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Some multiple integral relations involving general polynomials and multivariable Prathima's I-function

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ABSTRACT
In this paper, first we obtain a finite integral involving the classes of multivariable polynomials and the multivariable I-function defined by Prathima et al [4]. Next, with the application of this and a lemma to Srivastava et al ([7], 1981),, we obtain two general multiple integral relations involving a sequence of polynomials, a class of multivariable polynomials and the multivariable I-function and two arbitrary functions f and g. By suitable specializing the functions f and g occuring in the main integral relations, a number of multiple integrals are evaluated which are new and quite general nature.

Keywords: Multivariable I-function, multiple integrals, class of multivariable polynomials, sequence of polynomials, multivariable H-function, H-function of two variables.

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1.Introduction and preliminaries.

The generalized polynomials defined by Srivastava [6], is given in the following manner :

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1,\cdots,y_s] = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}$$

$$A[N_1,K_1;\cdots;N_s,K_s]y_1^{K_1}\cdots y_s^{K_s} \tag{1.1}$$

Where $M_1,\cdots,M_s$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_s,K_s]$ are arbitrary constants, real or complex.

The multivariable I-function defined by Prathima et al [4] is an extension of the multivariable H-function defined by Srivastava et al [8,9]. It is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1,\cdots,z_r) = I_{\mathbf{p},\mathbf{q}:p_1,q_1;\cdots;p_r,q_r}^{0,\mathbf{n}:m_1,n_1;\cdots;m_r,n_r}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_{1,p}: \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_{1,q}: \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,p_r} \\ (\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,q_r} \end{array}\right) \tag{1.2}$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.3}$$

where $\phi(s_1,\cdots,s_r),\theta_i(s_i), i=1,\cdots,r$ are given by :

$$\phi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{\mathbf{n}}\Gamma^{A_j}\left(1-aj+\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)}{\prod_{j=n+1}^{\mathbf{p}}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)\prod_{j=1}^{\mathbf{q}}\Gamma^{B_j}\left(1-bj+\sum_{i=1}^{r}\beta_j^{(i)}s_j\right)} \tag{1.4}$$

$$\theta_i(s_i)=\frac{\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}\left(1-c_j^{(i)}+\gamma_j^{(i)}s_i\right)\prod_{j=1}^{m_i}\Gamma^{D_j^{(i)}}\left(d_j^{(i)}-\delta_j^{(i)}s_i\right)}{\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}s_i\right)\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}+\delta_j^{(i)}s_i\right)} \tag{1.5}$$

For more details, see Prathima et al [4].
Following the result of Braaksma the I-function of r variables is analytic if :

$$U_i=\sum_{j=1}^{\mathbf{p}}A_j\alpha_j^{(i)}-\sum_{j=1}^{\mathbf{q}}B_j\beta_j^{(i)}+\sum_{j=1}^{p_i}C_j^{(i)}\gamma_j^{(i)}-\sum_{j=1}^{q_i}D_j^{(i)}\delta_j^{(i)}\leqslant 0, i=1,\cdots,r \tag{1.6}$$

The integral (2.1) converges absolutely if

$|arg(z_k)|<\frac{1}{2}\Delta_k\pi, k=1,\cdots,r$ where

$$\Delta_k=-\sum_{j=\mathbf{n}+1}^{\mathbf{p}}A_j\alpha_j^{(k)}-\sum_{j=1}^{\mathbf{q}}B_j\beta_j^{(k)}+\sum_{j=1}^{m_k}D_j^{(k)}\delta_j^{(k)}-\sum_{j=m_k+1}^{q_k}D_j^{(k)}\delta_j^{(k)}+\sum_{j=1}^{n_k}C_j^{(k)}\gamma_j^{(k)}-\sum_{j=n_k+1}^{p_k}C_j^{(k)}\gamma_j^{(k)}>0 \tag{1.7}$$

We shall note :

$$X=m_1,n_1;\cdots;m_r,n_r\ \ ;Y=p_1,q_1;\cdots;p_r,q_r \tag{1.8}$$

$$A=(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_{1,p}\ :\ (\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,p_r} \tag{1.9}$$

$$B=(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_{1,q}\ :\ (\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,q_r} \tag{1.10}$$

$$C_s=\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}A[N_1,K_1;\cdots;N_s,K_s] \tag{1.11}$$

$$C'_u=\frac{(-N'_1)_{M'_1K'_1}}{K'_1!}\cdots\frac{(-N'_u)_{M'_uK'_u}}{K'_u!}A[N'_1,K'_1;\cdots;N'_u,K'_u] \tag{1.12}$$

2. Required results;

The following results will be required in establishing our main integral relations :

Lemma 1 (Srivastava et al [7] ,1981) :

Let the functions $f(x)$ and $g(x)$ be integrable over the semi interval $(0,\infty)$ and define.

$$F(R)=\int_0^{\frac{\pi}{2}}h(R,\theta)\,\mathrm{d}\theta \tag{2.1}$$

where $h(R,\theta)$ is an integrable function of two variables in the rectangular region $0\leqslant R\leqslant\infty, 0\leqslant\theta\leqslant\frac{\pi}{2}$, then

$$\int_0^\infty \int_0^\infty f(x^2+y^2)h\left\{(x^2+y^2)^{1/2}tan^{-1}(y/x)\right\}\mathrm{d}x\mathrm{d}y = \frac{1}{2}\int_0^\infty f(t)F(\sqrt{t})\mathrm{d}t \tag{2.2}$$

and

$$\int_0^\infty \int_0^\infty \int_0^\infty (x^2+y^2)^{1/2}f(x^2+y^2+z^2)g\left[tan^{-1}\left\{(x^2+y^2)^{1/2}/z\right\}\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \int_0^\infty \int_0^\infty f(u^2+v^2)F\left\{(x^2+y^2)^{1/2}\right\}g\left\{tan^{-1}(v/u)\right\}\mathrm{d}u\mathrm{d}v \tag{2.3}$$

provided that the various integrals involved are absolutely convergent.

Lemma 2 (Kalla et al [3] ,1981) :

$$H'(at,bt) = H^{0,n'_1;1,n'_2;1,n'_3}_{p'_1,q'_1;p'_2,q'_2+1;p'_3,q'_3+1}\left(\begin{array}{c|c} \text{at} & (\text{a'}_j;A'_j,A''_j)_{1,p'_1} : (e_j,E_j)_{1,p'_2};(g_j,G_j)_{1,p'_3} \\ \cdot & \cdot \\ \cdot & \\ \text{bt} & (\text{b'}_j;B'_j,B''_j)_{1,q'_1} : (0,1),(f_j,F_j)_{1,q'_2};(0,1),(h_j,H_j)_{1,q'_3} \end{array}\right)$$

$$= \sum_{\mathbf{M}'=0}^{\infty}\phi(\mathbf{M}')\frac{(-at)^{\mathbf{M}'}}{\mathbf{M}'!} \tag{2.4}$$

where

$$\phi(\mathbf{M}') = \sum_{\mathbf{N'=0}}^{\mathbf{M}'}\phi'(\mathbf{M}'-\mathbf{N}',\mathbf{N}')\theta'_1(\mathbf{M}')\theta'_2(\mathbf{N}')(b/a)^{\mathbf{N}'}\binom{\mathbf{M}'}{\mathbf{N}'} \tag{2.5}$$

