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Some multiple integral relations involving general polynomials and multivariable Prasad's I-function

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ABSTRACT
In this paper, first we obtain a finite integral involving a sequence of polynomials, a class of multivariable polynomials and the multivariable I-function defined by Prasad [3]. Next, with the application of this and a lemma to Srivastava et al ([7], 1981),, we obtain two general multiple integral relations involving a sequence of polynomials, a class of multivariable polynomials and the multivariable I-function and two arbitrary functions f and g. By suitable specializing the functions f and g occuring in the main integral relations, a number of multiple integrals are evaluated which are new and quite general nature.

Keywords: Multivariable I-function, multiple integrals, class of multivariable polynomials, sequence of polynomials, multivariable H-function, H-function of two variables.

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1. Introduction and preliminaries.

The explicit form of the Raizada's generalized polynomial set ([4], Eq. ((2.3.4), page 71) will be defined and represented as given below.

$$S_n^{\alpha,\beta,0}[x;r,s,q,A,B,m,k,l]=B^{qn}x^{ln}\sum_{v=0}^{n}\sum_{u=0}^{v}\sum_{p=0}^{n}\sum_{e=0}^{p}\frac{(-)^p(-v)_u}{u!v!e!p!!}$$

$$\frac{(-p)_e(\alpha)_p(-\alpha-qn)_e}{(1-\alpha-p)_e}\left(\frac{e+k+ru}{l}\right)_n\left(\frac{Ax}{B}\right)^p \tag{1.1}$$

It may be pointed out here that the polynomial set defined by (1.1) is very general in nature and it unifies and extends a number of classical polynomials.
The generalized polynomials defined by Srivastava [6], is given in the following manner :

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1,\cdots,y_s] \;=\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}$$

$$A[N_1,K_1;\cdots;N_s,K_s]y_1^{K_1}\cdots y_s^{K_s} \tag{1.2}$$

Where $M_1,\cdots,M_s$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_s,K_s]$ are arbitrary constants, real or complex.

The multivariable I-function is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1,\cdots,z_r)=\;I_{p_2,q_2,p_3,q_3;\cdots;p_r,q_r:p',q';\cdots;p^{(r)},q^{(r)}}^{0,n_2;0,n_3;\cdots;0,n_r:m',n';\cdots;m^{(r)},n^{(r)}}\left(\begin{array}{c|c}\mathrm{z}_1 & (\mathrm{a}_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};\cdots;\\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};\cdots;\end{array}\right.$$

$$\left.\begin{matrix}(\mathrm{a}_{rj};\alpha'_{rj},\cdots,\alpha_{rj}^{(r)})_{1,p_r}:(a'_j,\alpha'_j)_{1,p'};\cdots;(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}}\\ \\ (\mathrm{b}_{rj};\beta'_{rj},\cdots,\beta_{rj}^{(r)})_{1,q_r}:(b'_j,\beta'_j)_{1,q'};\cdots;(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}}\end{matrix}\right) \tag{1.3}$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\xi(t_1,\cdots,t_r)\prod_{i=1}^{r}\phi_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.4}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [3].
The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_i|<\frac{1}{2}\Omega_i\pi$, where

$$\Omega_i=\sum_{k=1}^{n^{(i)}}\alpha_k^{(i)}-\sum_{k=n^{(i)}+1}^{p^{(i)}}\alpha_k^{(i)}+\sum_{k=1}^{m^{(i)}}\beta_k^{(i)}-\sum_{k=m^{(i)}+1}^{q^{(i)}}\beta_k^{(i)}+\left(\sum_{k=1}^{n_2}\alpha_{2k}^{(i)}-\sum_{k=n_2+1}^{p_2}\alpha_{2k}^{(i)}\right)+$$

$$+\left(\sum_{k=1}^{n_r}\alpha_{rk}^{(i)}-\sum_{k=n_r+1}^{p_r}\alpha_{rk}^{(i)}\right)-\left(\sum_{k=1}^{q_2}\beta_{2k}^{(i)}+\sum_{k=1}^{q_3}\beta_{3k}^{(i)}+\cdots+\sum_{k=1}^{q_r}\beta_{rk}^{(i)}\right) \tag{1.5}$$

where $i=1,\cdots,r$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.We may establish the the asymptotic expansion in the following convenient form :

$$I(z_1,\cdots,z_r)=0(\,|z_1|^{\gamma'_1},\cdots,|z_r|^{\gamma'_r}\,)\,,max(\,|z_1|,\cdots,|z_r|\,)\to 0$$

$$I(z_1,\cdots,z_r)\;=0(\,|z_1|,\cdots,|z_r|^{\beta'_s}\,)\,,min(\,|z_1|\,,\cdots,|z_r|\,)\to\infty$$

where $k=1,\cdots,z:\alpha'_k=min[Re(b_j^{(k)}/\beta_j^{(k)})],j=1,\cdots,m_k$ and

$$\beta'_k=max[Re((a_j^{(k)}-1)/\alpha_j^{(k)})],j=1,\cdots,n_k$$

We will use these following notations in this paper :

$$U=p_2,q_2;p_3,q_3;\cdots;p_{r-1},q_{r-1};V=0,n_2;0,n_3;\cdots;0,n_{s-1} \tag{1.6}$$

$$W=(p',q');\cdots;(p^{(r)},q^{(r)});X=(m',n');\cdots;(m^{(r)},n^{(r)}) \tag{1.7}$$

$$A=(a_{2k},\alpha'_{2k},\alpha''_{2k})_{1,p_2};\cdots;(a_{(r-1)k},\alpha'_{(r-1)k},\alpha''_{(r-1)k},\cdots,\alpha_{(r-1)k}^{(r-1)})_{1,p_{r-1}} \tag{1.8}$$

$$B=(b_{2k},\beta'_{2k},\beta''_{2k})_{1,q_2};\cdots;(b_{(r-1)k},\beta'_{(r-1)k},\beta''_{(r-1)k},\cdots,\beta_{(r-1)k}^{(r-1)})_{1,q_{r-1}} \tag{1.9}$$

$$\mathbb{A}=(a_{rk};\alpha'_{rk},\alpha''_{rk},\cdots,\alpha_{rk}^{(r)})_{1,p_r}:(a'_k,\alpha'_k)_{1,p'};\cdots;(a_k^{(r)},\alpha_k^{(r)})_{1,p^{(r)}} \tag{1.10}$$

$$\mathbb{B} = (b_{rk};\beta'{}_{rk},\beta''_{rk},\cdots,\beta^{(r)}_{rk})_{1,q_r}\text{: } (b'_k,\beta'_k)_{1,p'};\cdots;(b^{(r)}_k,\beta^{(r)}_k)_{1,q^{(r)}} \tag{1.11}$$

$$\mathbf{B}(e,p,u,v) = B^{qn-p}l^n \frac{(-)^p(-v)_u(-p)_e(\alpha)_p}{u!v!e!p!}\frac{(-\alpha-qn)_e}{(1-\alpha-p)_e}\left(\frac{e+k+ru}{l}\right)_n A^pB^v \tag{1.12}$$

$$C = \frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}A[N_1,K_1;\cdots;N_s,K_s] \tag{1.13}$$

$$L = ln + rv + p \tag{1.14}$$

$$\sum_{v=0}^{n}\sum_{u=0}^{v}\sum_{p=0}^{n}\sum_{e=0}^{p} = \sum_{e,p,u,v} \tag{1.15}$$

2. Required results;

The following results will be required in establishing our main integral relations :

Lemma 1 (Srivastava et al [7] ,1981) :

