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Multivariable Aleph-function with application to temperature distribution in a moving medium

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ABSTRACT
The Meijer's G-function has been obtained for the first times as a solution of a partial differential equation governing a heat conduction problem. The problem of temperature distribution in a moving medium between $x=-1$ and $x=1$and having variable velocity and variable thermal conductivity is considered. Due to a general character of the G-function, many known and unknown results may be derived as particular cases. The product of the class of multivariable polynomials, the Srivastava-Daoust function, the multivariable I-function defined by Prathima et al [6] and the multivariable Aleph-function to obtain a particular solution has been employed. Several particular cases will study in the end.

Keywords :Multivariable Aleph-function, , class of multivariable polynomials, Bessel function, expansion formula., Aleph-function of two variables, Srivastava-Daoust function, I-function of two variables, multivariable I-function, Meijer's function, temperature, velocity, thermal conductivity.

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1. Introduction

The generalized polynomials defined by Srivastava [13], is given in the following manner :

$$S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}[y_1,\cdots,y_u] = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_u)_{M_uK_u}}{K_u!}A[N_1,K_1;\cdots;N_u,K_u]y_1^{K_1}\cdots y_u^{K_u} \quad (1.1)$$

Where $M_1,\cdots,M_u$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_u,K_u]$ are arbitrary constants, real or complex.

The Srivastava-Daoust function is defined by (see [14]):

$$F_{\bar{C}:D^{(1)};\cdots;D^{(v)}}^{\bar{A}:B^{(1)};\cdots;B^{(v)}}\left(\begin{array}{c|c} \mathrm{z}_1 & [(\mathrm{a});\theta',\cdots,\theta^{(v)}]:[(b');\phi'];\cdots;[(b^{(v)});\phi^{(v)}] \\ \cdot & \\ \cdot & \cdot \\ \mathrm{z}_v & [(\mathrm{c});\psi',\cdots,\psi^{(v)}]:[(d');\delta'];\cdots;[(d^{(v)});\delta^{(v)}] \end{array}\right)$$

$$= \sum_{r_1,\cdots,r_v=0}^{\infty} A'\frac{z_1^{r_1}\cdots z_v^{r_v}}{r_1!\cdots r_v!} \quad (1.2)$$

where

$$A' = \frac{\prod_{j=1}^{\bar{A}}(a_j)_{r_1\theta_j'+\cdots+r_v\theta_j^{(v)}}\prod_{j=1}^{B^{(1)}}(b_j')_{r_1\phi_j'}\cdots\prod_{j=1}^{B^{(v)}}(b_j^{(v)})_{r_v\phi_j^{(v)}}}{\prod_{j=1}^{\bar{C}}(c_j)_{r_1\psi_j'+\cdots+r_v\psi_j^{(v)}}\prod_{j=1}^{D^{(1)}}(d_j')_{r_1\delta_j'}\cdots\prod_{j=1}^{D^{(v)}}(d_j^{(v)})_{r_v\delta_j^{(v)}}} \quad \mathbf{(1.3)}$$

The series given by (1.2) converges absolutely if

$$1+\sum_{j=1}^{\bar{C}}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{\bar{A}}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0; i=1,\cdots,v \quad (1.4)$$

The multivariable I-function defined by Prathima et al [7] is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1,\cdots,z_s) = I^{0,N:M_1,N_1;\cdots;M_s,N_s}_{P,Q:P_1,Q_1;\cdots;P_s,Q_s}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(s)};A_j)_{1,P}: \\ \cdot & \\ \cdot & \\ \cdot & (\mathrm{b}_j;\beta_j^{(1)},\cdots,\beta_j^{(s)};B_j)_{1,Q}: \\ \mathrm{z}_s & \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,N_1},(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{N_1+1,P_1};\cdots;(c_j^{(s)},\gamma_j^{(s)};C_j^{(s)})_{1,N_s},(c_j^{(s)},\gamma_j^{(s)};C_j^{(s)})_{N_s+1,P_s} \\ (\mathrm{d}_j^{(1)},\delta_j^{(1)};1)_{1,M_1},(d_j^{(1)},\delta_j^{(1)};D_1)_{M_1+1,Q_1};\cdots;(d_j^{(s)},\delta_j^{(s)};1)_{1,M_s},(d_j^{(s)},\delta_j^{(s)};D_s)_{M_s+1,Q_s} \end{array}\right) \tag{1.5}$$

$$= \frac{1}{(2\pi\omega)^s}\int_{L_1}\cdots\int_{L_s}\phi(t_1,\cdots,t_s)\prod_{i=1}^{s}\theta_i(t_i)z_i^{t_i}\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.6}$$

where $\phi(t_1,\cdots,t_s),\theta_i(t_i),i=1,\cdots,s$ are given by :