3. A useful integral

We obtain the following integral, which will be required in the next section :

$$\int_0^{\frac{\pi}{2}} \frac{(\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}}{(\cos^2\theta+c\sin^2\theta)^{\alpha+\beta}} S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left(\begin{array}{c} \text{y}_1R^{2\omega_1}\frac{(\sin\theta)^{2a_1}(\cos\theta)^{2b_1}}{(\cos^2\theta+c\sin^2\theta)^{a_1+b_1}} \\ \cdot \\ \cdot \\ \text{y}_sR^{2\omega_s}\frac{(\sin\theta)^{2a_s}(\cos\theta)^{2b_s}}{(\cos^2\theta+c\sin^2\theta)^{a_s+b_s}} \end{array}\right)$$

$$S^{M'_1,\cdots,M'_u}_{N'_1,\cdots,N'_u}\left(\begin{array}{c} \text{z}_1R^{2\omega'_1}\frac{(\sin\theta)^{2a'_1}(\cos\theta)^{2b'_1}}{(\cos^2\theta+c\sin^2\theta)^{a'_1+b'_1}} \\ \cdot \\ \cdot \\ \text{z}_uR^{2\omega'_u}\frac{(\sin\theta)^{2a'_u}(\cos\theta)^{2b'_u}}{(\cos^2\theta+c\sin^2\theta)^{a'_u+b'_u}} \end{array}\right) I\left(\begin{array}{c} \text{Z}_1R^{2\lambda_1}\frac{(\sin\theta)^{2h_1c_1}(\cos\theta)^{2h_1d_1}}{(\cos^2\theta+c\sin^2\theta)^{h_1(c_1+d_1)}} \\ \cdot \\ \cdot \\ \text{Z}_rR^{2\lambda_r}\frac{(\sin\theta)^{2c_rh_r}(\cos\theta)^{2h_rd_r}}{(\cos^2\theta+c\sin^2\theta)^{h_r(c_r+d_r)}} \end{array}\right) e^{-\mu R}\mathrm{d}\theta =$$

$$\frac{1}{2}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{K'_1=0}^{[N'_1/M'_1]}\cdots\sum_{K'_u=0}^{[N'_u/M'_u]} C_sC'_u\, y_1^{K_1}\cdots y_s^{K_s}\, z_1^{K'_1}\cdots z_u^{K'_u}R^{2(\sum_{j=0}^{s}K_j\omega_j+\sum_{j=0}^{u}K'_j\omega'_j)}\, e^{-\mu R}$$

$$c^{-(\alpha+\sum_{j=0}^{s}K_ja_j+\sum_{j=0}^{u}K'_ja'_j)}I_{\mathbf{p}+2,\mathbf{q}+1;Y}^{0,\mathbf{n}+2;X}\left(\begin{array}{c|c} \mathrm{z}_1R^{2\lambda_1}c^{-h_1c_1} & (1\text{-}\alpha-\sum_{j=1}^{s}K_ja_j-\sum_{j=1}^{u}K'_ja'_j;c_1h_1,\cdots,c_rh_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdots \\ \mathrm{z}_rR^{2\lambda_r}c^{-h_rc_r} & \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-}\beta-\sum_{j=1}^{s}K_jb_j-\sum_{j=1}^{u}K'_jb'_j;d_1h_1,\cdots,d_rh_r;1),A \\ \cdot \\ \cdot \\ (1\text{-}\alpha-\beta-\sum_{j=1}^{s}K_j(a_j+b_j)-\sum_{j=1}^{u}K'_j(a'_j+b'_j);(c_1+d_1)h_1,\cdots,(c_r+d_r)h_r;1),B \end{array}\right) \tag{3.1}$$

Provided that

$\min\{Re(\alpha),Re(\beta),a_i,b_i,\omega_i,a'_j,b'_j,\omega'_j,c_k,d_k,c,h_k,\lambda_k\}>0$ for $i=1,\cdots,s;j=1,\cdots,u;k=1,\cdots,r$

$$Re\left[\alpha+\sum_{j=1}^{s}K_ia_j+\sum_{j=1}^{u}K'_ia'_j+\sum_{i=1}^{r}c_ih_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right]>0 \text{ and}$$

$$Re\left[\beta+\sum_{j=1}^{s}K_ib_j+\sum_{j=1}^{u}K'_ib'_j+\sum_{i=1}^{r}d_ih_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right]>0$$

$|arg(Z_k)|<\dfrac{1}{2}\Delta_k\pi,k=1,\cdots,r$ where Δ_k is defined by (1.7).

Proof

To prove (3.1), we first express the classes of multivariable polynomials $S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[.]$ and $S_{N'_1,\cdots,N'_u}^{M'_1,\cdots,M'_u}[.]$ in series with the help of (1.1) and interchange the orders of summations and integrations (which is permissible under the conditions stated). Now, we express the multivariable I-function defined by Prathima et al [3] in Mellin-Barnes contour integral given by (1.3). Interchange the order of $\theta-$integral and $(s_1,\cdots,s_r)-$integrals. Now collect the power of $\cos\theta,\sin\theta$ and $(\cos^2\theta+k\sin^2\theta)$ and evaluate the $\theta-$integral with the help of result ([2], 1980, page 376, eq.(3.642)).

$$\int_0^{\frac{\pi}{2}}\frac{(\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}}{(\cos^2\theta+k\sin^2\theta)^{\alpha+\beta}}\mathrm{d}\theta=\frac{\Gamma(\alpha)\Gamma(\beta)}{2k^\alpha\Gamma(\alpha+\beta)} \tag{3.2}$$

where $k,Re(\alpha),\Re(\beta)>0$

Finally, interpret the Mellin-Barnes contour integral to multivariable I-function with the help of (1.4), we obtain the desired result.

4. The main integral

The following double and triple integral relations will be established in this section :

$$\int_0^\infty\int_0^\infty\frac{x^{2\beta-1}y^{2\alpha-1}(x^2+y^2)}{(x^2+cy^2)^{\alpha+\beta}}S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix}\mathrm{y}_1(x^2+y^2)^{\omega_1}\frac{y^{2a_1}x^{2b_1}}{(x^2+cy^2)^{a_1+b_1}} \\ \cdot \\ \cdot \\ \mathrm{y}_s(x^2+y^2)^{\omega_s}\frac{y^{2a_s}x^{2b_s}}{(x^2+cy^2)^{a_s+b_s}}\end{pmatrix}$$

$$S_{N_1',\cdots,N_u'}^{M_1',\cdots,M_u'}\begin{pmatrix} \mathrm{z}_1(x^2+y^2)^{\omega_1'}\frac{y^{2a_1'}x^{2b_1'}}{(x^2+cy^2)^{a_1'+b_1'}} \\ . \\ . \\ \mathrm{z}_u(x^2+y^2)^{\omega_u'}\frac{y^{2a_u'}x^{2b_u'}}{(x^2+cy^2)^{a_u'+b_u'}} \end{pmatrix} I\begin{pmatrix} \mathrm{Z}_1(x^2+y^2)^{\lambda_1}\frac{y^{2h_1c_1}x^{2h_1d_1}}{(x^2+cy^2)^{h_1(c_1+d_1)}} \\ . \\ . \\ \mathrm{Z}_r(x^2+y^2)^{\lambda_r}\frac{y^{2h_rc_r}x^{2h_rd_r}}{(x^2+cy^2)^{h_r(c_r+d_r)}} \end{pmatrix} e^{-\mu(x^2+y^2)^{1/2}}f(x^2+y^2)$$