Let the functions $f(x)$ and $g(x)$ be integrable over the semi interval $(0,\infty)$ and define.

$$F(R) = \int_0^{\frac{\pi}{2}} h(R,\theta)\,\mathrm{d}\theta \tag{2.1}$$

where $h(R,\theta)$ is an integrable function of two variables in the rectangular region $0\leqslant R\leqslant\infty, 0\leqslant\theta\leqslant\frac{\pi}{2}$, then

$$\int_0^\infty\int_0^\infty f(x^2+y^2)h\left\{(x^2+y^2)^{1/2}tan^{-1}(y/x)\right\}\mathrm{d}x\mathrm{d}y = \frac{1}{2}\int_0^\infty f(t)F(\sqrt{t})\mathrm{d}t \tag{2.2}$$

and

$$\int_0^\infty\int_0^\infty\int_0^\infty(x^2+y^2)^{1/2}f(x^2+y^2+z^2)g\left[tan^{-1}\left\{(x^2+y^2)^{1/2}/z\right\}\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \int_0^\infty\int_0^\infty f(u^2+v^2)F\left\{(x^2+y^2)^{1/2}\right\}g\left\{tan^{-1}(v/u)\right\}\mathrm{d}u\mathrm{d}v \tag{2.3}$$

provided that the various integrals involved are absolutely convergent.

Lemma 2 (Kalla et al [1] ,1981) :

$$H'(at,bt) = H^{0,n'_1;1,n'_2;1,n'_3}_{p'_1,q'_1;p'_2,q'_2+1;p'_3,q'_3+1}\left(\begin{array}{c|c} \text{at} & (\text{a'}_j;A'_j,A''_j)_{1,p'_1}:(e_j,E_j)_{1,p'_2};(g_j,G_j)_{1,p'_3} \\ \cdot & \cdot \\ \cdot & \\ \text{bt} & (\text{b'}_j;B'_j,B''_j)_{1,q'_1}:(0,1),(f_j,F_j)_{1,q'_2};(0,1),(h_j,H_j)_{1,q'_3} \end{array}\right)$$

$$= \sum_{\mathbf{M}'=0}^\infty \phi(\mathbf{M}')\frac{(-at)^{\mathbf{M}'}}{\mathbf{M}'!} \tag{2.4}$$

where

$$\phi(\mathbf{M}') = \sum_{\mathbf{N}'=\mathbf{0}}^{\mathbf{M}'} \phi'(\mathbf{M}'-\mathbf{N}',\mathbf{N}')\theta_1'(\mathbf{M}')\theta_2'(\mathbf{N}')(b/a)^{\mathbf{N}'}\binom{\mathbf{M}'}{\mathbf{N}'} \tag{2.5}$$

3. A useful integral

We obtain the following integral, which will be required in the next section :

$$\int_0^{\frac{\pi}{2}} e^{(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}e^{-\mu R}\, S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{y}_1 R^{2\omega_1'}e^{i(\lambda_1+\omega_1)\theta}(\sin\theta)^{\lambda_1}(\cos\theta)^{\omega_1} \\ . \\ . \\ \mathrm{y}_s R^{2\omega_s'}e^{i(\lambda_s+\omega_s)\theta}(\sin\theta)^{\lambda_s}(\cos\theta)^{\omega_s} \end{array}\right)$$

$$S_n^{\alpha,\beta,0}[YR^{2\rho'}e^{i(\alpha+\beta)}(\sin\theta)^{\zeta}(\cos\theta)^{\xi};r,s,q,A,B,m,k,l]H'\Big(ae^{i(\sigma+\rho)\theta}(\sin\theta)^{\sigma}(\cos\theta)^{\rho},be^{i(\sigma+\rho)\theta}(\sin\theta)^{\sigma}(\cos\theta)^{\rho}\Big)$$

$$I\left(\begin{array}{c} \mathrm{z}_1 R^{2\rho_1}e^{i(\sigma_1+\delta_1)\theta}(\sin\theta)^{\sigma_1}(\cos\theta)^{\delta_1} \\ . \\ . \\ \mathrm{z}_r R^{2\rho_r}e^{i(\sigma_r+\delta_r)\theta}(\sin\theta)^{\sigma_r}(\cos\theta)^{\delta_r} \end{array}\right)\mathrm{d}\theta = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{\mathbf{M}'=0}^{\infty}\sum_{e,p,u,v}\phi(\mathbf{M}')\frac{(-a)^{\mathbf{M}'}}{\mathbf{M}'!}C$$

$$\mathbf{B}(e,p,u,v)y_1^{K_1}\cdots y_s^{K_s}\,Y^L\,R^{2(\sum_{j=0}^{s}K_j\omega_j'+\rho' L)}e^{-\mu R}\,exp\left\{i\pi\big(\alpha+\sum_{j=1}^{s}K_j\lambda_j+\zeta L+\sigma\mathbf{M}'\big)/2\right\}$$

$$I_{U;p_r+2,q_r+1:W}^{V;0,n_r+2:X}\left(\begin{array}{c} \mathrm{z}_1 R^{2\rho_1}e^{i\pi\sigma_1/2} \\ . \\ . \\ . \\ \mathrm{z}_r R^{2\rho_r}e^{i\pi\sigma_r/2} \end{array}\middle| \begin{array}{c} \mathrm{A};\ (1\text{-}\alpha-\sum_{j=1}^{s}K_j\lambda_j-\zeta L-\sigma\mathbf{M}';\sigma_1,\cdots,\sigma_r), \\ . \\ . \\ \mathrm{B}; \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-}\beta-\sum_{j=1}^{s}K_j\omega_j-\xi L-\rho\mathbf{M}';\delta_1,\cdots,\delta_r),\mathbb{A} \\ . \\ . \\ (1\text{-}\alpha-\beta-\sum_{j=1}^{s}K_j(\lambda_j+\omega_j)-(\zeta+\xi)L-(\sigma+\rho)\mathbf{M}';\sigma_1+\delta_1,\cdots,\sigma_r+\delta_r),\mathbb{B} \end{array}\right) \tag{3.1}$$

Provided

$min\{\omega_j',\lambda_j,\omega_j,\rho_k,\sigma_k,\delta_k\}>0$ for $j=1,\cdots,s;k=1,\cdots,r$; $Re(\alpha)>0, Re(\beta)>0$

$Re\left[\alpha+\sum_{j=1}^{s}K_i\lambda_j+L\zeta+\mathbf{M}'\sigma+\sum_{i=1}^{r}\sigma_i\min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}\right]>0$ and

$$Re\left[\beta+\sum_{j=1}^{s}K_i\omega_j+L\xi+\mathbf{M}'\rho+\sum_{i=1}^{r}\delta_i\min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}\right]>0$$

The series occuring on the right hand side of (3.1) is absolutely convergent.

$|argz_i|<\frac{1}{2}\Omega_i\pi$, where Ω_i is defined by (1.5).

Proof

To prove (3.1), we first express the class of multivariable polynomials $S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[.]$, the sequence of polynomials $S_n^{\alpha,\beta,0}[.;r,s,q,A,B,m,k,l]$ and H-function of two variables in series with the help of (1.2), (1.1) and (2.4) respectively and interchange the orders of summations and integrations (which is permissible under the conditions stated). Now, we express the multivariable I-function defined by Prasad [3] in Mellin-Barnes contour integral given by (1.4). Interchange the order of $\theta-$integral and $(s_1,\cdots,s_r)-$integrals and collect the powers of $(\cos\theta),(\sin\theta)$ and $e^{i(\alpha+\beta)\theta}$. Now evaluate the $\theta-$integral with the help of result ([2],MacRobert, 1961)

$$\int_0^{\pi}e^{i(\alpha+\beta)\theta}(sin\theta)^{\alpha-1}(cos\theta)^{\beta-1}\mathrm{d}\theta=\frac{e^{i\frac{\pi}{2}\alpha}\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{3.2}$$

with $Re(\alpha)>0$ and $Re(\beta)>0$. Finally, interpret the Mellin-Barnes contour integral to multivariable I-function with the help of (1.4), we obtain the desired result.