$$\phi(t_1,\cdots,t_s) = \frac{\prod_{j=1}^{N}\Gamma^{A_j}\left(1-aj+\sum_{i=1}^{s}\alpha_j^{(i)}t_j\right)}{\prod_{j=N+1}^{P}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{s}\alpha_j^{(i)}t_j\right)\prod_{j=1}^{Q}\Gamma^{B_j}\left(1-bj+\sum_{i=1}^{s}\beta_j^{(i)}t_j\right)} \tag{1.7}$$

$$\phi_i(t_i) = \frac{\prod_{j=1}^{N_i}\Gamma^{C_j^{(i)}}\left(1-c_j^{(i)}+\gamma_j^{(i)}t_i\right)\prod_{j=1}^{M_i}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}t_i\right)}{\prod_{j=N_i+1}^{P_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}t_i\right)\prod_{j=M_i+1}^{Q_i}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}+\delta_j^{(i)}t_i\right)} \tag{1.8}$$

For more details, see Prathima et al [7].

We can obtain the series representation and behaviour for small values for the function $I(z_1,\cdots,z_t)$ defined and represented by (1.6). The series representation may be given as follows :

which is valid under the following conditions :

$$\delta_i^{(h)}[d_i^{(j)}+s] \neq \delta_i^{(j)}[d_i^{(h)}+\mu] \text{ for } j\neq h, j,h=1,\cdots,M_i\,, s,\mu=0,1,2,\cdots$$

$$U_i = \sum_{j=1}^{P}A_j\alpha_j^{(i)} - \sum_{j=1}^{Q}B_j\beta_j^{(i)} + \sum_{j=1}^{P_i}C_j^{(i)}\gamma_j^{(i)} - \sum_{j=M_i+1}^{Q_i}D_j^{(i)}\delta_j^{(i)} \leqslant 0, i=1,\cdots,s \quad \text{and} \quad z_i\neq 0$$

and if all the poles of (1.6) are simple ,then the integral (1.16) can be evaluated with the help of the Residue theorem to give

$$I(z_1,\cdots,z_s) = \sum_{G_i=1}^{M_i}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^{s}\phi_i z_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{s}g_i}}{\prod_{j=1}^{s}\delta_{G^{(i)}}^{(i)}\prod_{i=1}^{s}g_i!} \tag{1.9}$$

where ϕ_1 and ϕ_i are defined by

$$\phi_1 = \frac{\prod_{j=1}^{N} \Gamma^{A_j}\left(1 - aj + \sum_{i=1}^{s} \alpha_j^{(i)} \eta_{G_i,g_i}\right)}{\prod_{j=N+1}^{P} \Gamma^{A_j}\left(a_j - \sum_{i=1}^{s} \alpha_j^{(i)} \eta_{G_i,g_i}\right) \prod_{j=1}^{Q} \Gamma^{B_j}\left(1 - bj + \sum_{i=1}^{s} \beta_j^{(i)} \eta_{G_i,g_i}\right)} \tag{1.10}$$

and

$$\phi_i = \frac{\prod_{j=1}^{N_i} \Gamma^{C_j^{(i)}}\left(1 - c_j^{(i)} + \gamma_j^{(i)} \eta_{G_i,g_i}\right) \prod_{j=1}^{M_i} \Gamma\left(d_j^{(i)} - \delta_j^{(i)} \eta_{G_i,g_i}\right)}{\prod_{j=N_i+1}^{P_i} \Gamma^{C_j^{(i)}}\left(c_j^{(i)} - \gamma_j^{(i)} \eta_{G_i,g_i}\right) \prod_{j=M_i+1}^{Q_i} \Gamma^{(D_j^{(i)}}\left(1 - d_j^{(i)} + \delta_j^{(i)} \eta_{G_i,g_i}\right)}, i = 1, \cdots, s \tag{1.11}$$

where $\eta_{G_i,g_i} = \dfrac{d_{G^{(i)}}^{(i)} + g_i}{\delta_{G^{(i)}}^{(i)}}, i = 1, \cdots, s$

The multivariable Aleph-function is an extension of the multivariable I-function recently defined by C.K. Sharma and Ahmad [9] , itself is a generalization of the multivariable H-function defined by Srivastava et al [15, 16]. The multivariable Aleph-function is defined by means of the multiple contour integral :

We have : $\aleph(z_1, \cdots, z_r) = \aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i(1)},q_{i(1)},\tau_{i(1)};R^{(1)};\cdots;p_{i(r)},q_{i(r)};\tau_{i(r)};R^{(r)}} \left(\begin{array}{c|c} z_1 & \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r & \end{array}\right.$