$$\mathrm{d}x\mathrm{d}y=\frac{1}{4c^{\alpha}}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{K_1'=0}^{[N_1'/M_1']}\cdots\sum_{K_u'=0}^{[N_u'/M_u']} C_sC_u'\,y_1^{K_1}\cdots y_s^{K_s}\,z_1^{K_1'}\cdots z_u^{K_u'}\int_0^{\infty}t^{\sum_{j=1}^{s}K_j\omega_j+\sum_{j=1}^{u}K_j'\omega_j'}e^{-\mu\upsilon t}$$

$$c^{-(\sum_{i=1}^{s}K_ia_i+\sum_{i=1}^{u}K_i'a_i')}\ I_{\mathbf{p}+2,\mathbf{q}+1;Y}^{0,\mathbf{n}+2;X}\left(\begin{matrix} \mathrm{z}_1t^{\lambda_1}c^{-h_1c_1} \\ . \\ . \\ . \\ \mathrm{z}_rt^{\lambda_r}c^{-h_rc_r} \end{matrix}\left|\begin{matrix} (1\text{-}\alpha-\sum_{j=1}^{s}K_ja_j-\sum_{j=1}^{u}K_j'a_j';c_1h_1,\cdots,c_rh_r;1), \\ . \\ . \\ \cdots \end{matrix}\right.\right.$$

$$\left.\begin{matrix} (1\text{-}\beta-\sum_{j=1}^{s}K_jb_j-\sum_{j=1}^{u}K_j'b_j';d_1h_1,\cdots,d_rh_r;1),A \\ . \\ . \\ (1\text{-}\alpha-\beta-\sum_{j=1}^{s}K_j(a_j+b_j)-\sum_{j=1}^{u}K_j'(a_j'+b_j');(c_1+d_1)h_1,\cdots,(c_r+d_r)h_r;1),B \end{matrix}\right) f(t)\,\mathrm{d}t \tag{4.1}$$

under the same conditions that (3.1) and

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{x^{2\beta-1}y^{2\alpha-1}(x^2+y^2)^{1/2}}{(x^2+cy^2)^{\alpha+\beta}}\quad S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix} \mathrm{y}_1(x^2+y^2+z^2)^{\omega_1}\frac{y^{2a_1}x^{2b_1}}{(x^2+cy^2)^{a_1+b_1}} \\ . \\ . \\ \mathrm{y}_s(x^2+y^2+z^2)^{\omega_s}\frac{y^{2a_s}x^{2b_s}}{(x^2+cy^2)^{a_s+b_s}} \end{pmatrix}$$

$$S_{N_1',\cdots,N_u'}^{M_1',\cdots,M_u'}\begin{pmatrix} \mathrm{z}_1(x^2+y^2+z^2)^{\omega_1'}\frac{y^{2a_1'}x^{2b_1'}}{(x^2+cy^2)^{a_1'+b_1'}} \\ . \\ . \\ \mathrm{z}_u(x^2+y^2+z^2)^{\omega_u'}\frac{y^{2a_u'}x^{2b_u'}}{(x^2+cy^2)^{a_u'+b_u'}} \end{pmatrix} I\begin{pmatrix} \mathrm{Z}_1(x^2+y^2+z^2)^{\lambda_1}\frac{y^{2h_1c_1}x^{2h_1d_1}}{(x^2+cy^2)^{h_1(c_1+d_1)}} \\ . \\ . \\ \mathrm{Z}_r(x^2+y^2+z^2)^{\lambda_r}\frac{y^{2h_rc_r}x^{2h_rd_r}}{(x^2+cy^2)^{h_r(c_r+d_r)}} \end{pmatrix}$$

$$g\big[tan^{-1}\{(x^2+y^2)^{1/2}/z\}\big]e^{-\mu(x^2+y^2+z^2)^{1/2}}\ f(x^2+y^2+z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z=\frac{1}{2c^{\alpha}}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{K_1'=0}^{[N_1'/M_1']}\cdots\sum_{K_u'=0}^{[N_u'/M_u']}$$

$$c^{-(\sum_{i=1}^{s}K_ia_i+\sum_{i=1}^{u}K_i'a_i')}C_sC_u'\,y_1^{K_1}\cdots y_s^{K_s}\,z_1^{K_1'}\cdots z_u^{K_u'}\int_0^{\infty}\int_0^{\infty}f(u^2+v^2)g\left\{tan^{-1}(v/u)\right\}(u^2+v^2)^{\sum_{j=1}^{s}K_j\omega_j+\sum_{j=1}^{u}K_j'\omega_j'}$$

$$e^{-\mu(u^2+v^2)^{1/2}}\, I^{0,\mathbf{n}+2;X}_{\mathbf{p}+2,\mathbf{q}+1;Y}\left(\begin{array}{c} \mathrm{Z}_1(u^2+v^2)^{\lambda_1}c^{-h_1c_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Z}_r(u^2+v^2)^{\lambda_r}c^{-h_rc_r} \end{array}\middle| \begin{array}{c} (1\text{-}\alpha-\sum_{j=1}^s K_ja_j-\sum_{j=1}^u K'_ja'_j;c_1h_1,\cdots,c_rh_r;1), \\ \cdot \\ \cdot \\ \cdots \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-}\beta-\sum_{j=1}^s K_jb_j-\sum_{j=1}^u K'_jb'_j;d_1h_1,\cdots,d_rh_r;1),A \\ \cdot \\ \cdot \\ (1\text{-}\alpha-\beta-\sum_{j=1}^s K_j(a_j+b_j)-\sum_{j=1}^u K'_j(a'_j+b'_j);(c_1+d_1)h_1,\cdots,(c_r+d_r)h_r;1),B \end{array}\right)\mathrm{d}u\mathrm{d}v \tag{4.2}$$

under the same conditions that (3.1).

Proof of (4.1) and (4.2)

To establish the integral relations (4.1) and (4.2) ,we take in (2.1)

$$h(R,\theta)=\frac{(\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}}{(\cos^2\theta+c\sin^2\theta)^{\alpha+\beta}}\, S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left(\begin{array}{c} \mathrm{y}_1R^{2\omega_1}\frac{(\sin\theta)^{2a_1}(\cos\theta)^{2b_1}}{(\cos^2\theta+c\sin^2\theta)^{a_1+b_1}} \\ \cdot \\ \cdot \\ \mathrm{y}_sR^{2\omega_s}\frac{(\sin\theta)^{2a_s}(\cos\theta)^{2b_s}}{(\cos^2\theta+c\sin^2\theta)^{a_s+b_s}} \end{array}\right)$$

$$S^{M'_1,\cdots,M'_u}_{N'_1,\cdots,N'_u}\left(\begin{array}{c} \mathrm{z}_1R^{2\omega'_1}\frac{(\sin\theta)^{2a'_1}(\cos\theta)^{2b'_1}}{(\cos^2\theta+c\sin^2\theta)^{a'_1+b'_1}} \\ \cdot \\ \cdot \\ \mathrm{z}_uR^{2\omega'_u}\frac{(\sin\theta)^{2a'_u}(\cos\theta)^{2b'_u}}{(\cos^2\theta+c\sin^2\theta)^{a'_u+b'_u}} \end{array}\right) I\left(\begin{array}{c} \mathrm{Z}_1R^{2\lambda_1}\frac{(\sin\theta)^{2h_1c_1}(\cos\theta)^{2h_1d_1}}{(\cos^2\theta+c\sin^2\theta)^{h_1(c_1+d_1)}} \\ \cdot \\ \cdot \\ \mathrm{Z}_rR^{2\lambda_r}\frac{(\sin\theta)^{2c_rh_r}(\cos\theta)^{2h_rd_r}}{(\cos^2\theta+c\sin^2\theta)^{h_r(c_r+d_r)}} \end{array}\right)e^{-\mu R} \tag{4.3}$$

