k)}+\sum_{j=1}^{M_k}D_j^{(k)}\delta_j^{(k)}-\sum_{j=M_k+1}^{Q_k}D_j^{(k)}\delta_j^{(k)}+\sum_{j=1}^{N_k}C_j^{(k)}\gamma_j^{(k)}-\sum_{j=N_k+1}^{P_k}C_j^{(k)}\gamma_j^{(k)}>0$$

$$U_i^{(k)}=\ \tau_i\sum_{j=1}^{p_i}\alpha_{ji}^{(k)}\ +\tau_{i^{(k)}}\sum_{j=1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}\ -\tau_{i^{(k)}}\sum_{j=1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}<0$$

Proof

To prove (2.1), first expressing a class of multivariable polynomials defined by Srivastava [8] $S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[.]$, the

Srivastava-Daoust function [9] $F[.]$ and the series representation of the multivariable $\aleph$-function with the help of (1.1), (1.2) and (1.10) respectively and we interchange the order of summations and x-integral (which is permissible under the conditions stated). Expressing the I-function of s-variables defined by Prathima et al [5] in Mellin-Barnes contour integral with the help of (1.15) and interchange the order of integrations which is justifiable due to absolute convergence of the integrals involved in the process. Now collecting the powers of $(1-x)$ and $(1+x)$ and evaluating the inner x-integral with the help of the lemma . Interpreting the Mellin-Barnes contour integral in multivariable I-function, we obtain the desired result (2.2).

3. Boundary value problem on heat conduction in a finite bar

As an exemple of the application of the product of special functions in applied mathematics we shall consider the problem of determining a function $\theta(x,t)$ representing the temperature in a non-homogeneous bar with ends at $X=\pm 1$ in which the thermal diffusivity is proportional to $(1-x^2)$ and if the lateral surface on the bar is insulated, it satisfies the partial differential equation of heat conduction, see Churchill [3].

$$\frac{\partial\theta}{\partial t} = b\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial\theta}{\partial x}\right] \tag{3.1}$$

where b is a constant, provided thermal coefficient is constant. The boundary conditions of the problem are that both ends of a bar at $X=\pm 1$ are also insulated because the conductivity vanishes there and the initial conditions :

$$\theta(x,0) = f(x), -1 < x < 1 \tag{3.2}$$

Here we may assume the solution of the problem 1 in the form :

$$\theta(x,t) = \sum_{\mathbf{n}=0}^{\infty} A_{\mathbf{n}} e^{-k\mathbf{n}(\mathbf{n}+1)t} P_{\mathbf{n}}(x) \tag{3.3}$$

which is quite justified and for $t=0$, reduces to

$$\theta(x,0) = \sum_{\mathbf{n}=0}^{\infty} A_{\mathbf{n}} P_{\mathbf{n}}(x) \tag{3.4}$$

Now in (3.4), we may choose

$$f(x) = (1-x)^{\rho-1}(1+x)^{\sigma-1}\, S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}\begin{pmatrix} \mathrm{y}_1(1-x)^{f_1}(1+x)^{w_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{y}_u(1-x)^{f_u}(1+x)^{w_u} \end{pmatrix} F\begin{pmatrix} \mathrm{t}_1(1-x)^{a_1}(1+x)^{b_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{t}_v(1-x)^{a_v}(1+x)^{b_v} \end{pmatrix}$$

$$\aleph\begin{pmatrix} \mathrm{z}_1(1-x)^{c_1}(1+x)^{d_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{z}_r(1-x)^{c_r}(1+x)^{d_r} \end{pmatrix} I\begin{pmatrix} \mathrm{Z}_1(1-x)^{h_1}(1+x)^{k_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{Z}_s(1-x)^{h_s}(1+x)^{k_s} \end{pmatrix} \tag{3.5}$$

Therefore, by making an appeal to (3.4) and (3.5), we have

$$f(x)=(1-x)^{\rho-1}(1+x)^{\sigma-1}\,S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}\left(\begin{array}{c}\mathrm{y}_1(1-x)^{f_1}(1+x)^{w_1}\\ \cdots\\ \cdots\\ \mathrm{y}_u(1-x)^{f_u}(1+x)^{w_u}\end{array}\right)F\left(\begin{array}{c}\mathrm{t}_1(1-x)^{a_1}(1+x)^{b_1}\\ \cdots\\ \cdots\\ \mathrm{t}_v(1-x)^{a_v}(1+x)^{b_v}\end{array}\right)$$

$$\aleph\left(\begin{array}{c}\mathrm{z}_1(1-x)^{c_1}(1+x)^{d_1}\\ \cdots\\ \cdots\\ \mathrm{z}_r(1-x)^{c_r}(1+x)^{d_r}\end{array}\right)I\left(\begin{array}{c}\mathrm{Z}_1(1-x)^{h_1}(1+x)^{k_1}\\ \cdots\\ \cdots\\ \mathrm{Z}_s(1-x)^{h_s}(1+x)^{k_s}\end{array}\right)=\sum_{\mathbf{n}=0}^{\infty}A_{\mathbf{n}}P_{\mathbf{n}}(x) \tag{3.6}$$