4. The main integral relation

The following double integral and triple integral relations will be established in this section :

$$\int_0^{\infty}\int_0^{\infty}x^{\beta-1}y^{\alpha-1}(x^2+y^2)^{1-\frac{1}{2}(\alpha+\beta)}\,exp\left[i(\alpha+\beta)tan^{-1}(y/x)-\mu(x^2+y^2)^{1/2}\right]f(x^2+y^2)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c}\mathrm{y}_1exp\left[i(\lambda_1+\omega_1)tan^{-1}(y/x)\right]y^{\lambda_1}x^{\omega_1}(x^2+y^2)^{\omega_1'-\frac{1}{2}(\omega_1+\lambda_1)}\\ .\\ .\\ \mathrm{y}_sexp\left[i(\lambda_s+\omega_s)tan^{-1}(y/x)\right]y^{\lambda_s}x^{\omega_s}(x^2+y^2)^{\omega_s'-\frac{1}{2}(\omega_s+\lambda_s)}\end{array}\right)$$

$$S_n^{\alpha,\beta,0}[Yexp\left\{i(\zeta+\xi)tan^{-1}(y/x\right\}y^{\zeta}x^{\xi}(x^2+y^2)^{\rho-\frac{1}{2}(\zeta+\xi)};r,s,q,A,B,m,k,l]$$

$$H'\left(\begin{array}{c}\mathrm{a}\ \exp\left[i(\sigma+\rho)tan^{-1}(y/x)\right]y^{\sigma}x^{\rho}(x^2+y^2)^{-\frac{1}{2}(\sigma+\rho)}\\ .\\ .\\ \mathrm{b}\ \exp\left[i(\sigma+\rho)tan^{-1}(y/x)\right]y^{\sigma}x^{\rho}(x^2+y^2)^{-\frac{1}{2}(\sigma+\rho)}\end{array}\right)$$

$$I\left(\begin{array}{c}\mathrm{z}_1exp\left[i(\sigma_1+\delta_1)tan^{-1}(y/x)\right]y^{\sigma_1}x^{\delta_1}(x^2+y^2)^{\rho_1-\frac{1}{2}(\sigma_1+\delta_1)}\\ .\\ .\\ \mathrm{z}_rexp\left[i(\sigma_r+\delta_r)tan^{-1}(y/x)\right]y^{\sigma_r}x^{\delta_r}(x^2+y^2)^{\rho_r-\frac{1}{2}(\sigma_r+\delta_r)}\end{array}\right)\mathrm{d}x\mathrm{d}y=\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{\mathbf{M}'=0}^{\infty}\sum_{e,p,u,v}$$

$$\phi(\mathbf{M}')\frac{(-a)^{\mathbf{M}'}}{\mathbf{M}'!}\mathbf{B}(e,p,u,v)C\ y_1^{K_1}\cdots y_s^{K_s}\,Y^L\ \ exp\left\{i\pi\big(\alpha+\sum_{j=1}^{s}K_j\lambda_j+\zeta L+\sigma\mathbf{M}'\big)/2\right\}$$

$$\int_0^\infty t^{\sum_{j=1}^s K_j\omega_j'+\rho L}e^{-\mu vt}\,I_{U;p_r+2,q_r+1:W}^{V;0,n_r+2:X}\left(\begin{array}{c|c} \mathrm{z}_1 t^{\rho_1}e^{i\pi\sigma_1/2} & \mathrm{A};\ (1\text{-}\alpha-\sum_{j=1}^s K_j\lambda_j-\zeta L-\sigma\mathbf{M}';\sigma_1,\cdots,\sigma_r), \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}; \\ \mathrm{z}_r t^{\rho_r}e^{i\pi\sigma_r/2} & \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-}\beta-\sum_{j=1}^s K_j\omega_j-\xi L-\rho\mathbf{M}';\delta_1,\cdots,\delta_r),\mathbb{A} \\ \cdot \\ \cdot \\ (1\text{-}\alpha-\beta-\sum_{j=1}^s K_j(\lambda_j+\omega_j)-(\zeta+\xi)L-(\sigma+\rho)\mathbf{M}';\sigma_1+\delta_1,\cdots,\sigma_r+\delta_r),\mathbb{B} \end{array}\right)f(t)\,\mathrm{d}t \tag{4.1}$$

under the same conditions that (3.1)

$$\int_0^\infty\int_0^\infty\int_0^\infty x^{\beta-1}y^{\alpha-1}(x^2+y^2)^{(1-\alpha-\beta)/2}exp\Big\{i(\alpha+\beta)tan^{-1}(y/x)-\mu(x^2+y^2+z^2)^{1/2}\Big\}g\left[tan^{-1}\big\{(x^2+y^2)^{\frac{1}{2}}/z\big\}\right]$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{y}_1(x^2+y^2+z^2)^{\omega_1'}exp\left\{i(\lambda_1+\omega_1)tan^{-1}(y/x)\right\}x^{\omega_1}y^{\lambda_1}(x^2+y^2)^{-(\lambda_1+\omega_1)/2} \\ \cdot \\ \cdot \\ \mathrm{y}_s(x^2+y^2+z^2)^{\omega_s'}exp\left\{i(\lambda_s+\omega_s)tan^{-1}(y/x)\right\}x^{\omega_s}y^{\lambda_s}(x^2+y^2)^{-(\lambda_s+\omega_s)/2} \end{array}\right)$$

$$S_n^{\alpha,\beta,0}[Y(x^2+y^2+z^2))^{\rho}exp\big\{i(\zeta+\xi)tan^{-1}(y/x)\big\}x^{\xi}y^{\zeta}(x^2+y^2)^{-(\xi+\zeta)/2};r,s,q,A,B,m,k,l]$$

$$H'\left(\begin{array}{c} \mathrm{a}\ \exp\big[i(\rho+\sigma)tan^{-1}(y/x)\big]\,x^{\rho}y^{\sigma}(x^2+y^2)^{-(\rho+\sigma)/2} \\ \cdot \\ \cdot \\ \mathrm{b}\ \exp\big[i(\rho+\sigma)tan^{-1}(y/x)\big]\,x^{\rho}y^{\sigma}(x^2+y^2)^{-(\rho+\sigma)/2} \end{array}\right)$$

$$I\left(\begin{array}{c} \mathrm{z}_1(x^2+y^2+z^2)^{\rho_1}exp\left\{i(\delta_1+\sigma_1)tan^{-1}(y/x)\right\}x^{\delta_1}y^{\sigma_1}(x^2+y^2)^{-(\delta_1+\sigma_1)/2} \\ \cdot \\ \cdot \\ \mathrm{z}_r(x^2+y^2+z^2)^{\rho_r}exp\left\{i(\delta_r+\sigma_r)tan^{-1}(y/x)\right\}x^{\delta_r}y^{\sigma_r}(x^2+y^2)^{-(\delta_r+\sigma_r)/2} \end{array}\right)f(x^2+y^2+z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$=\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{\mathbf{M}'=0}^{\infty}\sum_{e,p,u,v}\phi(\mathbf{M}')\frac{(-a)^{\mathbf{M}'}}{\mathbf{M}'!}\mathbf{B}(e,p,u,v)C\ \ y_1^{K_1}\cdots y_s^{K_s}\,Y^L$$