Now, we evaluate the resulting integral by means of (3.1) and arrive at the following result :

$$F(R)=\ \frac{1}{2}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{K'_1=0}^{[N'_1/M'_1]}\cdots\sum_{K'_u=0}^{[N'_u/M'_u]} C_sC'_u\, y_1^{K_1}\cdots y_s^{K_s}\, z_1^{K'_1}\cdots z_u^{K'_u}R^{2(\sum_{j=0}^s K_j\omega_j+\sum_{j=0}^u K'_j\omega'_j)}$$

$$c^{-(\alpha+\sum_{j=0}^s K_ja_j+\sum_{j=0}^u K'_ja'_j)}I^{0,\mathbf{n}+2;X}_{\mathbf{p}+2,\mathbf{q}+1;Y}\left(\begin{array}{c} \mathrm{z}_1R^{2\lambda_1}c^{-h_1c_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_rR^{2\lambda_r}c^{-h_rc_r} \end{array}\middle| \begin{array}{c} (1\text{-}\alpha-\sum_{j=1}^s K_ja_j-\sum_{j=1}^u K'_ja'_j;c_1h_1,\cdots,c_rh_r;1), \\ \cdot \\ \cdot \\ \cdots \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-}\beta-\sum_{j=1}^s K_jb_j-\sum_{j=1}^u K'_jb'_j;d_1h_1,\cdots,d_rh_r;1),A \\ \cdot \\ \cdot \\ (1\text{-}\alpha-\beta-\sum_{j=1}^s K_j(a_j+b_j)-\sum_{j=1}^u K'_j(a'_j+b'_j);(c_1+d_1)h_1,\cdots,(c_r+d_r)h_r;1),B \end{array}\right)e^{-\mu R} \tag{4.4}$$

Now substituting the value of $h(R,\theta)$ and $F(R)$ as given by (4.3) and (4.4) respectively in (2.2) and (2.3) in succession, we obtain the integral relations (4.1) and (4.2) after algebric manipulations and simplification.

5. Special case

If we set $a_i = b_i = a'_j = b'_j = 0$ for $i = 1, \cdots, s; j = 1, \cdots, u;$ and $max\{c_1, d_1 \cdots, c_r, d_r\} \to 0$, we get

$$\int_0^\infty \int_0^\infty \frac{x^{2\beta-1}y^{2\alpha-1}(x^2+y^2)}{(x^2+cy^2)^{\alpha+\beta}} \quad S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix} \mathrm{y}_1(x^2+y^2)^{\omega_1} \\ . \\ . \\ \mathrm{y}_s(x^2+y^2)^{\omega_s} \end{pmatrix} S_{N'_1,\cdots,N'_u}^{M'_1,\cdots,M'_u}\begin{pmatrix} \mathrm{z}_1(x^2+y^2)^{\omega'_1} \\ . \\ . \\ \mathrm{z}_u(x^2+y^2)^{\omega'_u} \end{pmatrix}$$

$$I\begin{pmatrix} \mathrm{z}_1(x^2+y^2)^{\lambda_1}c^{-h_1c_1} \\ . \\ . \\ \mathrm{z}_r(x^2+y^2)^{\lambda_r}c^{-h_rc_r} \end{pmatrix} f(x^2+y^2)\,\mathrm{d}x\mathrm{d}y = \frac{1}{4c^\alpha}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{K'_1=0}^{[N'_1/M'_1]}\cdots\sum_{K'_u=0}^{[N'_u/M'_u]} C_sC'_u\, y_1^{K_1}\cdots y_s^{K_s}$$

$$z_1^{K'_1}\cdots z_u^{K'_u}\,\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\int_0^\infty t^{\sum_{j=1}^s K_j\omega_j+\sum_{j=1}^u K'_j\omega'_j} e^{-\mu vt}\; I_{\mathbf{p},\mathbf{q};Y}^{0,\mathbf{n};X}\left(\begin{array}{c|c} \mathrm{Z}_1t^{\lambda_1}c^{-h_1c_1} & \mathrm{A} \\ . & . \\ . & . \\ . & . \\ \mathrm{Z}_rt^{\lambda_r}c^{-h_rc_r} & \mathrm{B} \end{array}\right) f(t)\,\mathrm{d}t \tag{5.1}$$

6. Applications

By suitably choosing the functions f and g in the main integral relations, a large number of intersting double and triple integrals can be evaluated. We shall, however obtain her only one double and one triple integral by way illustration.

Thus if in (4.2) we set

$$g(t) = \frac{(\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}}{(\cos^2\theta + c\sin^2\theta)^{\alpha+\beta}} \tag{6.1}$$

we arrive at the following integral relation on making use of (5.1)

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{(xz)^{2\beta-1}y^{2\alpha-1}(x^2+y^2)^\alpha(x^2+y^2+z^2)}{(x^2+cy^2)^{\alpha+\beta}\left[z^2+c(x^2+y^2)\right]^{\alpha+\beta}} S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix} \mathrm{y}_1(x^2+y^2+z^2)^{\omega_1}\frac{y^{2a_1}x^{2b_1}}{(x^2+cy^2)^{a_1+b_1}} \\ . \\ . \\ \mathrm{y}_s(x^2+y^2+z^2)^{\omega_s}\frac{y^{2a_s}x^{2b_s}}{(x^2+cy^2)^{a_s+b_s}} \end{pmatrix}$$

$$S_{N'_1,\cdots,N'_u}^{M'_1,\cdots,M'_u}\begin{pmatrix} \mathrm{z}_1(x^2+y^2+z^2)^{\omega'_1}\frac{y^{2a'_1}x^{2b'_1}}{(x^2+cy^2)^{a'_1+b'_1}} \\ . \\ . \\ \mathrm{z}_u(x^2+y^2+z^2)^{\omega'_u}\frac{y^{2a'_u}x^{2b'_u}}{(x^2+cy^2)^{a'_u+b'_u}} \end{pmatrix} I\begin{pmatrix} \mathrm{Z}_1(x^2+y^2+z^2)^{\lambda_1}\frac{y^{2h_1c_1}x^{2h_1d_1}}{(x^2+cy^2)^{h_1(c_1+d_1)}} \\ . \\ . \\ \mathrm{Z}_r(x^2+y^2+z^2)^{\lambda_r}\frac{y^{2h_rc_r}x^{2h_rd_r}}{(x^2+cy^2)^{h_r(c_r+d_r)}} \end{pmatrix} e^{-\mu(x^2+y^2+z^2)^{1/2}}$$