Now making an appeal to the orthogonality property of Legendre polynomials, see Erdelyi (4, page 277, Eq.(13)); we derive

$$A_{\mathbf{m}}=(2\mathbf{m}+1)2^{\rho+\sigma-2}\sum_{\mathbf{p}=0}^{\mathbf{m}}\frac{(-\mathbf{m})_{\mathbf{p}}(\mathbf{m}+1)_{\mathbf{p}}2^{\mathbf{p}}}{\mathbf{p}!\Gamma(\mathbf{p})}\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m'_1}\cdots\sum_{g_r=0}^{m'_r}\sum_{m_1,\cdots,m_v=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}A'B'C$$

$$\frac{t_1^{m_1}\cdots t_v^{m_v}}{m_1!\cdots m_v!}y_1^{K_1}\cdots y_u^{K_u}\,z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}\;2^{\sum_{i=1}^{u}K_i(f_i+w_i)+\sum_{i=1}^{v}(a_i+b_i)m_i+\sum_{i=1}^{r}\eta_{G_i,g_i}(c_i+d_i)}$$

$$I_{P+2,Q+1;Y}^{0,N+2;X}\left(\begin{array}{c}\mathrm{Z}_1 2^{h_1+k_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{Z}_s 2^{h_s+k_s}\end{array}\right|\begin{array}{c}(1\text{-}\rho-\mathbf{p}-\sum_{i=1}^{u}f_iK_i-\sum_{i=1}^{v}a_im_i-\sum_{i=1}^{r}c_i\eta_{G_i,g_i}:h_1,\cdots,h_s;1),\\ \cdot\\ \cdot\\ \cdots\end{array}$$

$$\left.\begin{array}{c}(1\text{-}\sigma-\sum_{i=1}^{u}w_iK_i-\sum_{i=1}^{v}b_im_i-\sum_{i=1}^{r}d_i\eta_{G_i,g_i}:k_1,\cdots,k_s;1),A\\ \cdot\\ \cdot\\ (1\text{-}\sigma-\rho-\mathbf{p}-\sum_{i=1}^{u}(f_i+w_i)K_i-\sum_{i=1}^{v}(a_i+b_i)m_i-\sum_{i=1}^{r}(c_i+d_i)\eta_{G_i,g_i}:h_1+k_1,\cdots,h_s+k_s;1),B\end{array}\right) \tag{3.7}$$

under the same validity conditions that (2.2). Now substituting the value of $A_{\mathbf{n}}$ from (3.7) in (3.3), we obtain the following solution of the prolem 1 :

$$\theta(x,t)=2^{\rho+\sigma-2}\sum_{\mathbf{m}=0}^{\infty}(2\mathbf{m}+1)e^{-k\mathbf{m}(\mathbf{m}+1)t}P_{\mathbf{m}}(x)\sum_{\mathbf{p}=0}^{\mathbf{m}}\frac{(-\mathbf{m})_{\mathbf{p}}(\mathbf{m}+1)_{\mathbf{p}}2^{\mathbf{p}}}{\mathbf{p}!\Gamma(\mathbf{p})}$$

$$\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m'_1}\cdots\sum_{g_r=0}^{m'_r}\sum_{m_1,\cdots,m_v=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}A'B'C\,\frac{t_1^{m_1}\cdots t_v^{m_v}}{m_1!\cdots m_v!}y_1^{K_1}\cdots y_u^{K_u}\,z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}$$

$$2^{\sum_{i=1}^u K_i(f_i+w_i)+\sum_{i=1}^v(a_i+b_i)m_i+\sum_{i=1}^r \eta_{G_i,g_i}(c_i+d_i)}\ I_{P+2,Q+1;Y}^{0,N+2;X}\left(\begin{array}{c} \mathrm{Z}_1 2^{h_1+k_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Z}_s 2^{h_s+k_s} \end{array}\ \middle|\ \begin{array}{c} (1\text{-}\rho-\mathbf{p}-\sum_{i=1}^u f_iK_i-\sum_{i=1}^v a_im_i-\sum_{i=1}^r c_i\eta_{G_i,g_i}:h_1,\cdots,h_s;1), \\ \cdot \\ \cdot \\ \cdot\ \cdot\ \cdot \\ (1\text{-}\sigma-\sum_{i=1}^u w_iK_i-\sum_{i=1}^v b_im_i-\sum_{i=1}^r d_i\eta_{G_i,g_i}:k_1,\cdots,k_s;1), A \\ \cdot \\ \cdot \\ (1\text{-}\sigma-\rho-\mathbf{p}-\sum_{i=1}^u(f_i+w_i)K_i-\sum_{i=1}^v(a_i+b_i)m_i-\sum_{i=1}^r(c_i+d_i)\eta_{G_i,g_i}:h_1+k_1,\cdots,h_s+k_s;1), B \end{array}\right) \quad (3.8)$$

under the same validity conditions that (2.2).