$$exp\left\{i\pi\big(\alpha+\sum_{j=1}^{s}K_j\lambda_j+\zeta L+\sigma\mathbf{M}'\big)/2\right\}\int_0^\infty\int_0^\infty f(u^2+v^2)g\left\{tan^{-1}(v/u)\right\}(u^2+v^2)^{\sum_{j=1}^{s}K_j\omega_j'+\rho L}$$

$$e^{-\mu(u^2+v^2)}I_{U;p_r+2,q_r+1:W}^{V;0,n_r+2:X}\left(\begin{array}{c|c} \mathrm{z}_1'u^2+v^2)^{\rho_1}e^{i\pi\sigma_1/2} & \mathrm{A};\ (1\text{-}\alpha-\sum_{j=1}^{s}K_j\lambda_j-\zeta L-\sigma\mathbf{M}';\sigma_1,\cdots,\sigma_r), \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}; \\ \mathrm{z}_r(u^2+v^2)^{\rho_r}e^{i\pi\sigma_r/2} & \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-}\beta-\sum_{j=1}^{s}K_j\omega_j-\xi L-\rho\mathbf{M}';\delta_1,\cdots,\delta_r),\mathbb{A} \\ \cdot \\ \cdot \\ (1\text{-}\alpha-\beta-\sum_{j=1}^{s}K_j(\lambda_j+\omega_j)-(\zeta+\xi)L-(\sigma+\rho)\mathbf{M}';\sigma_1+\delta_1,\cdots,\sigma_r+\delta_r),\mathbb{B} \end{array}\right)\mathrm{d}u\mathrm{d}v \tag{4.2}$$

under the same conditions that (3.1)

Proofs

To prove the integral relations (4.1) and (4.2) we take in (2.1)

$$h(R,\theta)=e^{(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}e^{-\mu R}\,S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{y}_1R^{2\omega_1'}e^{i(\lambda_1+\omega_1)\theta}(\sin\theta)^{\lambda_1}(\cos\theta)^{\omega_1} \\ \cdot \\ \cdot \\ \mathrm{y}_sR^{2\omega_s'}e^{i(\lambda_s+\omega_s)\theta}(\sin\theta)^{\lambda_s}(\cos\theta)^{\omega_s} \end{array}\right)$$

$$S_n^{\alpha,\beta,0}[YR^{2\rho'}e^{i(\alpha+\beta)}(\sin\theta)^{\zeta}(\cos\theta)^{\xi};r,s,q,A,B,m,k,l]H'\Big(ae^{i(\sigma+\rho)\theta}(\sin\theta)^{\sigma}(\cos\theta)^{\rho},be^{i(\sigma+\rho)\theta}(\sin\theta)^{\sigma}(\cos\theta)^{\rho}\Big)$$

$$I\left(\begin{array}{c} \mathrm{z}_1R^{2\rho_1}e^{i(\sigma_1+\delta_1)\theta}(\sin\theta)^{\sigma_1}(\cos\theta)^{\delta_1} \\ \cdot \\ \cdot \\ \mathrm{z}_rR^{2\rho_r}e^{i(\sigma_r+\delta_r)\theta}(\sin\theta)^{\sigma_r}(\cos\theta)^{\delta_r} \end{array}\right) \tag{4.3}$$

Now, we evaluate the resulting integral by means of (3.1) and arrive at the following result :

$$F(R)=\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{\mathbf{M}'=0}^{\infty}\sum_{e,p,u,v}\phi(\mathbf{M}')\frac{(-a)^{\mathbf{M}'}}{\mathbf{M}'!}C$$

$$\mathbf{B}(e,p,u,v)y_1^{K_1}\cdots y_s^{K_s}Y^L\,R^{2(\sum_{j=0}^{s}K_j\omega_j'+\rho'L)}e^{-\mu R}\,exp\left\{i\pi\big(\alpha+\sum_{j=1}^{s}K_j\lambda_j+\zeta L+\sigma\mathbf{M}'\big)/2\right\}$$

$$I_{U;p_r+2,q_r+1:W}^{V;0,n_r+2:X}\left(\begin{array}{c|c} \mathrm{z}_1 R^{2\rho_1} e^{i\pi\sigma_1/2} & \mathrm{A};\ (1\text{-}\alpha-\sum_{j=1}^s K_j\lambda_j-\zeta L-\sigma\mathbf{M}';\sigma_1,\cdots,\sigma_r), \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{z}_r R^{2\rho_r} e^{i\pi\sigma_r/2} & \mathrm{B}; \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-}\beta-\sum_{j=1}^s K_j\omega_j-\xi L-\rho\mathbf{M}';\delta_1,\cdots,\delta_r),\mathbb{A} \\ \cdot \\ \cdot \\ (1\text{-}\alpha-\beta-\sum_{j=1}^s K_j(\lambda_j+\omega_j)-(\zeta+\xi)L-(\sigma+\rho)\mathbf{M}';\sigma_1+\delta_1,\cdots,\sigma_r+\delta_r),\mathbb{B} \end{array}\right) \tag{4.4}$$

Now substituting the value of $h(R,\theta)$ and $F(R)$ as given by (4.3) and (4.4) respectively in (2.2) and (2.3) in succession, we obtain the integral relations (4.1) and (4.2) after algebric manipulations and simplification.

5. Particular case

The integral relations (4.1) and (4.2) are quite general in nature . A very large number of (known and new integrals can be derived as specials cases. We mention below a special case of our main result.

If we set $\omega_j=\lambda_j=\xi=\zeta=0$ for $j=1,\cdots,s$ and $Max\{\sigma_k,\delta_k\}\rightarrow 0$ for $k=1,\cdots,r$, we get

$$\int_0^\infty\int_0^\infty x^{\beta-1}y^{\alpha-1}(x^2+y^2)^{1-\frac{1}{2}(\alpha+\beta)}\,exp\left[i(\alpha+\beta)tan^{-1}(y/x)-\mu(x^2+y^2)^{1/2}\right]f(x^2+y^2)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{y}_1(x^2+y^2)^{\omega_1'} \\ \cdot \\ \cdot \\ \mathrm{y}_s(x^2+y^2)^{\omega_s'} \end{array}\right) S_n^{\alpha,\beta,0}[Y(x^2+y^2)^\rho;r,s,q,A,B,m,k,l]$$

$$H'\left(\begin{array}{c} \mathrm{a}\ \exp\{i(\sigma+\rho)tan^{-1}(y/x)\}\,y^\sigma x^\rho(x^2+y^2)^{-\frac{1}{2}(\sigma+\rho)} \\ \cdot \\ \cdot \\ \mathrm{b}\ \exp\{i(\sigma+\rho)tan^{-1}(y/x)\}\,y^\sigma x^\rho(x^2+y^2)^{-\frac{1}{2}(\sigma+\rho)} \end{array}\right) I\left(\begin{array}{c} \mathrm{z}_1(x^2+y^2)^{\rho_1} \\ \cdot \\ \cdot \\ \mathrm{z}_r(x^2+y^2)^{\rho_r} \end{array}\right)\mathrm{d}x\mathrm{d}y$$

$$=\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{\mathbf{M}'=0}^{\infty}\sum_{e,p,u,v}\phi(\mathbf{M}')\frac{(-a)^{\mathbf{M}'}}{\mathbf{M}'!}\mathbf{B}(e,p,u,v)C\ \ y_1^{K_1}\cdots y_s^{K_s}\,Y^L\ \ exp\left\{i\pi\big(\alpha+\sigma\mathbf{M}'\big)/2\right\}$$

$$\frac{\Gamma(\alpha+\sigma\mathbf{M}')\Gamma(\beta+\rho\mathbf{M}')}{\Gamma(\alpha+\beta+(\sigma+\rho)\mathbf{M}')}\int_0^\infty t^{\sum_{j=1}^s K_j\omega_j'+\rho L}e^{-\mu v t}I_{U;p_r,q_r:W}^{V;0,n_r:X}\left(\begin{array}{c|c} \mathrm{z}_1 t^{\rho_1} & \mathrm{A}\ ;\ \mathbb{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{z}_r t^{\rho_r} & \mathrm{B}\ ;\mathbb{B} \end{array}\right)f(t)\,\mathrm{d}t \tag{5.1}$$