$$f(x^2+y^2+z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{2c^{2\alpha}}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{K'_1=0}^{[N'_1/M'_1]}\cdots\sum_{K'_u=0}^{[N'_u/M'_u]} C'_sC'_u\, y_1^{K_1}\cdots y_s^{K_s}z_1^{K'_1}\cdots z_u^{K'_u}c^{-(\sum_{i=1}^s K_ia_i+\sum_{i=1}^u K'_ia'_i)}$$

$$\int_0^\infty t^{\sum_{j=1}^s K_j\omega_j+\sum_{j=1}^u K'_j\omega'_j} e^{-\mu\sqrt{t}}\, I^{0,\mathbf{n}+2;X}_{\mathbf{p}+2,\mathbf{q}+1;Y}\left(\begin{array}{c} \mathrm{z}_1 t^{\lambda_1} c^{-h_1c_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r t^{\lambda_r} c^{-h_rc_r}\end{array}\right| \begin{array}{c}(1\text{-}\alpha-\sum_{j=1}^s K_ja_j-\sum_{j=1}^u K'_ja'_j; c_1h_1,\cdots,c_rh_r;1),\\ \cdot\\ \cdot\\ \cdots\end{array}$$

$$\left.\begin{array}{c}(1\text{-}\beta-\sum_{j=1}^s K_jb_j-\sum_{j=1}^u K'_jb'_j; d_1h_1,\cdots,d_rh_r;1), A\\ \cdot\\ \cdot\\ (1\text{-}\alpha-\beta-\sum_{j=1}^s K_j(a_j+b_j)-\sum_{j=1}^u K'_j(a'_j+b'_j);(c_1+d_1)h_1,\cdots,(c_r+d_r)h_r;1),B\end{array}\right)\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}f(t)\,\mathrm{d}t \tag{6.1}$$

under the same conditions that (3.1). Next, we take

$$f(t)=H'(dt,ft) \tag{6.2}$$

and evaluate the $t-$integral occuring on the right hand sides of (4.1) and (6.2) with the help of the Gamma-function definition and arrive at the following multiple integrals after algebric manipulations and simplififcation :

$$\int_0^\infty\int_0^\infty \frac{x^{2\beta-1}y^{2\alpha-1}(x^2+y^2)}{(x^2+cy^2)^{\alpha+\beta}} S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left(\begin{array}{c}\mathrm{y}_1(x^2+y^2)^{\omega_1}\frac{y^{2a_1}x^{2b_1}}{(x^2+cy^2)^{a_1+b_1}}\\ \cdot\\ \cdot\\ \mathrm{y}_s(x^2+y^2)^{\omega_s}\frac{y^{2a_s}x^{2b_s}}{(x^2+cy^2)^{a_s+b_s}}\end{array}\right) H\big[d(x^2+y^2),f(x^2+y^2)\big]$$

$$S^{M'_1,\cdots,M'_u}_{N'_1,\cdots,N'_u}\left(\begin{array}{c}\mathrm{z}_1(x^2+y^2)^{\omega'_1}\frac{y^{2a'_1}x^{2b'_1}}{(x^2+cy^2)^{a'_1+b'_1}}\\ \cdot\\ \cdot\\ \mathrm{z}_u(x^2+y^2)^{\omega'_u}\frac{y^{2a'_u}x^{2b'_u}}{(x^2+cy^2)^{a'_u+b'_u}}\end{array}\right) I\left(\begin{array}{c}\mathrm{Z}_1(x^2+y^2)^{\lambda_1}\frac{y^{2h_1c_1}x^{2h_1d_1}}{(x^2+cy^2)^{h_1(c_1+d_1)}}\\ \cdot\\ \cdot\\ \mathrm{Z}_r(x^2+y^2)^{\lambda_r}\frac{y^{2h_rc_r}x^{2h_rd_r}}{(x^2+cy^2)^{h_r(c_r+d_r)}}\end{array}\right) e^{-\mu(x^2+y^2)^{1/2}}\mathrm{d}x\mathrm{d}y=$$

$$\frac{1}{2c^{2\alpha}}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{K'_1=0}^{[N'_1/M'_1]}\cdots\sum_{K'_u=0}^{[N'_u/M'_u]}\sum_{\mathbf{M}'=0}^{\infty} C_sC'_u\,\phi(\mathbf{M}')\frac{(-d)^{\mathbf{M}'}}{\mathbf{M}'!}\,y_1^{K_1}\cdots y_s^{K_s}\,z_1^{K'_1}\cdots z_u^{K'_u}c^{-(\sum_{i=1}^s K_ia_i+\sum_{i=1}^u K'_ia'_i)}$$

$$\mu^{-2(\sum_{j=1}^s K_j\omega_j+\sum_{j=1}^u K'_j\omega'_j+\mathbf{M}'+1)} I^{0,\mathbf{n}+3;X}_{\mathbf{p}+3,\mathbf{q}+1;Y}\left(\begin{array}{c} \mathrm{z}_1 t^{\lambda_1} c^{-h_1c_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r t^{\lambda_r} c^{-h_rc_r}\end{array}\right| \begin{array}{c}(1\text{-}\alpha-\sum_{j=1}^s K_ja_j-\sum_{j=1}^u K'_ja'_j; c_1h_1,\cdots,c_rh_r;1),\\ \cdot\\ \cdot\\ \cdots\end{array}$$

$$\left.\begin{array}{c}(\text{-}1\text{-}2(\sum_{i=1}^s K_i\omega_i+\sum_{i=1}^u K'_i\omega'+\mathbf{M}');2\lambda_1,\cdots,2\lambda_r;1),(1-\beta-\sum_{j=1}^s K_jb_j-\sum_{j=1}^u K'_jb'_j;d_1h_1,\cdots,d_rh_r;1),A\\ \cdot\\ \cdot\\ (1\text{-}\alpha-\beta-\sum_{j=1}^s K_j(a_j+b_j)-\sum_{j=1}^u K'_j(a'_j+b'_j);(c_1+d_1)h_1,\cdots,(c_r+d_r)h_r;1),B\end{array}\right) \tag{6.3}$$

under the same conditions that (3.1) and $Re(2\sum_{j=1}^{s} K_j\omega_j + 2\sum_{j=1}^{u} K'_j\omega'_j + 2\mathbf{M}' + 2\sum_{j=1}^{r}\rho_j s_j) > 0$ and