4. Expansion formula

Now making an appeal to (3.4), (3.5) and (3.7), we obtain the following expansion formula

$$(1-x)^{\rho-1}(1+x)^{\sigma-1}\quad S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}\left(\begin{array}{c} \mathrm{y}_1(1-x)^{f_1}(1+x)^{w_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{y}_u(1-x)^{f_u}(1+x)^{w_u} \end{array}\right)\ F\left(\begin{array}{c} \mathrm{t}_1(1-x)^{a_1}(1+x)^{b_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{t}_v(1-x)^{a_v}(1+x)^{b_v} \end{array}\right)$$

$$\aleph\left(\begin{array}{c} \mathrm{z}_1(1-x)^{c_1}(1+x)^{d_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{z}_r(1-x)^{c_r}(1+x)^{d_r} \end{array}\right)\ I\left(\begin{array}{c} \mathrm{Z}_1(1-x)^{h_1}(1+x)^{k_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{Z}_s(1-x)^{h_s}(1+x)^{k_s} \end{array}\right)$$

$$= 2^{\rho+\sigma-2}\sum_{\mathbf{m}=0}^{\infty}(2\mathbf{m}+1)P_{\mathbf{m}}(x)\sum_{\mathbf{p}=0}^{\mathbf{m}}\frac{(-\mathbf{m})_{\mathbf{p}}(\mathbf{m}+1)_{\mathbf{p}}2^{\mathbf{P}}}{(\mathbf{p}!)^2}$$

$$\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m_1'}\cdots\sum_{g_r=0}^{m_r'}\sum_{m_1,\cdots,m_v=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}A'B'C\,\frac{t_1^{m_1}\cdots t_v^{m_v}}{m_1!\cdots m_v!}\,y_1^{K_1}\cdots y_u^{K_u}\,z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}$$

$$2^{\sum_{i=1}^u K_i(f_i+w_i)+\sum_{i=1}^v(a_i+b_i)m_i+\sum_{i=1}^r \eta_{G_i,g_i}(c_i+d_i)}\ I_{P+2,Q+1;Y}^{0,N+2;X}\left(\begin{array}{c} \mathrm{Z}_1 2^{h_1+k_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Z}_s 2^{h_s+k_s} \end{array}\ \middle|\right.$$

$$(1\text{-}\rho-\mathbf{p}-\sum_{i=1}^{u}f_iK_i-\sum_{i=1}^{v}a_im_i-\sum_{i=1}^{r}c_i\eta_{G_i,g_i}:h_1,\cdots,h_s;1),$$

$$\vdots$$

$$\left.\begin{array}{c}(1\text{-}\sigma-\sum_{i=1}^{u}w_iK_i-\sum_{i=1}^{v}b_im_i-\sum_{i=1}^{r}d_i\eta_{G_i,g_i}:k_1,\cdots,k_s;1),A\\ \vdots \\ (1\text{-}\sigma-\rho-\mathbf{p}-\sum_{i=1}^{u}(f_i+w_i)K_i-\sum_{i=1}^{v}(a_i+b_i)m_i-\sum_{i=1}^{r}(c_i+d_i)\eta_{G_i,g_i}:h_1+k_1,\cdots,h_s+k_s;1),B\end{array}\right) \tag{4.1}$$

under the same validity conditions that (2.2).

5. Problem on electrostatics potential in spherical regions

Here in this section, we shall make applications of the product of multivariable Aleph-function, the class of multivariable polynomials [8], the Srivastava-Daoust function [9] and the multivariable I-function [5] to obtain the harmonic function V representing the electrostatics potential in the domain $R<c$ such that V assumes a prescribed value $F(\theta)$ on the spherical surface $R=c$, where R,θ,ϕ are the spherical polar coordinates and V is independent of ϕ. Thus V satisfies the following partial differential equation

$$R\frac{\partial^2(rV)}{\partial R^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0 \tag{5.1}$$

in the domain $R<c,\, 0\leqslant\theta<\pi$ and under the condition

$$\lim_{R\to c}V(R,\theta)=F(\theta),(0\leqslant\theta<\pi,R=c) \tag{5.2}$$

where V and its derivatives of first and second orders are assumed to be continuous throughout the interior of the sphere :

$$(0\leqslant R<c,0\leqslant\theta<\pi) \tag{5.3}$$

If we take $x=\cos\theta$ $(0\leqslant\theta<\pi)$, the equation (5.1) reduces to :

$$R\frac{\partial^2(rV)}{\partial R^2}+\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial V}{\partial x}\right)=0 \tag{5.4}$$

where $(R<c,-1<x<1)$ and V is continuous every where interior to the sphere and bounded when $(0\leqslant R<c)$.

$$\lim_{R\to\infty}W(R,x)=0 \tag{5.5}$$

where W is harmonic function in the bounded domain $R>c$, exterior to the spherical surface and RV is bounded for large values of R and all $x(-1\leqslant x\leqslant 1)$.