6. Applications

By suitably choosing the functions f and g in the main integral relations, a large number of intersting double and triple integrals can be evaluated. We shall, however obtain her only one double and one triple integral by way illustration.

Thus if in (4.2) $g(t) = e^{i(\alpha+\beta)t}(\sin t)^{\alpha-1}(\cos t)^{\beta-1}H'\left[ae^{i(\sigma+\rho)t}(\sin t)^{\sigma}(\cos t)^{\rho}, be^{i(\sigma+\rho)t}(\sin t)^{\sigma}(\cos t)^{\rho}\right]$, we arrive at the following integral relation on making of (5.1) :

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}(xz)^{\beta-1}y^{\alpha-1}(x^2+y^2)^{-\beta/2}(x^2+y^2+z^2)^{1-(1/2)(\alpha+\beta)}f(x^2+y^2+z^2)$$

$$exp\{i(\alpha+\beta)[tan^{-1}(y/x)+tan^{-1}\{(x^2+y^2)^{\frac{1}{2}}/z\}]\}$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{y}_1(x^2+y^2+z^2)^{\omega_1'}exp\left\{i(\lambda_1+\omega_1)tan^{-1}(y/x)\right\}x^{\omega_1}y^{\lambda_1}(x^2+y^2)^{-(\lambda_1+\omega_1)/2} \\ . \\ . \\ \mathrm{y}_s(x^2+y^2+z^2)^{\omega_s'}exp\left\{i(\lambda_s+\omega_s)tan^{-1}\right\}(y/x)x^{\omega_s}y^{\lambda_s}(x^2+y^2)^{-(\lambda_s+\omega_s)/2} \end{array}\right)$$

$$S_n^{\alpha,\beta,0}[Y(x^2+y^2+z^2))^{\rho}exp\{i(\zeta+\xi)tan^{-1}(y/x)\}x^{\xi}y^{\zeta}(x^2+y^2)^{-(\xi+\zeta)/2};r,s,q,A,B,m,k,l]$$

$$H'\left(\begin{array}{c} \mathrm{a}\ \exp\left[i(\rho+\sigma)tan^{-1}(y/x)\right]x^{\rho}y^{\sigma}(x^2+y^2)^{-(\rho+\sigma)/2} \\ . \\ . \\ \mathrm{b}\ \exp\left[i(\rho+\sigma)tan^{-1}(y/x)\right]x^{\rho}y^{\sigma}(x^2+y^2)^{-(\rho+\sigma)/2} \end{array}\right)$$

$$H'\left(\begin{array}{c} \mathrm{a}\ \exp\left\{i(\sigma+\rho)tan^{-1}\{(x^2+y^2)^{\frac{1}{2}}/z\}\right\}(x^2+y^2)^{\frac{\sigma}{2}}(x^2+y^2+z^2)^{-(\sigma+\rho)/2}z^{\rho} \\ . \\ . \\ \mathrm{b}\ \exp\left\{i(\sigma+\rho)tan^{-1}\{(x^2+y^2)^{\frac{1}{2}}/z\}\right\}(x^2+y^2)^{\frac{\sigma}{2}}(x^2+y^2+z^2)^{-(\sigma+\rho)/2}z^{\rho} \end{array}\right)$$

$$I\left(\begin{array}{c} \mathrm{z}_1(x^2+y^2+z^2)^{\rho_1}exp\left\{i(\delta_1+\sigma_1)tan^{-1}(y/x)\right\}x^{\delta_1}y^{\sigma_1}(x^2+y^2)^{-(\delta_1+\sigma_1)/2} \\ . \\ . \\ \mathrm{z}_r(x^2+y^2+z^2)^{\rho_r}exp\left\{i(\delta_r+\sigma_r)tan^{-1}(y/x)\right\}x^{\delta_r}y^{\sigma_r}(x^2+y^2)^{-(\delta_r+\sigma_r)/2} \end{array}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$=\frac{1}{2}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{\mathbf{M}',\mathbf{M}''=0}^{\infty}\sum_{e,p,u,\upsilon}\phi(\mathbf{M}')\phi(\mathbf{M}'')\frac{(-a)^{\mathbf{M}'+\mathbf{M}''}}{\mathbf{M}'!\mathbf{M}''!}\mathbf{B}(e,p,u,v)C\ \ y_1^{K_1}\cdots y_s^{K_s}\,Y^L$$

$$exp\left\{i\pi\big(2\alpha+\sum_{j=1}^{s}K_j\lambda_j+\zeta L+\sigma(\mathbf{M}'+\mathbf{M}'')\big)/2\right\}\frac{\Gamma(\alpha+\sigma\mathbf{M}'')\Gamma(\beta+\rho\mathbf{M}'')}{\Gamma(\alpha+\beta+(\sigma+\rho)\mathbf{M}'')}\int_0^{\infty}t^{\sum_{j=1}^{s}K_j\omega_j'+\rho L}e^{-\mu\upsilon t}$$

$$I_{U;p_r+2,q_r+1:W}^{V;0,n_r+2:X}\left(\begin{array}{c} \mathrm{z}_1 t^{\rho_1} e^{i\pi\sigma_1/2} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r t^{\rho_r} e^{i\pi\sigma_r/2} \end{array}\right| \begin{array}{c} \mathrm{A};\ (1\text{-}\alpha-\sum_{j=1}^s K_j\lambda_j-\zeta L-\sigma\mathbf{M}';\sigma_1,\cdots,\sigma_r), \\ \cdot \\ \cdot \\ \mathrm{B}; \end{array}$$

$$\begin{array}{c} (1\text{-}\beta-\sum_{j=1}^s K_j\omega_j-\xi L-\rho\mathbf{M}';\delta_1,\cdots,\delta_r),\mathbb{A} \\ \cdot \\ \cdot \\ (1\text{-}\alpha-\beta-\sum_{j=1}^s K_j(\lambda_j+\omega_j)-(\zeta+\xi)L-(\sigma+\rho)\mathbf{M}';\sigma_1+\delta_1,\cdots,\sigma_r+\delta_r),\mathbb{B} \end{array}\right) f(t)\,\mathrm{d}t \tag{6.1}$$

under the same conditions that (3.1).