$[(\mathrm{a}_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,\mathfrak{n}}]$ $,[\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:$
.... $,[\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{m+1,q_i}]:$

$$\left.\begin{array}{l} [(\mathrm{c}_j^{(1)}), \gamma_j^{(1)})_{1,n_1}], [\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}, \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}]; \cdots; [(\mathrm{c}_j^{(r)}), \gamma_j^{(r)})_{1,n_r}], [\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}, \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ [(\mathrm{d}_j^{(1)}), \delta_j^{(1)})_{1,m_1}], [\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}]; \cdots; [(\mathrm{d}_j^{(r)}), \delta_j^{(r)})_{1,m_r}], [\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}, \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \cdots, s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \, \mathrm{d}s_1 \cdots \mathrm{d}s_r \tag{1.12}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1, \cdots, s_r) = \frac{\prod_{j=1}^{\mathfrak{n}} \Gamma(1 - a_j + \sum_{k=1}^{r} \alpha_j^{(k)} s_k)}{\sum_{i=1}^{R} [\tau_i \prod_{j=\mathfrak{n}+1}^{p_i} \Gamma(a_{ji} - \sum_{k=1}^{r} \alpha_{ji}^{(k)} s_k) \prod_{j=1}^{q_i} \Gamma(1 - b_{ji} + \sum_{k=1}^{r} \beta_{ji}^{(k)} s_k)]} \tag{1.13}$$

and $\theta_k(s_k) = \dfrac{\prod_{j=1}^{m_k} \Gamma(d_j^{(k)} - \delta_j^{(k)} s_k) \prod_{j=1}^{n_k} \Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}} [\tau_{i^{(k)}} \prod_{j=m_k+1}^{q_{i^{(k)}}} \Gamma(1 - d_{ji^{(k)}}^{(k)} + \delta_{ji^{(k)}}^{(k)} s_k) \prod_{j=n_k+1}^{p_{i^{(k)}}} \Gamma(c_{ji^{(k)}}^{(k)} - \gamma_{ji^{(k)}}^{(k)} s_k)]}$ (1.14)

For more details, see Ayant [1]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|arg z_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)} - \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i(k)}}\delta_{ji^{(k)}}^{(k)} > 0, \text{ with } k=1,\cdots,r, i=1,\cdots,R\ , i^{(k)}=1,\cdots,R^{(k)} \tag{1.15}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function. We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,)\,, max(\,|z_1|,\cdots,|z_r|\,) \rightarrow 0$$

$$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,)\,, min(\,|z_1|,\cdots,|z_r|\,) \rightarrow \infty$$

where $k=1,\cdots,r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j=1,\cdots,m_k$ and

$$\beta_k = max[Re((c_j^{(k)}-1)/\gamma_j^{(k)})], j=1,\cdots,n_k$$

For convenience, we will use the following notations in this paper.

$$V = m_1,n_1;\cdots;m_r,n_r \tag{1.16}$$

$$\mathrm{W} = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)} \tag{1.17}$$

$$A = \{(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,n}\},\{\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{n+1,p_i}\}:\{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\}$$

$$\{\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\};\cdots;\{(c_j^{(r)};\gamma_j^{(r)})_{1,n_r}\},\{\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.18}$$

$$B = \{\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}\}:\{(d_j^{(1)};\delta_j^{(1)})_{1,m_1}\},\{\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}\ ;\cdots;$$

$$\{(d_j^{(r)};\delta_j^{(r)})_{1,m_r}\},\{\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.19}$$

$$B' = \frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_u)_{M_uK_u}}{K_u!}A[N_1,K_1;\cdots;N_u,K_u] \tag{1.20}$$

$$F = F_{\bar{C}:D^{(1)};\cdots;D^{(v)}}^{\bar{A}:B^{(1)};\cdots;B^{(v)}} \tag{1.21}$$

2. Statement of the problem and governing equations

Let us consider a medium (say bar) moving in the direction of its lenth ($x-$axis) between the limits $x=-1$ to $x=1$. the bar is supposed to be so thin that the temperature at all points of the section may be taken to be the same. We also assume that conductivity of the medium is a function of position and is proportional to $(1-x^2)$. Further, the velocity of the medium is supposed to be a function of position and proportional to $[\alpha+\gamma-\beta+(\alpha+\beta-\gamma)x]$. Lateral surface of the medium and its two ends $(x=\pm 1)$ are supposed to be insulated.

The partial differential equation satisfied by the temperature $\theta(x,t)$ at any time t in a solid medium with conductivity K , dendity ρ,specific heat C and with no generation of heat within the medium is given [2] as

$$\rho C\frac{\partial\theta}{\partial t}+\frac{\partial}{\partial x}\left(-K\frac{\partial\theta}{\partial x}\right)=0 \tag{2.1}$$

But in our problem the medium is moving with velocity $\mathbf{u}$ in the direction of $x-$axis and, therefore, while calculating the rate at which heat crosses any plane, a convection term $\rho C\theta\mathbf{u}$ must be added to the part due to conduction. The equation (2.1) then assumes the form

$$\rho C\frac{\partial\theta}{\partial t}+\frac{\partial}{\partial x}\left(-K\frac{\partial\theta}{\partial x}+\rho C\mathbf{u}\right)=0 \tag{2.2}$$