We shall obtain the solution of the above boundary value problem for

$$f(x)=(1-x)^{\rho-1}(1+x)^{\sigma-1}\ S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}\begin{pmatrix}\mathrm{y}_1(1-x)^{f_1}(1+x)^{w_1}\\ \cdots\\ \cdots\\ \mathrm{y}_u(1-x)^{f_u}(1+x)^{w_u}\end{pmatrix}\ F\begin{pmatrix}\mathrm{t}_1(1-x)^{a_1}(1+x)^{b_1}\\ \cdots\\ \cdots\\ \mathrm{t}_v(1-x)^{a_v}(1+x)^{b_v}\end{pmatrix}$$

$$\aleph\begin{pmatrix} \mathrm{z}_1(1-x)^{c_1}(1+x)^{d_1} \\ \cdots \\ \cdots \\ \mathrm{z}_r(1-x)^{c_r}(1+x)^{d_r} \end{pmatrix} I\begin{pmatrix} \mathrm{Z}_1(1-x)^{h_1}(1+x)^{k_1} \\ \cdots \\ \cdots \\ \mathrm{Z}_s(1-x)^{h_s}(1+x)^{k_s} \end{pmatrix} \tag{5.6}$$

Case 1 (solution for $V(R,x)$ when $R < c$)

The solution of the problem 2 is :

$$V(R,x) = \sum_{\mathbf{m}=0}^{\infty} A_{\mathbf{m}}\,(R/c)^{\mathbf{m}}\,P_{\mathbf{m}}(x) \tag{5.7}$$

which by appeal to (5.5) reduces to

$$f(x) = \sum_{\mathbf{m}=0}^{\infty} A_{\mathbf{m}} P_{\mathbf{m}}(x) \tag{5.8}$$

where $A_{\mathbf{m}}$ is given by (3.7). Substituting the value of $A_{\mathbf{m}}$ in (5.7), we obtain the solution of the problem 2

$$V(R,x) = 2^{\rho+\sigma-2}\sum_{\mathbf{m}=0}^{\infty}(2\mathbf{m}+1)\,(R/c)^{\mathbf{m}}\,P_{\mathbf{m}}(x)\sum_{\mathbf{p}=0}^{\mathbf{m}}\frac{(-\mathbf{m})_{\mathbf{p}}(\mathbf{m}+1)_{\mathbf{p}}2^{\mathbf{p}}}{(\mathbf{p}!)^2}$$

$$\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m'_1}\cdots\sum_{g_r=0}^{m'_r}\sum_{m_1,\cdots,m_v=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]} A'B'C\,\frac{t_1^{m_1}\cdots t_v^{m_v}}{m_1!\cdots m_v!}\,y_1^{K_1}\cdots y_u^{K_u}\,z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}$$

$$2^{\sum_{i=1}^{u}K_i(f_i+w_i)+\sum_{i=1}^{v}(a_i+b_i)m_i+\sum_{i=1}^{r}\eta_{G_i,g_i}(c_i+d_i)}\; I^{0,N+2;X}_{P+2,Q+1;Y}\left(\begin{array}{c|} \mathrm{Z}_1 2^{h_1+k_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Z}_s 2^{h_s+k_s} \end{array}\right.$$

$$(1\text{-}\rho-\mathbf{p}-\textstyle\sum_{i=1}^{u}f_iK_i-\sum_{i=1}^{v}a_im_i-\sum_{i=1}^{r}c_i\eta_{G_i,g_i}:h_1,\cdots,h_s;1),$$

$$\cdot$$
$$\cdot$$
$$\cdots$$

$$\left.\begin{array}{c} (1\text{-}\sigma-\sum_{i=1}^{u}w_iK_i-\sum_{i=1}^{v}b_im_i-\sum_{i=1}^{r}d_i\eta_{G_i,g_i}:k_1,\cdots,k_s;1),A \\ \cdot \\ \cdot \\ (1\text{-}\sigma-\rho-\mathbf{p}-\sum_{i=1}^{u}(f_i+w_i)K_i-\sum_{i=1}^{v}(a_i+b_i)m_i-\sum_{i=1}^{r}(c_i+d_i)\eta_{G_i,g_i}:h_1+k_1,\cdots,h_s+k_s;1),B \end{array}\right) \tag{5.9}$$

under the same validity conditions that (2.2) and $R < c$

Case 2 (solution for $W(R,x)$ when $R \geqslant c$)

In this situation, in view of Churchil [3, page 219, Eq.(12)], we can write the following solution of the problem 2

$$W(R,x) = \sum_{\mathbf{m}=0}^{\infty} A_{\mathbf{m}} \, (c/R)^{\mathbf{m}+1} \, P_{\mathbf{m}}(x) \tag{5.10}$$