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{(xz)^{2\beta-1}y^{2\alpha-1}(x^2+y^2)^\alpha(x^2+y^2+z^2)}{(x^2+cy^2)^{\alpha+\beta}\big[z^2+c(x^2+y^2)\big]^{\alpha+\beta}} S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix} \mathrm{y}_1(x^2+y^2+z^2)^{\omega_1}\frac{y^{2a_1}x^{2b_1}}{(x^2+cy^2)^{a_1+b_1}} \\ \cdot \\ \cdot \\ \mathrm{y}_s(x^2+y^2+z^2)^{\omega_s}\frac{y^{2a_s}x^{2b_s}}{(x^2+cy^2)^{a_s+b_s}}\end{pmatrix}$$

$$S_{N'_1,\cdots,N'_u}^{M'_1,\cdots,M'_u}\begin{pmatrix} \mathrm{z}_1(x^2+y^2+z^2)^{\omega'_1}\frac{y^{2a'_1}x^{2b'_1}}{(x^2+cy^2)^{a'_1+b'_1}} \\ \cdot \\ \cdot \\ \mathrm{z}_u(x^2+y^2+z^2)^{\omega'_u}\frac{y^{2a'_u}x^{2b'_u}}{(x^2+cy^2)^{a'_u+b'_u}}\end{pmatrix} I\begin{pmatrix} \mathrm{Z}_1(x^2+y^2+z^2)^{\lambda_1}\frac{y^{2h_1c_1}x^{2h_1d_1}}{(x^2+cy^2)^{h_1(c_1+d_1)}} \\ \cdot \\ \cdot \\ \mathrm{Z}_r(x^2+y^2+z^2)^{\lambda_r}\frac{y^{2h_rc_r}x^{2h_rd_r}}{(x^2+cy^2)^{h_r(c_r+d_r)}}\end{pmatrix} e^{-\mu(x^2+y^2+z^2)^{1/2}}$$

$$H\big[d(x^2+y^2+z^2), f(x^2+y^2+z^2)\big]\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{4c^{2\alpha}}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{K'_1=0}^{[N'_1/M'_1]}\cdots\sum_{K'_u=0}^{[N'_u/M'_u]}\sum_{\mathbf{M}'=0}^{\infty} C_s C'_u$$

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} c^{-(\sum_{i=1}^{s}K_ia_i+\sum_{i=1}^{u}K'_ia'_i)}\ \phi(\mathbf{M}')\frac{(-d)^{\mathbf{M}'}}{\mathbf{M}'!}y_1^{K_1}\cdots y_s^{K_s}\, z_1^{K'_1}\cdots z_u^{K'_u}$$

$$\mu^{-2(\sum_{j=1}^{s}K_j\omega_j+\sum_{j=1}^{u}K'_j\omega'_j+\mathbf{M}'+1)} I_{\mathbf{p}+3,\mathbf{q}+1;Y}^{0,\mathbf{n}+3;X}\left(\begin{array}{c|c} \mathrm{z}_1 t^{\lambda_1}c^{-h_1c_1} & (1\text{-}\alpha-\sum_{j=1}^{s}K_ja_j-\sum_{j=1}^{u}K'_ja'_j; c_1h_1,\cdots,c_rh_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdots \\ \mathrm{z}_r t^{\lambda_r}c^{-h_rc_r} & \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-}1\text{-}2(\sum_{i=1}^{s}K_i\omega_i+\sum_{i=1}^{u}K'_i\omega'+\mathbf{M}');2\lambda_1,\cdots,2\lambda_r;1),(1-\beta-\sum_{j=1}^{s}K_jb_j-\sum_{j=1}^{u}K'_jb'_j;d_1h_1,\cdots,d_rh_r;1),A \\ \cdot \\ \cdot \\ (1\text{-}\alpha-\beta-\sum_{j=1}^{s}K_j(a_j+b_j)-\sum_{j=1}^{u}K'_j(a'_j+b'_j);(c_1+d_1)h_1,\cdots,(c_r+d_r)h_r;1),B \end{array}\right) \quad (6.4)$$

under the same conditions that (3.1) and $Re(2\sum_{j=1}^{s} K_j\omega_j + 2\sum_{j=1}^{u} K'_j\omega'_j + 2\mathbf{M}' + 2\sum_{j=1}^{r}\rho_j s_j) > 0$

7. Multivariable H-function

If $A_j = B_j = C_j^{(i)} = D_j^{(i)} = 1$, the multivariable I-function defined by Prathima et al [] reduces to multivariable H-function defined by Srivastava et al [8,9] and we have

$$\int_0^\infty\int_0^\infty \frac{x^{2\beta-1}y^{2\alpha-1}(x^2+y^2)}{(x^2+cy^2)^{\alpha+\beta}} S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix} \mathrm{y}_1(x^2+y^2)^{\omega_1}\frac{y^{2a_1}x^{2b_1}}{(x^2+cy^2)^{a_1+b_1}} \\ \cdot \\ \cdot \\ \mathrm{y}_s(x^2+y^2)^{\omega_s}\frac{y^{2a_s}x^{2b_s}}{(x^2+cy^2)^{a_s+b_s}}\end{pmatrix} H\big[d(x^2+y^2), f(x^2+y^2)\big]$$

$$S^{M'_1,\cdots,M'_u}_{N'_1,\cdots,N'_u}\begin{pmatrix} \mathrm{z}_1(x^2+y^2)^{\omega'_1}\frac{y^{2a'_1}x^{2b'_1}}{(x^2+cy^2)^{a'_1+b'_1}} \\ . \\ . \\ \mathrm{z}_u(x^2+y^2)^{\omega'_u}\frac{y^{2a'_u}x^{2b'_u}}{(x^2+cy^2)^{a'_u+b'_u}} \end{pmatrix} H\begin{pmatrix} \mathrm{Z}_1(x^2+y^2)^{\lambda_1}\frac{y^{2h_1c_1}x^{2h_1d_1}}{(x^2+cy^2)^{h_1(c_1+d_1)}} \\ . \\ . \\ \mathrm{Z}_r(x^2+y^2)^{\lambda_r}\frac{y^{2h_rc_r}x^{2h_rd_r}}{(x^2+cy^2)^{h_r(c_r+d_r)}} \end{pmatrix} e^{-\mu(x^2+y^2)^{1/2}}\mathrm{d}x\mathrm{d}y =$$

$$\frac{1}{2c^{2\alpha}}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{K'_1=0}^{[N'_1/M'_1]}\cdots\sum_{K'_u=0}^{[N'_u/M'_u]}\sum_{\mathbf{M}'=0}^{\infty} C_sC'_u\,\phi(\mathbf{M}')\frac{(-d)^{\mathbf{M}'}}{\mathbf{M}'!}\,y_1^{K_1}\cdots y_s^{K_s}\,z_1^{K'_1}\cdots z_u^{K'_u}c^{-(\sum_{i=1}^s K_ia_i+\sum_{i=1}^u K'_ia'_i)}$$

$$\mu^{-2(\sum_{j=1}^s K_j\omega_j+\sum_{j=1}^u K'_j\omega'_j+\mathbf{M}'+1)}H^{0,\mathbf{n}+3;X}_{\mathbf{p}+3,\mathbf{q}+1;Y}\left(\begin{array}{c|c} \mathrm{z}_1t^{\lambda_1}c^{-h_1c_1} & (1\text{-}\alpha-\sum_{j=1}^s K_ja_j-\sum_{j=1}^u K'_ja'_j;c_1h_1,\cdots,c_rh_r), \\ . & . \\ . & . \\ . & \cdots \\ \mathrm{z}_rt^{\lambda_r}c^{-h_rc_r} & \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-}1\text{-}2(\sum_{i=1}^s K_i\omega_i+\sum_{i=1}^u K'_i\omega'+\mathbf{M}');2\lambda_1,\cdots,2\lambda_r;1),(1-\beta-\sum_{j=1}^s K_jb_j-\sum_{j=1}^u K'_jb'_j;d_1h_1,\cdots,d_rh_r;1),A \\ . \\ . \\ (1\text{-}\alpha-\beta-\sum_{j=1}^s K_j(a_j+b_j)-\sum_{j=1}^u K'_j(a'_j+b'_j);(c_1+d_1)h_1,\cdots,(c_r+d_r)h_r;1),B \end{array}\right) \tag{7.1}$$