If both conductivity K and velocity $\mathbf{u}$ of the medium are functions of positions and (or) temperature , the equation (2.2) becomes

$$\rho C\frac{\partial\theta}{\partial t}+\rho c\mathbf{u}\frac{\partial\theta}{\partial x}+\rho C\theta\frac{\partial\theta}{\partial x}+K\frac{\partial^2\theta}{\partial x^2}-\frac{\partial\theta}{\partial x}\frac{\partial K}{\partial x}=0 \tag{2.3}$$

If

$$K=C_1(1-x^2) \tag{2.4}$$

and

$$u=C_2\left[\alpha+\gamma-\beta+(\alpha+\beta-\gamma)x\right] \tag{2.5}$$

where C_1 and C_2 are constants of proportionality, $\alpha>-1,\beta>\gamma-1$, then the equation (2.5) reduces to

$$\frac{\partial\theta}{\partial t}+C_2[\alpha+\gamma-\beta+(\alpha-\gamma+\beta)x\frac{\partial\theta}{\partial x}+\frac{2C_1}{\rho C}\frac{\partial\theta}{\partial x}+C_2(\alpha-\gamma+\beta)\theta-\frac{C_1}{\rho C}(1-x^2)\frac{\partial^2\theta}{\partial x^2}=0 \tag{2.6}$$

Let us, for further simplicity, assume $C_2=\dfrac{C_1}{\rho C}$, which transforms the equation (2.6) into

$$\frac{1}{C_2}\frac{\partial\theta}{\partial t}+[\alpha+\gamma-\beta+(\alpha-\gamma+\beta+2)x]\frac{\partial\theta}{\partial x}+(\alpha-\gamma+\beta)\theta-(1-x^2)\frac{\partial^2\theta}{\partial x^2}=0 \tag{2.7}$$

Let the initial temperature distribution in the medium be given by

$$\theta(x,0)=f(x) \tag{2.8}$$

3. Required integrals

Lemma 1

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}G_{2,2}^{1,2}\left[\ \tfrac{1}{2}(\text{x-1})\ \middle|\ \begin{matrix}1+\upsilon,-\upsilon-\alpha-\beta\\ .\\ 0,\text{ -}\alpha\end{matrix}\ \right]\mathrm{d}x$$

$$=\frac{2^{\rho+\sigma+1}\Gamma(\sigma+1)\Gamma(\upsilon+1)}{\Gamma(\upsilon+\alpha+1)}\sum_{N=0}^{\upsilon}\frac{\Gamma(-\upsilon+N)\Gamma(\upsilon+\alpha+\beta+1+N)\Gamma(\rho+1+N)}{N!(\alpha+1)_N\Gamma(\rho+\sigma+N+2)} \tag{3.1}$$

Provided that $Re(\rho)>-1,Re(\sigma)>-1.$

Lemma 2

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}G_{2,2}^{1,2}\left[\begin{array}{c|c} \frac{1}{2}(\text{x-1}) & \begin{array}{c} 1+\upsilon,-\upsilon-\alpha-\beta \\ . \\ 0,\text{ -}\alpha \end{array}\end{array}\right]G_{2,2}^{1,2}\left[\begin{array}{c|c} \frac{1}{2}(\text{x-1}) & \begin{array}{c} 1+\mu,-\mu-\alpha-\beta \\ . \\ 0,\text{ -}\alpha \end{array}\end{array}\right]\text{d}x=0 \quad (3.2)$$

provided that $Re(\alpha)>-1, Re(\beta)>-1.$

Lemma 3

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\beta}G_{2,2}^{1,2}\left[\begin{array}{c|c} \frac{1}{2}(\text{x-1}) & \begin{array}{c} 1+\upsilon,-\upsilon-\alpha-\beta \\ . \\ 0,\text{ -}\alpha \end{array}\end{array}\right]G_{2,2}^{1,2}\left[\begin{array}{c|c} \frac{1}{2}(\text{x-1}) & \begin{array}{c} 1+\upsilon,-\upsilon-\alpha-\beta \\ . \\ 0,\text{ -}\rho \end{array}\end{array}\right]\text{d}x$$

$$=\frac{2^{\rho+\beta+1}\Gamma(\beta+\upsilon+1)\Gamma(\alpha+\beta+2\upsilon+1)\Gamma(\rho+\beta+\upsilon+1)[\Gamma(-\upsilon)\Gamma(\upsilon+1)]^2}{\upsilon!\Gamma(\rho+\beta+2\upsilon+2)\Gamma(\upsilon+\alpha+1)} \quad (3.3)$$

Provided that $Re(\rho)>-1, Re(\beta)>-1.$

Proofs of the results (3.1)-(3.3) are based upon the hypergeometric serie representation [5] of the G-function occuting in the integrand and appeal to the relationships [4].