Substituting the value of $A_{\mathbf{m}}$ in (5.10), we obtain the solution of the problem 2

$$W(R,x) = 2^{\rho+\sigma-2} \sum_{\mathbf{m}=0}^{\infty} (2\mathbf{m}+1) \, (c/R)^{\mathbf{m}+1} \, P_{\mathbf{m}}(x) \sum_{\mathbf{p}=0}^{\mathbf{m}} \frac{(-\mathbf{m})_{\mathbf{p}}(\mathbf{m}+1)_{\mathbf{p}} 2^{\mathbf{p}}}{(\mathbf{p}!)^2}$$

$$\sum_{G_1,\cdots,G_r=0}^{\infty} \sum_{g_1=0}^{m'_1} \cdots \sum_{g_r=0}^{m'_r} \sum_{m_1,\cdots,m_v=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_u=0}^{[N_u/M_u]} A'B'C \, \frac{t_1^{m_1} \cdots t_v^{m_v}}{m_1! \cdots m_v!} \, y_1^{K_1} \cdots y_u^{K_u} \, z_1^{\eta_{G_1,g_1}} \cdots z_r^{\eta_{G_r,g_r}}$$

$$2^{\sum_{i=1}^{u} K_i(f_i+w_i) + \sum_{i=1}^{v}(a_i+b_i)m_i + \sum_{i=1}^{r} \eta_{G_i,g_i}(c_i+d_i)} \, I_{P+2,Q+1;Y}^{0,N+2;X} \left(\begin{array}{c} \mathrm{Z}_1 2^{h_1+k_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Z}_s 2^{h_s+k_s} \end{array} \right|$$

$$(1\text{-}\rho - \mathbf{p} - \textstyle\sum_{i=1}^{u} f_i K_i - \sum_{i=1}^{v} a_i m_i - \sum_{i=1}^{r} c_i \eta_{G_i,g_i} : h_1, \cdots, h_s; 1),$$
$$\cdot$$
$$\cdot$$
$$\cdot \;\; \cdot \;\; \cdot$$

$$\left. \begin{array}{c} (1\text{-}\sigma - \sum_{i=1}^{u} w_i K_i - \sum_{i=1}^{v} b_i m_i - \sum_{i=1}^{r} d_i \eta_{G_i,g_i} : k_1, \cdots, k_s; 1), A \\ \cdot \\ \cdot \\ (1\text{-}\sigma - \rho - \mathbf{p} - \sum_{i=1}^{u}(f_i+w_i)K_i - \sum_{i=1}^{v}(a_i+b_i)m_i - \sum_{i=1}^{r}(c_i+d_i)\eta_{G_i,g_i} : h_1+k_1, \cdots, h_s+k_s; 1), B \end{array} \right) \tag{5.11}$$

under the same validity conditions that (2.2) and $R \geqslant c$

6. Multivariable H-function

If $A_j = B_j = C_j^{(1)} = \cdots = C_j^{(s)} = D_j^{(1)} = \cdots = D_j^{(s)} = 1$, the multivariable I-function reduces to multivariable H-function defined by Srivastava et al [10,11]. We have the solutions of two problems.

Problem 1 (problem of conduction)

$$\theta(x,t) = 2^{\rho+\sigma-2} \sum_{\mathbf{m}=0}^{\infty} (2\mathbf{m}+1) e^{-k\mathbf{m}(\mathbf{m}+1)t} P_{\mathbf{m}}(x) \sum_{\mathbf{p}=0}^{\mathbf{m}} \frac{(-\mathbf{m})_{\mathbf{p}}(\mathbf{m}+1)_{\mathbf{p}} 2^{\mathbf{p}}}{\mathbf{p}!\Gamma(\mathbf{p})}$$

$$\sum_{G_1,\cdots,G_r=0}^{\infty} \sum_{g_1=0}^{m'_1} \cdots \sum_{g_r=0}^{m'_r} \sum_{m_1,\cdots,m_v=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_u=0}^{[N_u/M_u]} A'B'C \, \frac{t_1^{m_1} \cdots t_v^{m_v}}{m_1! \cdots m_v!} \, y_1^{K_1} \cdots y_u^{K_u} \, z_1^{\eta_{G_1,g_1}} \cdots z_r^{\eta_{G_r,g_r}}$$

$$2^{\sum_{i=1}^u K_i(f_i+w_i)+\sum_{i=1}^v (a_i+b_i)m_i+\sum_{i=1}^r \eta_{G_i,g_i}(c_i+d_i)}\ H_{P+2,Q+1;Y}^{0,N+2;X}\left(\begin{array}{c} \mathrm{Z}_1 2^{h_1+k_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Z}_s 2^{h_s+k_s} \end{array}\right|$$

$$(1\text{-}\rho-\mathbf{p}-\textstyle\sum_{i=1}^u f_iK_i-\sum_{i=1}^v a_im_i-\sum_{i=1}^r c_i\eta_{G_i,g_i}:h_1,\cdots,h_s),$$
$$\cdot$$
$$\cdot$$
$$\cdots$$