under the same conditions that (3.1) and $Re(2\sum_{j=1}^s K_j\omega_j+2\sum_{j=1}^u K'_j\omega'_j+2\mathbf{M}'+2\sum_{j=1}^r \rho_js_j)>0$ and $A_j=B_j=C_j^{(i)}=D_j^{(i)}=1$ and

$$\int_0^\infty\int_0^\infty\int_0^\infty\frac{(xz)^{2\beta-1}y^{2\alpha-1}(x^2+y^2)^\alpha(x^2+y^2+z^2)^{1-1/2(\alpha+\beta)}}{(x^2+cy^2)^{\alpha+\beta}\left[z^2+c(x^2+y^2)\right]^{\alpha+\beta}}S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\begin{pmatrix} \mathrm{y}_1(x^2+y^2+z^2)^{\omega_1}\frac{y^{2a_1}x^{2b_1}}{(x^2+cy^2)^{a_1+b_1}} \\ . \\ . \\ \mathrm{y}_s(x^2+y^2+z^2)^{\omega_s}\frac{y^{2a_s}x^{2b_s}}{(x^2+cy^2)^{a_s+b_s}} \end{pmatrix}$$

$$S^{M'_1,\cdots,M'_u}_{N'_1,\cdots,N'_u}\begin{pmatrix} \mathrm{z}_1(x^2+y^2+z^2)^{\omega'_1}\frac{y^{2a'_1}x^{2b'_1}}{(x^2+cy^2)^{a'_1+b'_1}} \\ . \\ . \\ \mathrm{z}_u(x^2+y^2+z^2)^{\omega'_u}\frac{y^{2a'_u}x^{2b'_u}}{(x^2+cy^2)^{a'_u+b'_u}} \end{pmatrix} H\begin{pmatrix} \mathrm{Z}_1(x^2+y^2+z^2)^{\lambda_1}\frac{y^{2h_1c_1}x^{2h_1d_1}}{(x^2+cy^2)^{h_1(c_1+d_1)}} \\ . \\ . \\ \mathrm{Z}_r(x^2+y^2+z^2)^{\lambda_r}\frac{y^{2h_rc_r}x^{2h_rd_r}}{(x^2+cy^2)^{h_r(c_r+d_r)}} \end{pmatrix} e^{-\mu(x^2+y^2+z^2)^{1/2}}$$

$$H\left[d(x^2+y^2+z^2),f(x^2+y^2+z^2)\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z=\frac{1}{4c^{2\alpha}}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{K'_1=0}^{[N'_1/M'_1]}\cdots\sum_{K'_u=0}^{[N'_u/M'_u]}\sum_{\mathbf{M}'=0}^{\infty}C_sC'_u$$

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\, c^{-(\sum_{i=1}^{s} K_i a_i+\sum_{i=1}^{u} K'_i a'_i)}\ \phi(\mathbf{M}')\frac{(-d)^{\mathbf{M}'}}{\mathbf{M}'!}y_1^{K_1}\cdots y_s^{K_s}\, z_1^{K'_1}\cdots z_u^{K'_u}$$

$$\mu^{-2(\sum_{j=1}^{s} K_j\omega_j+\sum_{j=1}^{u} K'_j\omega'_j+\mathbf{M}'+1)}H^{0,\mathbf{n}+3;X}_{\mathbf{p}+3,\mathbf{q}+1;Y}\left(\begin{array}{c|c} \mathrm{z}_1 t^{\lambda_1}c^{-h_1c_1} & (1\text{-}\alpha-\sum_{j=1}^{s}K_ja_j-\sum_{j=1}^{u}K'_ja'_j;c_1h_1,\cdots,c_rh_r), \\ . & . \\ . & . \\ . & \cdots \\ \mathrm{z}_r t^{\lambda_r}c^{-h_rc_r} & \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-}1\text{-}2(\sum_{i=1}^{s}K_i\omega_i+\sum_{i=1}^{u}K'_i\omega'+\mathbf{M}');2\lambda_1,\cdots,2\lambda_r;1),(1-\beta-\sum_{j=1}^{s}K_jb_j-\sum_{j=1}^{u}K'_jb'_j;d_1h_1,\cdots,d_rh_r;1),A \\ . \\ . \\ (1\text{-}\alpha-\beta-\sum_{j=1}^{s}K_j(a_j+b_j)-\sum_{j=1}^{u}K'_j(a'_j+b'_j);(c_1+d_1)h_1,\cdots,(c_r+d_r)h_r;1),B \end{array}\right) \quad (7.2)$$

under the same conditions that (3.1) and $Re(2\sum_{j=1}^{s}K_j\omega_j+2\sum_{j=1}^{u}K'_j\omega'_j+2\mathbf{M}'+2\sum_{j=1}^{r}\rho_js_j)>0$ and $A_j=B_j=C_j^{(i)}=D_j^{(i)}=1$

8. I-function of two variables

If $r=2$, the multivariable I-function reduces to I-function defined by Rathie et al [5] and we have the following integrals :

$$\int_0^\infty\int_0^\infty \frac{x^{2\beta-1}y^{2\alpha-1}(x^2+y^2)}{(x^2+cy^2)^{\alpha+\beta}}S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left(\begin{array}{c} \mathrm{y}_1(x^2+y^2)^{\omega_1}\frac{y^{2a_1}x^{2b_1}}{(x^2+cy^2)^{a_1+b_1}} \\ . \\ . \\ \mathrm{y}_s(x^2+y^2)^{\omega_s}\frac{y^{2a_s}x^{2b_s}}{(x^2+cy^2)^{a_s+b_s}} \end{array}\right)H\left[d(x^2+y^2),f(x^2+y^2)\right]$$

$$S^{M'_1,\cdots,M'_u}_{N'_1,\cdots,N'_u}\left(\begin{array}{c} \mathrm{z}_1(x^2+y^2)^{\omega'_1}\frac{y^{2a'_1}x^{2b'_1}}{(x^2+cy^2)^{a'_1+b'_1}} \\ . \\ . \\ \mathrm{z}_u(x^2+y^2)^{\omega'_u}\frac{y^{2a'_u}x^{2b'_u}}{(x^2+cy^2)^{a'_u+b'_u}} \end{array}\right)I\left(\begin{array}{c} \mathrm{Z}_1(x^2+y^2)^{\lambda_1}\frac{y^{2h_1c_1}x^{2h_1d_1}}{(x^2+cy^2)^{h_1(c_1+d_1)}} \\ . \\ . \\ \mathrm{Z}_2(x^2+y^2)^{\lambda_2}\frac{y^{2h_2c_2}x^{2h_2d_2}}{(x^2+cy^2)^{h_2(c_2+d_2)}} \end{array}\right)e^{-\mu(x^2+y^2)^{1/2}}\mathrm{d}x\mathrm{d}y=$$