4. Main integral

Theorem

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}G_{2,2}^{1,2}\left[\begin{array}{c|c} \frac{1}{2}(\text{x-1}) & \begin{array}{c} 1+\upsilon,-\upsilon-\alpha-\beta \\ . \\ 0,\text{ -}\alpha \end{array}\end{array}\right]S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}\left(\begin{array}{c} \text{y}_1(1-x)^{f_1}(1+x)^{w_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \text{y}_u(1-x)^{f_u}(1+x)^{w_u} \end{array}\right)$$

$$F\left(\begin{array}{c} \text{t}_1(1-x)^{a_1}(1+x)^{b_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \text{t}_v(1-x)^{a_v}(1+x)^{b_v} \end{array}\right)I\left(\begin{array}{c} \text{z}_1(1-x)^{c_1}(1+x)^{d_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \text{z}_s(1-x)^{c_s}(1+x)^{d_s} \end{array}\right)\aleph\left(\begin{array}{c} \text{Z}_1(1-x)^{h_1}(1+x)^{k_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \text{Z}_r(1-x)^{h_r}(1+x)^{k_r} \end{array}\right)\text{d}x$$

$$=\frac{2^{\rho+\sigma+1}\Gamma(\upsilon+1)}{\Gamma(\upsilon+\alpha+1)}\sum_{N=0}^{\upsilon}\frac{\Gamma(-\upsilon+N)\Gamma(\upsilon+\alpha+\beta+1+N)}{N!(\alpha+1)_N}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{r_1,\cdots,r_v=0}^{\infty}A'B'$$

$$\sum_{G_i=1}^{M_i}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^{s}\phi_i z_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{s}g_i}}{\prod_{j=1}^{s}\delta_{G^{(i)}}^{(i)}\prod_{i=1}^{s}g_i!}\frac{y_1^{K_1}\cdots y_u^{K_u}t_1^{r_1}\cdots t_v^{r_v}}{r_1!\cdots r_v!}F(\rho,\sigma;N) \quad (4.1)$$

where $F(\rho,\sigma;N)=\aleph_{p_i+2,q_i+1,\tau_i;R:W}^{0,\mathbf{n}+2:V}\left(\begin{array}{c} \text{Z}_1 2^{h_1+k_1} \\ . \\ . \\ . \\ \text{Z}_2 2^{h_2+k_2} \end{array}\middle| \begin{array}{c} (\text{-}\sigma-\sum_{i=1}^{u}f_iK_i-\sum_{i=1}^{v}a_ir_i-\sum_{i=1}^{s}c_i\eta_{G_i,g_i}:h_1,\cdots,h_r), \\ . \\ . \\ \cdot\ \cdot\ \cdot \end{array}\right.$

$$\left.\begin{array}{r}(\text{-}\rho - N - \sum_{i=1}^{u} w_i K_i - \sum_{i=1}^{v} b_i r_i - \sum_{i=1}^{s} d_i \eta_{G_i,g_i} : k_1, \cdots, k_r), A \\ \cdot \\ \cdot \\ (\text{-1-}\sigma - \rho - N - \sum_{i=1}^{u}(f_i + w_i)K_i - \sum_{i=1}^{v}(a_i + b_i)r_i - \sum_{i=1}^{s}(c_i + d_i)\eta_{G_i,g_i} : h_1 + k_1, \cdots, h_r + k_r), B\end{array}\right) \quad (4.2)$$

provided that

$min\{f_i, w_i, a_j, b_j, c_k, d_k, h_l, k_l\} > 0$ for $i = 1, \cdots, v; j = 1, \cdots, u; k = 1, \cdots, s; l = 1, \cdots, r$

$$1 + \sum_{j=1}^{\bar{C}} \psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{\bar{A}} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} > 0; i = 1, \cdots, v$$

$$U_i = \sum_{j=1}^{P} A_j \alpha_j^{(i)} - \sum_{j=1}^{Q} B_j \beta_j^{(i)} + \sum_{j=1}^{P_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=M_i+1}^{Q_i} D_j^{(i)} \delta'^{(i)}_j \leqslant 0, i = 1, \cdots, s$$

$$Re\left[\rho + \sum_{i=1}^{u} K_i f_i + \sum_{i=1}^{v} r_i a_i + \sum_{i=1}^{s} c_i \eta_{G_i,g_i} + \sum_{i=1}^{r} h_i \min_{1 \leqslant j \leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}\right] > -1 \text{ and}$$

$$Re\left[\sigma + \sum_{i=1}^{u} K_i w_i + \sum_{i=1}^{v} r_i b_i + \sum_{i=1}^{s} d_i \eta_{G_i,g_i} + \sum_{i=1}^{r} k_i \min_{1 \leqslant j \leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}\right] > -1$$

$|argZ_k| < \frac{1}{2} A_i^{(k)} \pi$, where $A_i^{(k)}$ is defined by (1.15).