If we take $f(t)=H'(ct,dt)$ and evaluate the $t-$integrals occuring on the right hand sides of (5.1) and (5.2) with the help of the Gamma-function definition and arrive at the following multiple integrals after algebric manipulations and simplififcation :

$$\int_0^\infty\int_0^\infty x^{\beta-1}y^{\alpha-1}(x^2+y^2)^{1-\frac{1}{2}(\alpha+\beta)}\,exp\left[i(\alpha+\beta)tan^{-1}(y/x)-\mu(x^2+y^2)^{1/2}\right] f(x^2+y^2)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{y}_1 exp\left[i(\lambda_1+\omega_1)tan^{-1}(y/x)\right] y^{\lambda_1}x^{\omega_1}(x^2+y^2)^{\omega_1'-\frac{1}{2}(\omega_1+\lambda_1)} \\ \cdot \\ \cdot \\ \mathrm{y}_s exp\left[i(\lambda_s+\omega_s)tan^{-1}(y/x)\right] y^{\lambda_s}x^{\omega_s}(x^2+y^2)^{\omega_s'-\frac{1}{2}(\omega_s+\lambda_s)} \end{array}\right)$$

$$S_n^{\alpha,\beta,0}[Y exp\left\{i(\zeta+\xi)tan^{-1}(y/x\right\} y^\zeta x^\xi (x^2+y^2)^{\rho-\frac{1}{2}(\zeta+\xi)};r,s,q,A,B,m,k,l]$$

$$H'\left(\begin{array}{c} \mathrm{a}\ \exp\left[i(\sigma+\rho)tan^{-1}(y/x)\right] y^\sigma x^\rho (x^2+y^2)^{-\frac{1}{2}(\sigma+\rho)} \\ \cdot \\ \cdot \\ \mathrm{b}\ \exp\left[i(\sigma+\rho)tan^{-1}(y/x)\right] y^\sigma x^\rho (x^2+y^2)^{-\frac{1}{2}(\sigma+\rho)} \end{array}\right)\ H'\left(c(x^2+y^2),d(x^2+y^2)\right)$$

$$I\left(\begin{array}{c} \mathrm{z}_1 exp\left[i(\sigma_1+\delta_1)tan^{-1}(y/x)\right] y^{\sigma_1}x^{\delta_1}(x^2+y^2)^{\rho_1-\frac{1}{2}(\sigma_1+\delta_1)} \\ \cdot \\ \cdot \\ \mathrm{z}_r exp\left[i(\sigma_r+\delta_r)tan^{-1}(y/x)\right] y^{\sigma_r}x^{\delta_r}(x^2+y^2)^{\rho_r-\frac{1}{2}(\sigma_r+\delta_r)} \end{array}\right)\mathrm{d}x\mathrm{d}y=\frac{1}{2}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{\mathbf{M}',\mathbf{M}'''=0}^{\infty}\sum_{e,p,u,v}$$

$$\phi(\mathbf{M}')\phi(\mathbf{M}''')\frac{(-a)^{\mathbf{M}'}(-c)^{\mathbf{M}'''}}{\mathbf{M}'!\mathbf{M}'''!}\mathbf{B}(e,p,u,v)C\ \ y_1^{K_1}\cdots y_s^{K_s}\,Y^L\ \frac{2}{\mu^{2(\sum_{j=1}^s K_j\omega_j'+\rho L+\mathbf{M}'''+1)}}$$

$$exp\left\{i\pi\big(\alpha+\sum_{j=1}^s K_j\lambda_j+\zeta L+\sigma\mathbf{M}'\big)/2\right\}\ I_{U;p_r+3,q_r+1:W}^{V;0,n_r+3:X}\left(\begin{array}{c} \mathrm{z}_1\mu^{-2\rho_1}e^{i\pi\sigma_1/2} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r\mu^{-2\rho_r}e^{i\pi\sigma_r/2} \end{array}\right|$$

$$\begin{matrix} \mathrm{A};\ (1\text{-}\alpha-\sum_{j=1}^{s}K_j\lambda_j-\zeta L-\sigma\mathbf{M}';\sigma_1,\cdots,\sigma_r),(-1-2(\sum_{j=1}^{s}K_j\omega_j'+\rho L+\mathbf{M}''');2\rho_1,\cdots,2\rho_r), \\ \cdot \\ \cdot \\ \mathrm{B}; \end{matrix}$$

$$\left.\begin{matrix} (1\text{-}\beta-\sum_{j=1}^{s}K_j\omega_j-\xi L-\rho\mathbf{M}';\delta_1,\cdots,\delta_r),\mathbb{A} \\ \cdot \\ \cdot \\ (1\text{-}\alpha-\beta-\sum_{j=1}^{s}K_j(\lambda_j+\omega_j)-(\zeta+\xi)L-(\sigma+\rho)\mathbf{M}';\sigma_1+\delta_1,\cdots,\sigma_r+\delta_r),\mathbb{B} \end{matrix}\right) \tag{6.2}$$

under the same conditions that (3.1) and $Re\big(2(\sum_{j=1}^{s}K_j\omega_j'+\rho L+\mathbf{M}'''+\sum_{j=1}^{r}\rho_j s_j)\big)>0$

and

$$\int_0^\infty\int_0^\infty\int_0^\infty (xz)^{\beta-1}y^{\alpha-1}(x^2+y^2)^{-\beta/2}(x^2+y^2+z^2)^{1-(1/2)(\alpha+\beta)}$$

$$exp\big\{i(\alpha+\beta)[tan^{-1}(y/x)+tan^{-1}\big\{(x^2+y^2)^{\frac{1}{2}}/z-\mu(x^2+y^2+z^2)^{1/2}\big\}]\big\}$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{matrix} \mathrm{y}_1(x^2+y^2+z^2)^{\omega_1'}exp\left\{i(\lambda_1+\omega_1)tan^{-1}(y/x)\right\}x^{\omega_1}y^{\lambda_1}(x^2+y^2)^{-(\lambda_1+\omega_1)/2} \\ \cdot \\ \cdot \\ \mathrm{y}_s(x^2+y^2+z^2)^{\omega_s'}exp\left\{i(\lambda_s+\omega_s)tan^{-1}\big(y/x)\right\}x^{\omega_s}y^{\lambda_s}(x^2+y^2)^{-(\lambda_s+\omega_s)/2} \end{matrix}\right)$$

$$S_n^{\alpha,\beta,0}[Y(x^2+y^2+z^2))^{\rho}exp\big\{i(\zeta+\xi)tan^{-1}(y/x)\big\}x^{\xi}y^{\zeta}(x^2+y^2)^{-(\xi+\zeta)/2};r,s,q,A,B,m,k,l]$$

$$H'\left(\begin{matrix} \mathrm{a}\ \exp\big[i(\rho+\sigma)tan^{-1}(y/x)\big]\,x^{\rho}y^{\sigma}(x^2+y^2)^{-(\rho+\sigma)/2} \\ \cdot \\ \cdot \\ \mathrm{b}\ \exp\big[i(\rho+\sigma)tan^{-1}(y/x)\big]\,x^{\rho}y^{\sigma}(x^2+y^2)^{-(\rho+\sigma)/2} \end{matrix}\right)H\left(c(x^2+y^2+z^2),d(x^2+y^2+z^2)\right)$$

$$H'\left(\begin{matrix} \mathrm{a}\ \exp\Big\{i(\sigma+\rho)tan^{-1}\big\{(x^2+y^2)^{\frac{1}{2}}\Big\}\,(x^2+y^2)^{\frac{\sigma}{2}}(x^2+y^2+z^2)^{-(\sigma+\rho)/2}z^{\rho} \\ \cdot \\ \cdot \\ \mathrm{b}\ \exp\Big\{i(\sigma+\rho)tan^{-1}\big\{(x^2+y^2)^{\frac{1}{2}}\Big\}\,(x^2+y^2)^{\frac{\sigma}{2}}(x^2+y^2+z^2)^{-(\sigma+\rho)/2}z^{\rho} \end{matrix}\right)$$