$$\left.\begin{array}{c} (1\text{-}\sigma-\sum_{i=1}^u w_iK_i-\sum_{i=1}^v b_im_i-\sum_{i=1}^r d_i\eta_{G_i,g_i}:k_1,\cdots,k_s),A \\ \cdot \\ \cdot \\ (1\text{-}\sigma-\rho-\mathbf{p}-\sum_{i=1}^u (f_i+w_i)K_i-\sum_{i=1}^v (a_i+b_i)m_i-\sum_{i=1}^r (c_i+d_i)\eta_{G_i,g_i}:h_1+k_1,\cdots,h_s+k_s),B \end{array}\right) \tag{6.1}$$

under the same validity conditions that (2.2) with $A_j=B_j=C_j^{(1)}=\cdots=C_j^{(s)}=D_j^{(1)}=\cdots=D_j^{(s)}=1$

problem 2 (electrostatics problem)

case 1(solution for $V(R,x)$ when $R<c$)

$$V(R,x)=2^{\rho+\sigma-2}\sum_{\mathbf{m}=0}^{\infty}(2\mathbf{m}+1)\,(R/c)^{\mathbf{m}}\,P_{\mathbf{m}}(x)\sum_{\mathbf{p}=0}^{\mathbf{m}}\frac{(-\mathbf{m})_{\mathbf{p}}(\mathbf{m}+1)_{\mathbf{p}}2^{\mathbf{p}}}{(\mathbf{p}!)^2}$$

$$\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m_1'}\cdots\sum_{g_r=0}^{m_r'}\sum_{m_1,\cdots,m_v=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}A'B'C\,\frac{t_1^{m_1}\cdots t_v^{m_v}}{m_1!\cdots m_v!}\,y_1^{K_1}\cdots y_u^{K_u}\,z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}$$

$$2^{\sum_{i=1}^u K_i(f_i+w_i)+\sum_{i=1}^v (a_i+b_i)m_i+\sum_{i=1}^r \eta_{G_i,g_i}(c_i+d_i)}\ H_{P+2,Q+1;Y}^{0,N+2;X}\left(\begin{array}{c} \mathrm{Z}_1 2^{h_1+k_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Z}_s 2^{h_s+k_s} \end{array}\right|$$

$$(1\text{-}\rho-\mathbf{p}-\textstyle\sum_{i=1}^u f_iK_i-\sum_{i=1}^v a_im_i-\sum_{i=1}^r c_i\eta_{G_i,g_i}:h_1,\cdots,h_s),$$
$$\cdot$$
$$\cdot$$
$$\cdots$$

$$\left.\begin{array}{c} (1\text{-}\sigma-\sum_{i=1}^u w_iK_i-\sum_{i=1}^v b_im_i-\sum_{i=1}^r d_i\eta_{G_i,g_i}:k_1,\cdots,k_s),A \\ \cdot \\ \cdot \\ (1\text{-}\sigma-\rho-\mathbf{p}-\sum_{i=1}^u (f_i+w_i)K_i-\sum_{i=1}^v (a_i+b_i)m_i-\sum_{i=1}^r (c_i+d_i)\eta_{G_i,g_i}:h_1+k_1,\cdots,h_s+k_s),B \end{array}\right) \tag{6.2}$$

under the same validity conditions that (2.2), $R < c$ and $A_j = B_j = C_j^{(1)} = \cdots = C_j^{(s)} = D_j^{(1)} = \cdots = D_j^{(s)} = 1$
Case 2 (solution for $W(R,x)$ when $R \geqslant c$)

$$W(R,x) = 2^{\rho+\sigma-2} \sum_{\mathbf{m}=0}^{\infty} (2\mathbf{m}+1)\,(c/R)^{\mathbf{m}+1}\, P_{\mathbf{m}}(x) \sum_{\mathbf{p}=0}^{\mathbf{m}} \frac{(-\mathbf{m})_{\mathbf{p}}(\mathbf{m}+1)_{\mathbf{p}} 2^{\mathbf{p}}}{(\mathbf{p}!)^2}$$

$$\sum_{G_1,\cdots,G_r=0}^{\infty} \sum_{g_1=0}^{m'_1} \cdots \sum_{g_r=0}^{m'_r} \sum_{m_1,\cdots,m_v=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_u=0}^{[N_u/M_u]} A'B'C \frac{t_1^{m_1}\cdots t_v^{m_v}}{m_1!\cdots m_v!} y_1^{K_1}\cdots y_u^{K_u}\, z_1^{\eta_{G_1,g_1}} \cdots z_r^{\eta_{G_r,g_r}}$$

$$2^{\sum_{i=1}^{u} K_i(f_i+w_i)+\sum_{i=1}^{v}(a_i+b_i)m_i+\sum_{i=1}^{r}\eta_{G_i,g_i}(c_i+d_i)}\; H_{P+2,Q+1;Y}^{0,N+2;X} \left(\begin{array}{c} \mathrm{Z}_1 2^{h_1+k_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Z}_s 2^{h_s+k_s} \end{array} \right|$$