Proof

To prove (2.1), first expressing a class of multivariable polynomials defined by Srivastava [13] $S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[.]$, the Srivastava-Daoust function [14] $F[.]$ and the multivariable $\bar{I}$-function defined by Prathima et al [7] in series with the help of (1.1), (1.2),and (1.9) respectively, expressing the generalized hypergeometric in series and we interchange the order of summations and x-integral (which is permissible under the conditions stated). Expressing the $\aleph$-function of r-variables in Mellin-Barnes contour integral with the help of (1.12) and interchange the order of integrations which is justifiable due to absolute convergence of the integrals involved in the process. Now collecting the powers of $(1-x)$ and $(1+x)$ and evaluating the inner x-integral with the help of the lemma 1. Interpreting the Mellin-Barnes contour integral in multivariable $\aleph$-function, we obtain the desired result (4.2).

5. Solution of problem

Assuming the solution of the partial differential equation (2.7) as $\theta(x,t) = X(x)T(t)$, the equation (2.7) reduces to

$$\frac{1}{C_2 T}\frac{dt}{dt} = (1-x^2)\frac{1}{X}\frac{D^2 X}{dX^2} + \frac{1}{x}[\beta - \alpha - \gamma - (\alpha - \gamma + \beta + 2)x]\frac{dX}{dx} + (\gamma - \alpha - \beta) \quad (5.1)$$

Each side of the equation (5.1) equals a constant. Assuming the constant to be $(v+1)(\gamma - \alpha - \beta - v)$, the equations yields

$$(1-x^2)\frac{d^2 X}{dx^2} + [\beta - \alpha - \gamma - (\alpha - \gamma + \beta + 2)x]\frac{dX}{dx} + v(v + \alpha + \beta - \gamma + 1)X = 0 \quad (5.2)$$

and

$$\frac{dT}{dt} - C_2(\upsilon+1)(\gamma-\alpha-\beta-\upsilon)T = 0 \tag{5.3}$$

The solution of the equation (5.2) is, see [5]

$$X(x) = C_3 G^{1,2}_{2,2}\left[\ \tfrac{1}{2}(\text{x-1}) \ \middle|\ \begin{matrix} 1{+}\upsilon, \gamma-\upsilon-\alpha-\beta \\ . \\ 0,\ \text{-}\alpha \end{matrix}\ \right] \tag{5.4}$$

and the solution of the equation (5.3) is

$$T(t) = C_4 exp[C_2(\upsilon+1)(\gamma-\alpha-\beta-\upsilon)t] \tag{5.5}$$

where C_3 and C_4 are contants of integration.

A general solution may, therefore, be given as

$$\theta(x,t) = \sum_{\upsilon=0}^{\infty} A_\upsilon exp[C_2(\upsilon+1)(\gamma-\alpha-\beta-\upsilon)t] G^{1,2}_{2,2}\left[\ \tfrac{1}{2}(\text{x-1}) \ \middle|\ \begin{matrix} 1{+}\upsilon, \gamma-\upsilon-\alpha-\beta \\ . \\ 0,\ \text{-}\alpha \end{matrix}\ \right] \tag{5.6}$$

which on appeal to the initial condition $\theta(x,0) = f(x)$, we obtain

$$f(x) = \sum_{\upsilon=0}^{\infty} A_\upsilon G^{1,2}_{2,2}\left[\ \tfrac{1}{2}(\text{x-1}) \ \middle|\ \begin{matrix} 1{+}\upsilon, \gamma-\upsilon-\alpha-\beta \\ . \\ 0,\ \text{-}\alpha \end{matrix}\ \right] \tag{5.7}$$

To determine A_υ, we multiply both sides of the equation (5.7) by $(1-x)^\alpha(1+x)^{\beta-\gamma} G^{1,2}_{2,2}\left[\ \tfrac{1}{2}(\text{x-1}) \ \middle|\ \begin{matrix} 1{+}\upsilon, \gamma-\upsilon-\alpha-\beta \\ . \\ 0,\ \text{-}\alpha \end{matrix}\ \right]$

and integrate, thereafter, with respect to x between the limits $x=-1$ to $x=1$. In the right hand side we interchange the order of summation and integration which is justifiable due to absolute convergence of the integral and the serie involved therein. Finally, appeal to result (3.2) and (3.3), consequently, demands

$$A_\upsilon = \frac{2^{\gamma-\alpha-\beta-1}(\alpha-\gamma+\beta+2\upsilon+1)\Gamma(\alpha+\upsilon+1)}{\upsilon!\Gamma(\upsilon+\alpha-\gamma+\beta+1)\Gamma^2(-\upsilon)\Gamma(\upsilon+\beta+1)}$$