$$\frac{1}{2c^{2\alpha}}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{K'_1=0}^{[N'_1/M'_1]}\cdots\sum_{K'_u=0}^{[N'_u/M'_u]}\sum_{\mathbf{M}'=0}^{\infty}C_sC'_u\,\phi(\mathbf{M}')\frac{(-d)^{\mathbf{M}'}}{\mathbf{M}'!}\,y_1^{K_1}\cdots y_s^{K_s}\,z_1^{K'_1}\cdots z_u^{K'_u}c^{-(\sum_{i=1}^{s}K_ia_i+\sum_{i=1}^{u}K'_ia'_i)}$$

$$\mu^{-2(\sum_{j=1}^{s} K_j\omega_j+\sum_{j=1}^{u} K'_j\omega'_j+\mathbf{M}'+1)}I^{0,\mathbf{n}+3;X}_{\mathbf{p}+3,\mathbf{q}+1;Y}\left(\begin{array}{c|c} \mathrm{z}_1 t^{\lambda_1}c^{-h_1c_1} & (1\text{-}\alpha-\sum_{j=1}^{s}K_ja_j-\sum_{j=1}^{u}K'_ja'_j;c_1h_1,c_2h_2;1), \\ . & . \\ . & . \\ . & \cdots \\ \mathrm{z}_2 t^{\lambda_2}c^{-h_2c_2} & \end{array}\right.$$

$$(\text{-1-2}(\textstyle\sum_{i=1}^{s} K_i\omega_i + \sum_{i=1}^{u} K'_i\omega' + \mathbf{M}'); 2\lambda_1, 2\lambda_2; 1), (1-\beta-\sum_{j=1}^{s} K_j b_j - \sum_{j=1}^{u} K'_j b'_j; d_1h_1, d_2h_2; 1), A \quad \cdot \quad \cdot \quad (1\text{-}\alpha-\beta-\sum_{j=1}^{s} K_j(a_j+b_j) - \sum_{j=1}^{u} K'_j(a'_j+b'_j); (c_1+d_1)h_1, (c_2+d_2)h_r; 1), B \Bigg) \tag{8.1}$$

under the same conditions that (3.1) and $Re(2\sum_{j=1}^{s} K_j\omega_j + 2\sum_{j=1}^{u} K'_j\omega'_j + 2\mathbf{M}' + 2\sum_{j=1}^{r}\rho_j s_j) > 0$ and $r = 2$

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{(xz)^{2\beta-1}y^{2\alpha-1}(x^2+y^2)^\alpha(x^2+y^2+z^2)^{1-1/2(\alpha+\beta)}}{(x^2+cy^2)^{\alpha+\beta}\big[z^2+c(x^2+y^2)\big]^{\alpha+\beta}} S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{y}_1(x^2+y^2+z^2)^{\omega_1}\frac{y^{2a_1}x^{2b_1}}{(x^2+cy^2)^{a_1+b_1}} \\ \cdot \\ \cdot \\ \mathrm{y}_s(x^2+y^2+z^2)^{\omega_s}\frac{y^{2a_s}x^{2b_s}}{(x^2+cy^2)^{a_s+b_s}} \end{array}\right)$$

$$S_{N'_1,\cdots,N'_u}^{M'_1,\cdots,M'_u}\left(\begin{array}{c} \mathrm{z}_1(x^2+y^2+z^2)^{\omega'_1}\frac{y^{2a'_1}x^{2b'_1}}{(x^2+cy^2)^{a'_1+b'_1}} \\ \cdot \\ \cdot \\ \mathrm{z}_u(x^2+y^2+z^2)^{\omega'_u}\frac{y^{2a'_u}x^{2b'_u}}{(x^2+cy^2)^{a'_u+b'_u}} \end{array}\right) I\left(\begin{array}{c} \mathrm{Z}_1(x^2+y^2+z^2)^{\lambda_1}\frac{y^{2h_1c_1}x^{2h_1d_1}}{(x^2+cy^2)^{h_1(c_1+d_1)}} \\ \cdot \\ \cdot \\ \mathrm{Z}_2(x^2+y^2+z^2)^{\lambda_2}\frac{y^{2h_2c_2}x^{2h_2d_2}}{(x^2+cy^2)^{h_2(c_2+d_2)}} \end{array}\right) e^{-\mu(x^2+y^2+z^2)^{1/2}}$$

$$H\big[d(x^2+y^2+z^2), f(x^2+y^2+z^2)\big]\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{4c^{2\alpha}}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{K'_1=0}^{[N'_1/M'_1]}\cdots\sum_{K'_u=0}^{[N'_u/M'_u]}\sum_{\mathbf{M}'=0}^{\infty} C_s C'_u$$

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} c^{-(\sum_{i=1}^{s} K_i a_i + \sum_{i=1}^{u} K'_i a'_i)} \ \phi(\mathbf{M}')\frac{(-d)^{\mathbf{M}'}}{\mathbf{M}'!} y_1^{K_1}\cdots y_s^{K_s}\, z_1^{K'_1}\cdots z_u^{K'_u}$$

$$\mu^{-2(\sum_{j=1}^{s} K_j\omega_j + \sum_{j=1}^{u} K'_j\omega'_j + \mathbf{M}'+1)} I_{\mathbf{p}+3,\mathbf{q}+1;Y}^{0,\mathbf{n}+3;X}\left(\begin{array}{c} \mathrm{z}_1 t^{\lambda_1}c^{-h_1c_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_2 t^{\lambda_2}c^{-h_2c_2} \end{array}\right| \begin{array}{c} (1\text{-}\alpha-\sum_{j=1}^{s} K_j a_j - \sum_{j=1}^{u} K'_j a'_j; c_1h_1, c_rh_2; 1), \\ \cdot \\ \cdot \\ \cdots \end{array}$$

$$(\text{-1-2}(\textstyle\sum_{i=1}^{s} K_i\omega_i + \sum_{i=1}^{u} K'_i\omega' + \mathbf{M}'); 2\lambda_1, 2\lambda_2; 1), (1-\beta-\sum_{j=1}^{s} K_j b_j - \sum_{j=1}^{u} K'_j b'_j; d_1h_1, d_2h_2; 1), A \quad \cdot \quad \cdot \quad (1\text{-}\alpha-\beta-\sum_{j=1}^{s} K_j(a_j+b_j) - \sum_{j=1}^{u} K'_j(a'_j+b'_j); (c_1+d_1)h_1, (c_2+d_2)h_r; 1), B \Bigg) \tag{8.2}$$

under the same conditions that (3.1) and $Re(2\sum_{j=1}^{s} K_j\omega_j + 2\sum_{j=1}^{u} K'_j\omega'_j + 2\mathbf{M}' + 2\sum_{j=1}^{r}\rho_j s_j) > 0$ and $r = 2$

Remark

If $S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[.] \rightarrow S_N^M[.]$, $S_{N'_1,\cdots,N'_u}^{M'_1,\cdots,M'_u}[.] \rightarrow S_{N'}^{M'}$ and the multivariable I-function is replaced by the multivariable H-function, we obtain the the results of Garg et al [1].

9. Conclusion

The integral relations (4.1) and (4.2) are quite general in nature on account of the arbitrary nature of the functions f and g and also on account of the presence of the general classes of multivariable polynomials $S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[.]$ and $S_{N'_1,\cdots,N'_u}^{M'_1,\cdots,M'_u}[.]$ the multivariable I-function defined by Prathima et al [4]. A very large number of (known and new) integrals can be derived as specials cases.

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