$$\begin{array}{c} (1\text{-}\rho - \mathbf{p} - \sum_{i=1}^{u} f_i K_i - \sum_{i=1}^{v} a_i m_i - \sum_{i=1}^{r} c_i \eta_{G_i,g_i} : h_1,\cdots,h_s), \\ \cdot \\ \cdot \\ \cdot\ \cdot\ \cdot \end{array}$$

$$\left. \begin{array}{c} (1\text{-}\sigma - \sum_{i=1}^{u} w_i K_i - \sum_{i=1}^{v} b_i m_i - \sum_{i=1}^{r} d_i \eta_{G_i,g_i} : k_1,\cdots,k_s), A \\ \cdot \\ \cdot \\ (1\text{-}\sigma - \rho - \mathbf{p} - \sum_{i=1}^{u}(f_i+w_i)K_i - \sum_{i=1}^{v}(a_i+b_i)m_i - \sum_{i=1}^{r}(c_i+d_i)\eta_{G_i,g_i} : h_1+k_1,\cdots,h_s+k_s), B \end{array} \right) \quad (6.3)$$

under the same validity conditions that (2.2), $R \geqslant c$ and $A_j = B_j = C_j^{(1)} = \cdots = C_j^{(s)} = D_j^{(1)} = \cdots = D_j^{(s)} = 1$

7. I-function of two variables

If $s = 2$, the multivariable I-function defined by Prathima et al [5] reduces to I-function of two variables defined by Rathie et al [6]. We have the solutions of two problems.

Problem 1 (problem of conduction)

$$\theta(x,t) = 2^{\rho+\sigma-2} \sum_{\mathbf{m}=0}^{\infty} (2\mathbf{m}+1) e^{-k\mathbf{m}(\mathbf{m}+1)t} P_{\mathbf{m}}(x) \sum_{\mathbf{p}=0}^{\mathbf{m}} \frac{(-\mathbf{m})_{\mathbf{p}}(\mathbf{m}+1)_{\mathbf{p}} 2^{\mathbf{p}}}{\mathbf{p}!\Gamma(\mathbf{p})}$$

$$\sum_{G_1,\cdots,G_r=0}^{\infty} \sum_{g_1=0}^{m'_1} \cdots \sum_{g_r=0}^{m'_r} \sum_{m_1,\cdots,m_v=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_u=0}^{[N_u/M_u]} A'B'C \frac{t_1^{m_1}\cdots t_v^{m_v}}{m_1!\cdots m_v!} y_1^{K_1}\cdots y_u^{K_u}\, z_1^{\eta_{G_1,g_1}} \cdots z_r^{\eta_{G_r,g_r}}$$

$$2^{\sum_{i=1}^{u} K_i(f_i+w_i)+\sum_{i=1}^{v}(a_i+b_i)m_i+\sum_{i=1}^{r}\eta_{G_i,g_i}(c_i+d_i)}\; I_{P+2,Q+1;Y}^{0,N+2;X} \left(\begin{array}{c} \mathrm{Z}_1 2^{h_1+k_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Z}_2 2^{h_2+k_2} \end{array} \right|$$

$$(1\text{-}\rho - \mathbf{p} - \sum_{i=1}^{u} f_i K_i - \sum_{i=1}^{v} a_i m_i - \sum_{i=1}^{r} c_i \eta_{G_i,g_i} : h_1, h_2; 1), \\ \cdot \\ \cdot \\ \cdots$$

$$\left. \begin{array}{c} (1\text{-}\sigma - \sum_{i=1}^{u} w_i K_i - \sum_{i=1}^{v} b_i m_i - \sum_{i=1}^{r} d_i \eta_{G_i,g_i} : k_1, k_2; 1), A \\ \cdot \\ \cdot \\ (1\text{-}\sigma - \rho - \mathbf{p} - \sum_{i=1}^{u} (f_i + w_i) K_i - \sum_{i=1}^{v} (a_i + b_i) m_i - \sum_{i=1}^{r} (c_i + d_i) \eta_{G_i,g_i} : h_1 + k_1, h_2 + k_2; 1), B \end{array} \right) \quad (7.1)$$

under the same validity conditions that (2.2) with $s = 2$.