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^{\beta-\gamma} G^{1,2}_{2,2}\left[\ \tfrac{1}{2}(\text{x-1}) \ \middle|\ \begin{matrix} 1{+}\upsilon, \gamma-\upsilon-\alpha-\beta \\ . \\ 0,\ \text{-}\alpha \end{matrix}\ \right] f(x)\mathrm{d}x \tag{5.8}$$

Use the equation (5.6) and (5.8), we obtain a general solution as

$$\theta(x,t) = 2^{\gamma-\alpha-\beta-1}\sum_{\upsilon=0}^{\infty}\sum_{N=0}^{\upsilon} \frac{(\alpha-\gamma+\beta+2\upsilon+1)\Gamma(\alpha+\upsilon+1)}{N!(\alpha+1)_N\upsilon!\Gamma(\upsilon+\alpha-\gamma+\beta+1)\Gamma^2(-\upsilon)\Gamma(\upsilon-\gamma+\beta+1)}$$

$$exp[C_2(\upsilon+1)(\gamma-\alpha-\beta-\upsilon)t]G_{2,2}^{1,2}\left[\begin{array}{c|c} \frac{1}{2}(\text{x-1}) & \begin{array}{c} 1+\upsilon,\gamma-\upsilon-\alpha-\beta \\ . \\ 0, -\alpha \end{array}\end{array}\right]$$

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta-\gamma}G_{2,2}^{1,2}\left[\begin{array}{c|c} \frac{1}{2}(\text{x-1}) & \begin{array}{c} 1+\upsilon,\gamma-\upsilon-\alpha-\beta \\ . \\ 0, -\alpha \end{array}\end{array}\right]f(x)\mathrm{d}x \tag{5.9}$$

6. Example

If the initial temperature distribution in the medium is given by :

$$f(x)=(1-x)^{\rho}(1+x)^{\sigma}G_{2,2}^{1,2}\left[\begin{array}{c|c} \frac{1}{2}(\text{x-1}) & \begin{array}{c} 1+\upsilon,-\upsilon-\alpha-\beta \\ . \\ 0, -\alpha \end{array}\end{array}\right]S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}\left(\begin{array}{c} \mathrm{y}_1(1-x)^{f_1}(1+x)^{w_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{y}_u(1-x)^{f_u}(1+x)^{w_u} \end{array}\right)$$

$$F\left(\begin{array}{c} \mathrm{t}_1(1-x)^{a_1}(1+x)^{b_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{t}_v(1-x)^{a_v}(1+x)^{b_v} \end{array}\right)\bar{I}\left(\begin{array}{c} \mathrm{z}_1(1-x)^{c_1}(1+x)^{d_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{z}_s(1-x)^{c_s}(1+x)^{d_s} \end{array}\right)\aleph\left(\begin{array}{c} \mathrm{Z}_1(1-x)^{h_1}(1+x)^{k_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{Z}_r(1-x)^{h_r}(1+x)^{k_r} \end{array}\right) \tag{6.1}$$

substituting this value for $f(x)$ in the equation (5.9) and utilizing the formula (4.1), the solution to the problem is given by :

$$\theta(x,t)=2^{\rho+\sigma}\sum_{\upsilon=0}^{\infty}\sum_{N=0}^{\upsilon}\frac{(\alpha-\gamma+\beta+2\upsilon+1)\Gamma(\alpha+\upsilon+1)}{N!(\alpha+1)_N\upsilon!\Gamma(\upsilon+\alpha-\gamma+\beta+1)\Gamma^2(-\upsilon)\Gamma(\upsilon-\gamma+\beta+1)}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{r_1,\cdots,r_v=0}^{\infty}$$

$$\sum_{G_i=1}^{M_i}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^{s}\phi_i z_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{s}g_i}}{\prod_{j=1}^{s}\delta_{G^{(i)}}^{(i)}\prod_{i=1}^{s}g_i!}\frac{y_1^{K_1}\cdots y_u^{K_u}t_1^{r_1}\cdots t_v^{r_v}}{r_1!\cdots r_v!}exp[C_2(\upsilon+1)(\gamma-\alpha-\beta-\upsilon)t]A'B'$$

$$G_{2,2}^{1,2}\left[\begin{array}{c|c} \frac{1}{2}(\text{x-1}) & \begin{array}{c} 1+\upsilon,\gamma-\upsilon-\alpha-\beta \\ . \\ 0, -\alpha \end{array}\end{array}\right]\ F(\rho+\alpha,\sigma+\beta;N) \tag{6.2}$$

where $F(\rho+\alpha,\sigma+\beta;N)$ is given by the equation (4.2).