$$I\left(\begin{matrix} \mathrm{z}_1(x^2+y^2+z^2)^{\rho_1}exp\left\{i(\delta_1+\sigma_1)tan^{-1}(y/x)\right\}x^{\delta_1}y^{\sigma_1}(x^2+y^2)^{\rho_1-(\delta_1+\sigma_1)/2} \\ \cdot \\ \cdot \\ \mathrm{z}_r(x^2+y^2+z^2)^{\rho_r}exp\left\{i(\delta_r+\sigma_r)tan^{-1}(y/x)\right\}x^{\delta_r}y^{\sigma_r}(x^2+y^2)^{(\rho_r-(\delta_r+\sigma_r)/2} \end{matrix}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \frac{1}{2}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{\mathbf{M}',\mathbf{M}'',\mathbf{M}'''=0}^{\infty}\sum_{e,p,u,v}\phi(\mathbf{M}')\phi(\mathbf{M}'')\phi(\mathbf{M}''')\frac{(-a)^{\mathbf{M}'+\mathbf{M}''}(-c)^{\mathbf{M}'''}}{\mathbf{M}'!\mathbf{M}''!\mathbf{M}'''!}\mathbf{B}(e,p,u,v)C\, y_1^{K_1}\cdots y_s^{K_s}\,Y^L$$

$$\frac{\Gamma(\alpha+\sigma\mathbf{M}'')\Gamma(\beta+\rho\mathbf{M}'')}{\Gamma(\alpha+\beta+(\sigma+\rho)\mathbf{M}'')}\ exp\left\{i\pi\big(\alpha+\sum_{j=1}^{s}K_j\lambda_j+\zeta L+\sigma(\mathbf{M}'+\mathbf{M}'')\big)/2\right\}\frac{2}{\mu^{2(\sum_{j=1}^{s}K_j\omega'_j+\rho L+\mathbf{M}'''+1)}}$$

$$I_{U;p_r+2,q_r+1:W}^{V;0,n_r+2:X}\left(\begin{array}{c|c} \mathrm{z}_1\mu^{-2\rho_1}e^{i\pi\sigma_1/2} & \mathrm{A};\ (1\text{-}\alpha-\sum_{j=1}^{s}K_j\lambda_j-\zeta L-\sigma\mathbf{M}';\sigma_1,\cdots,\sigma_r), \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B}; \\ \mathrm{z}_r\mu^{-2\rho_r}e^{i\pi\sigma_r/2} & \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-}1\text{-}2(\sum_{j=1}^{s}K_j\omega'_j-2\rho L-\mathbf{M}''');2\rho_1,\cdots,2\rho_r),(1-\beta-\sum_{j=1}^{s}K_j\omega_j-\xi L-\rho\mathbf{M}';\delta_1,\cdots,\delta_r),\mathbb{A} \\ \cdot \\ \cdot \\ (1\text{-}\alpha-\beta-\sum_{j=1}^{s}K_j(\lambda_j+\omega_j)-(\zeta+\xi)L-(\sigma+\rho)\mathbf{M}';\sigma_1+\delta_1,\cdots,\sigma_r+\delta_r),\mathbb{B} \end{array}\right) \tag{6.3}$$

under the same conditions that (3.1) and $Re\big(2(\sum_{j=1}^{s}K_j\omega'_j+\rho L+\mathbf{M}'''+\sum_{j=1}^{r}\rho_j s_j)\big)>0$

7. Multivariable H-function.

If $U=V=A=B=0$, the multivariable I-function defined by Prasad [3] reduces to multivariable H-function defined by Srivastava et al [8,9]. We obtain the two following multiple integral relations.

$$\int_0^\infty\int_0^\infty x^{\beta-1}y^{\alpha-1}(x^2+y^2)^{1-\frac{1}{2}(\alpha+\beta)}\,exp\left[i(\alpha+\beta)tan^{-1}(y/x)-\mu(x^2+y^2)^{1/2}\right]f(x^2+y^2)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{y}_1exp\left[i(\lambda_1+\omega_1)tan^{-1}(y/x)\right]y^{\lambda_1}x^{\omega_1}(x^2+y^2)^{\omega'_1-\frac{1}{2}(\omega_1+\lambda_1)} \\ \cdot \\ \cdot \\ \mathrm{y}_s exp\left[i(\lambda_s+\omega_s)tan^{-1}(y/x)\right]y^{\lambda_s}x^{\omega_s}(x^2+y^2)^{\omega'_s-\frac{1}{2}(\omega_s+\lambda_s)} \end{array}\right)$$

$$S_n^{\alpha,\beta,0}[Yexp\left\{i(\zeta+\xi)tan^{-1}(y/x\right\}y^{\zeta}x^{\xi}(x^2+y^2)^{\rho-\frac{1}{2}(\zeta+\xi)};r,s,q,A,B,m,k,l]$$

$$H'\left(\begin{array}{c} \mathrm{a}\ \exp\left[i(\sigma+\rho)tan^{-1}(y/x)\right]y^{\sigma}x^{\rho}(x^2+y^2)^{-\frac{1}{2}(\sigma+\rho)} \\ \cdot \\ \cdot \\ \mathrm{b}\ \exp\left[i(\sigma+\rho)tan^{-1}(y/x)\right]y^{\sigma}x^{\rho}(x^2+y^2)^{-\frac{1}{2}(\sigma+\rho)} \end{array}\right)\ H'\left(c(x^2+y^2),d(x^2+y^2)\right)$$

$$H\left(\begin{array}{c} \mathrm{z}_1 exp\left[i(\sigma_1+\delta_1)tan^{-1}(y/x)\right] y^{\sigma_1}x^{\delta_1}(x^2+y^2)^{\rho_1-\frac{1}{2}(\sigma_1+\delta_1)} \\ . \\ . \\ \mathrm{z}_r exp\left[i(\sigma_r+\delta_r)tan^{-1}(y/x)\right] y^{\sigma_r}x^{\delta_r}(x^2+y^2)^{\rho_r-\frac{1}{2}(\sigma_r+\delta_r)} \end{array}\right)\mathrm{d}x\mathrm{d}y=\frac{1}{2}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{\mathbf{M}',\mathbf{M}'''=0}^{\infty}\sum_{e,p,u,v}$$

$$\phi(\mathbf{M}')\phi(\mathbf{M}''')\frac{(-a)^{\mathbf{M}'}(-c)^{\mathbf{M}'''}}{\mathbf{M}'!\mathbf{M}'''!}\mathbf{B}(e,p,u,v)C\ \ y_1^{K_1}\cdots y_s^{K_s}\,Y^L\ \frac{2}{\mu^{2(\sum_{j=1}^s K_j\omega_j'+\rho L+\mathbf{M}'''+1)}}$$

$$exp\left\{i\pi\big(\alpha+\sum_{j=1}^s K_j\lambda_j+\zeta L+\sigma\mathbf{M}'\big)/2\right\} H^{0,n_r+3:X}_{p_r+3,q_r+1:W}\left(\begin{array}{c} \mathrm{z}_1\mu^{-2\rho_1}e^{i\pi\sigma_1/2} \\ . \\ . \\ . \\ \mathrm{z}_r\mu^{-2\rho_r}e^{i\pi\sigma_r/2} \end{array}\right|$$

$$(1\text{-}\alpha-\textstyle\sum_{j=1}^s K_j\lambda_j-\zeta L-\sigma\mathbf{M}';\sigma_1,\cdots,\sigma_r),(-1-2(\sum_{j=1}^s K_j\omega_j'+\rho L+\mathbf{M}''');2\rho_1,\cdots,2\rho_r),$$
$$\begin{array}{c} . \\ . \\ .\ .\ . \end{array}$$

$$\left.\begin{array}{c} (1\text{-}\beta-\sum_{j=1}^s K_j\omega_j-\xi L-\rho\mathbf{M}';\delta_1,\cdots,\delta_r),\mathbb{A} \\ . \\ . \\ (1\text{-}\alpha-\beta-\sum_{j=1}^s K_j(\lambda_j+\omega_j)-(\zeta+\xi)L-(\sigma+\rho)\mathbf{M}';\sigma_1+\delta_1,\cdots,\sigma_r+\delta_r),\mathbb{B} \end{array}\right) \tag{7.1}$$

under the same conditions that (6.2) with $U=V=A=B=0$.