problem 2 (electrostatics problem)

case 1(solution for $V(R,x)$ when $R < c$)

$$V(R,x) = 2^{\rho+\sigma-2} \sum_{\mathbf{m}=0}^{\infty} (2\mathbf{m}+1) \, (R/c)^{\mathbf{m}} \, P_{\mathbf{m}}(x) \sum_{\mathbf{p}=0}^{\mathbf{m}} \frac{(-\mathbf{m})_{\mathbf{p}} (\mathbf{m}+1)_{\mathbf{p}} 2^{\mathbf{p}}}{(\mathbf{p}!)^2}$$

$$\sum_{G_1,\cdots,G_r=0}^{\infty} \sum_{g_1=0}^{m'_1} \cdots \sum_{g_r=0}^{m'_r} \sum_{m_1,\cdots,m_v=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_u=0}^{[N_u/M_u]} A'B'C \, \frac{t_1^{m_1} \cdots t_v^{m_v}}{m_1! \cdots m_v!} \, y_1^{K_1} \cdots y_u^{K_u} \, z_1^{\eta_{G_1,g_1}} \cdots z_r^{\eta_{G_r,g_r}}$$

$$2^{\sum_{i=1}^{u} K_i (f_i + w_i) + \sum_{i=1}^{v} (a_i + b_i) m_i + \sum_{i=1}^{r} \eta_{G_i,g_i} (c_i + d_i)} \; I_{P+2,Q+1;Y}^{0,N+2;X} \left(\begin{array}{c} \mathrm{Z}_1 2^{h_1+k_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Z}_2 2^{h_2+k_2} \end{array} \right|$$

$$(1\text{-}\rho - \mathbf{p} - \sum_{i=1}^{u} f_i K_i - \sum_{i=1}^{v} a_i m_i - \sum_{i=1}^{r} c_i \eta_{G_i,g_i} : h_1, h_2; 1), \\ \cdot \\ \cdot \\ \cdots$$

$$\left. \begin{array}{c} (1\text{-}\sigma - \sum_{i=1}^{u} w_i K_i - \sum_{i=1}^{v} b_i m_i - \sum_{i=1}^{r} d_i \eta_{G_i,g_i} : k_1, k_2; 1), A \\ \cdot \\ \cdot \\ (1\text{-}\sigma - \rho - \mathbf{p} - \sum_{i=1}^{u} (f_i + w_i) K_i - \sum_{i=1}^{v} (a_i + b_i) m_i - \sum_{i=1}^{r} (c_i + d_i) \eta_{G_i,g_i} : h_1 + k_1, h_2 + k_2; 1), B \end{array} \right) \quad (7.2)$$

under the same validity conditions that (2.2), $R < c$ and $s = 2$.

Case 2 (solution for $W(R,x)$ when $R \geqslant c$)

$$W(R,x) = 2^{\rho+\sigma-2} \sum_{\mathbf{m}=0}^{\infty} (2\mathbf{m}+1) \, (c/R)^{\mathbf{m}+1} \, P_{\mathbf{m}}(x) \sum_{\mathbf{p}=0}^{\mathbf{m}} \frac{(-\mathbf{m})_{\mathbf{p}} (\mathbf{m}+1)_{\mathbf{p}} 2^{\mathbf{p}}}{(\mathbf{p}!)^2}$$

$$\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m'_1}\cdots\sum_{g_r=0}^{m'_r}\sum_{m_1,\cdots,m_v=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]} A'B'C\,\frac{t_1^{m_1}\cdots t_v^{m_v}}{m_1!\cdots m_v!}\,y_1^{K_1}\cdots y_u^{K_u}\,z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}$$

$$2^{\sum_{i=1}^u K_i(f_i+w_i)+\sum_{i=1}^v (a_i+b_i)m_i+\sum_{i=1}^r \eta_{G_i,g_i}(c_i+d_i)}\, I_{P+2,Q+1;Y}^{0,N+2;X}\left(\begin{array}{c} \mathrm{Z}_1 2^{h_1+k_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{Z}_2 2^{h_2+k_2}\end{array}\middle|\right.$$

$$(1\text{-}\rho-\mathbf{p}-\sum_{i=1}^u g_iK_i-\sum_{i=1}^v a_im_i-\sum_{i=1}^r c_i\eta_{G_i,g_i}:h_1,h_2;1),\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$\left.\begin{array}{c}(1\text{-}\sigma-\sum_{i=1}^u w_iK_i-\sum_{i=1}^v b_im_i-\sum_{i=1}^r d_i\eta_{G_i,g_i}:k_1,k_2;1),A\\ \cdot\\ \cdot\\ (1\text{-}\sigma-\rho-\mathbf{p}-\sum_{i=1}^u (f_i+w_i)K_i-\sum_{i=1}^v (a_i+b_i)m_i-\sum_{i=1}^r (c_i+d_i)\eta_{G_i,g_i}:h_1+k_1,h_2+k_2;1),B\end{array}\right) \quad (7.3)$$

under the same validity conditions that (2.2) with $R \geqslant c$ and $s = 2$.

8. Conclusion

Similarly, specializing the coefficient $A[N_1, K_1; \cdots ; N_u, K_u]$ of the generalized multivariable polynomials [8] and parameters of multivariable I-function defined by Prathima et al [5], the multivariable aleph-function and the Srivastava-Daoust function [9], we can obtain large number of results involving various special functions of one and several variables useful in Mathematics analysis, Applied Mathematics, Physics and Mechanics, for example, the distribution of temperature in a non-homogeneous bar and the problem on electrostatics potential in spherical regions.

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