and

$$\int_0^\infty\int_0^\infty\int_0^\infty (xz)^{\beta-1}y^{\alpha-1}(x^2+y^2)^{-\beta/2}(x^2+y^2+z^2)^{1-(1/2)(\alpha+\beta)}$$

$$exp\big\{i(\alpha+\beta)[tan^{-1}(y/x)+tan^{-1}\big\{(x^2+y^2)^{\frac{1}{2}}/z-\mu(x^2+y^2+z^2)^{1/2}\big\}]\big\}$$

$$S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left(\begin{array}{c} \mathrm{y}_1(x^2+y^2+z^2)^{\omega_1'}exp\left\{i(\lambda_1+\omega_1)tan^{-1}(y/x)\right\}x^{\omega_1}y^{\lambda_1}(x^2+y^2)^{-(\lambda_1+\omega_1)/2} \\ . \\ . \\ \mathrm{y}_s(x^2+y^2+z^2)^{\omega_s'}exp\left\{i(\lambda_s+\omega_s)tan^{-1}\right\}(y/x)x^{\omega_s}y^{\lambda_s}(x^2+y^2)^{-(\lambda_s+\omega_s)/2} \end{array}\right)$$

$$S_n^{\alpha,\beta,0}[Y(x^2+y^2+z^2))^{\rho}exp\big\{i(\zeta+\xi)tan^{-1}(y/x)\big\}x^{\xi}y^{\zeta}(x^2+y^2)^{-(\xi+\zeta)/2};r,s,q,A,B,m,k,l]$$

$$H'\left(\begin{array}{c}\text{a } \exp\big[i(\rho+\sigma)tan^{-1}(y/x)\big]\, x^{\rho}y^{\sigma}(x^2+y^2)^{-(\rho+\sigma)/2}\\ \cdot\\ \cdot\\ \text{b } \exp\big[i(\rho+\sigma)tan^{-1}(y/x)\big]\, x^{\rho}y^{\sigma}(x^2+y^2)^{-(\rho+\sigma)/2}\end{array}\right)H\left(c(x^2+y^2+z^2),d(x^2+y^2+z^2)\right)$$

$$H'\left(\begin{array}{c}\text{a } \exp\Big\{i(\sigma+\rho)tan^{-1}\big\{(x^2+y^2)^{\frac{1}{2}}\Big\}\,(x^2+y^2)^{\frac{\sigma}{2}}(x^2+y^2+z^2)^{-(\sigma+\rho)/2}z^{\rho}\\ \cdot\\ \cdot\\ \text{b } \exp\Big\{i(\sigma+\rho)tan^{-1}\big\{(x^2+y^2)^{\frac{1}{2}}\Big\}\,(x^2+y^2)^{\frac{\sigma}{2}}(x^2+y^2+z^2)^{-(\sigma+\rho)/2}z^{\rho}\end{array}\right)$$

$$H\left(\begin{array}{c}\text{z}_1(x^2+y^2+z^2)^{\rho_1}exp\left\{i(\delta_1+\sigma_1)tan^{-1}(y/x)\right\}x^{\delta_1}y^{\sigma_1}(x^2+y^2)^{-(\delta_1+\sigma_1)/2}\\ \cdot\\ \cdot\\ \text{z}_r(x^2+y^2+z^2)^{\rho_r}exp\left\{i(\delta_r+\sigma_r)tan^{-1}(y/x)\right\}x^{\delta_r}y^{\sigma_r}(x^2+y^2)^{-(\delta_r+\sigma_r)/2}\end{array}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$=\frac{1}{2}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{\mathbf{M}',\mathbf{M}'',\mathbf{M}'''=0}^{\infty}\sum_{e,p,u,v}\phi(\mathbf{M}')\phi(\mathbf{M}'')\phi(\mathbf{M}''')\frac{(-a)^{\mathbf{M}'+\mathbf{M}''}(-c)^{\mathbf{M}'''}}{\mathbf{M}'!\mathbf{M}''!\mathbf{M}'''!}\mathbf{B}(e,p,u,v)C\,y_1^{K_1}\cdots y_s^{K_s}\,Y^L$$

$$exp\left\{i\pi\big(\alpha+\sum_{j=1}^{s}K_j\lambda_j+\zeta l+\sigma(\mathbf{M}'+\mathbf{M}'')\big)/2\right\}\frac{\Gamma(\alpha+\sigma\mathbf{M}'')\Gamma(\beta+\rho\mathbf{M}'')}{\Gamma(\alpha+\beta+(\sigma+\rho)\mathbf{M}'')}\frac{2}{\mu^{2(\sum_{j=1}^{s}K_j\omega_j'+\rho L+\mathbf{M}'''+1)}}$$

$$H^{0,n_r+2:X}_{p_r+2,q_r+1:W}\left(\begin{array}{c}\text{z}_1\mu^{-2\rho_1}e^{i\pi\sigma_1/2}\\ \cdot\\ \cdot\\ \cdot\\ \text{z}_r\mu^{-2\rho_r}e^{i\pi\sigma_r/2}\end{array}\left|\begin{array}{c}(1\text{-}\alpha-\sum_{j=1}^{s}K_j\lambda_j-\zeta L-\sigma\mathbf{M}';\sigma_1,\cdots,\sigma_r),\\ \cdot\\ \cdot\\ \cdot\;\cdot\;\cdot\end{array}\right.\right.$$

$$\left.\begin{array}{c}(\text{-}1\text{-}2(\sum_{j=1}^{s}K_j\omega_j'-2\rho L-\mathbf{M}''');2\rho_1,\cdots,2\rho_r),(1-\beta-\sum_{j=1}^{s}K_j\omega_j-\xi L-\rho\mathbf{M}';\delta_1,\cdots,\delta_r),\mathbb{A}\\ \cdot\\ \cdot\\ (1\text{-}\alpha-\beta-\sum_{j=1}^{s}K_j(\lambda_j+\omega_j)-(\zeta+\xi)L-(\sigma+\rho)\mathbf{M}';\sigma_1+\delta_1,\cdots,\sigma_r+\delta_r),\mathbb{B}\end{array}\right) \tag{7.2}$$

under the same conditions that (6.3) with $U=V=A=B=0$.

Remark

If $S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}[y_1,\cdots,y_s]\rightarrow S^M_N(y)$, we obtain the results of Sharma et al [5].

8. Conclusion

The integral relations (4.1) and (4.2) are quite general in nature on account of the arbitrary nature of the functions f and g and also on account of the presence of the class of multivariable polynomials $S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[.]$, the sequence of polynomials $S_n^{\alpha,\beta,0}[.;r,s,q,A,B,m,k,l]$ and the multivariable I-function defined by Prasad [3]. A very large number of (known and new) integrals can be derived as specials cases.

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