Provided that

$min\{f_i,w_i,a_j,b_j,c_k,d_k,h_l,k_l\}>0$ for $i=1,\cdots,v;j=1,\cdots,u;k=1,\cdots,s;l=1,\cdots,r$

$$1+\sum_{j=1}^{\bar{C}}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{\bar{A}}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0;i=1,\cdots,v$$

$$U_i=\sum_{j=1}^{P}A_j\alpha_j^{(i)}-\sum_{j=1}^{Q}B_j\beta_j^{(i)}+\sum_{j=1}^{P_i}C_j^{(i)}\gamma_j^{(i)}-\sum_{j=M_i+1}^{Q_i}D_j^{(i)}{\delta'}_j^{(i)}\leqslant 0, i=1,\cdots,s$$

and $Re\left[\rho+\alpha+\sum_{i=1}^{u}K_if_i+\sum_{i=1}^{v}r_ia_i+\sum_{i=1}^{s}c_i\eta_{G_i,g_i}+\sum_{i=1}^{r}h_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right]>-1$

$$Re\left[\sigma+\beta+\sum_{i=1}^{u}K_iw_i+\sum_{i=1}^{v}r_ib_i+\sum_{i=1}^{s}d_i\eta_{G_i,g_i}+\sum_{i=1}^{r}k_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right]>-1$$

$|argZ_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.15).

7. Particular cases

(i) When the medium moves with uniform velocity $2C_2\alpha$, we have $\gamma=\alpha+\beta$, because the velocity is independent of position and time, the partial differential equation (2.7) reduces to

$$\frac{1}{C_2}\frac{\partial\theta}{\partial t}=(1-x^2)\frac{\partial^2\theta}{\partial x^2}-2(x+\alpha)\frac{\partial\theta}{\partial x}=0 \tag{7.1}$$

and the solution of (7.1) is given by

$$\theta(x,t)=2^{\rho+\sigma}\sum_{\upsilon=0}^{\infty}\sum_{N=0}^{\upsilon}\frac{\Gamma(2\upsilon+2)(-\upsilon)_N(\upsilon+1)_N}{N!(\alpha+1)_N\Gamma(\upsilon-\alpha+1)\Gamma(-\upsilon)\Gamma(2\upsilon+1)}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{r_1,\cdots,r_v=0}^{\infty}$$

$$\sum_{h_i=1}^{M_i}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{j=1}^{s}\phi_j z_i^{\eta_{h_j,g_j}}(-)^{\sum_{j=1}^{s}g_j}}{\prod_{j=1}^{s}\delta_{h^{(i)}}^{(j)}\prod_{i=1}^{s}g_i!}\frac{y_1^{K_1}\cdots y_u^{K_u}t_1^{r_1}\cdots t_v^{r_v}}{r_1!\cdots r_v!}exp[-C_2(\upsilon+1)\upsilon t]A'B'$$

$$G_{2,2}^{1,2}\left[\ \tfrac{1}{2}(\text{x-1})\ \middle|\ \begin{matrix}1+\upsilon,-\upsilon\\ \cdot\\ 0,\text{ -}\alpha\end{matrix}\right]\ F(\rho+\alpha,\sigma+\beta;N) \tag{7.2}$$

under the same conditions that (6.2) with $\gamma=\alpha+\beta$.

(ii) When the medium is stationary ($\mathbf{u}=0$), we have $\alpha=0$ and $\beta=\gamma$, the partial differential equation (2.7) reduces to

$$\frac{1}{C_2}\frac{\partial\theta}{\partial t}=(1-x^2)\frac{\partial^2\theta}{\partial x^2}-2x\frac{\partial\theta}{\partial x}=0 \tag{7.3}$$

and the solution of (7.3) is given by

$$\theta(x,t)=2^{\rho+\sigma}\sum_{\upsilon=0}^{\infty}\sum_{N=0}^{\upsilon}\frac{\Gamma(2\upsilon+2)(-\upsilon)_N(\upsilon+1)_N}{(N!)^2\Gamma(\upsilon+1)\Gamma(-\upsilon)\Gamma(2\upsilon+1)}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{r_1,\cdots,r_v=0}^{\infty}$$

$$\sum_{G_i=1}^{M_i}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^{s}\phi_i z_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{s}g_i}}{\prod_{j=1}^{s}\delta_{G^{(i)}}^{(i)}\prod_{i=1}^{s}g_i!}\frac{y_1^{K_1}\cdots y_u^{K_u}t_1^{r_1}\cdots t_v^{r_v}}{r_1!\cdots r_v!}exp[-C_2(\upsilon+1)\upsilon t]A'B'$$

$$G_{2,2}^{1,2}\left[\begin{array}{c|c}\frac{1}{2}(\text{x-1}) & \begin{array}{c}1+\upsilon,-\upsilon\\ .\\ 0,\ 0\end{array}\end{array}\right]F(\rho,\sigma;N) \tag{7.4}$$

under the same conditions that (6.2) with $\alpha=0$ and $\beta=\gamma$.

(iii) If we let $\gamma=0$, the partial differential equation (2.7) reduces to

$$\frac{1}{C_2}\frac{\partial\theta}{\partial t}=(1-x^2)\frac{\partial^2\theta}{\partial x^2}+[\beta-\alpha-(\alpha+\beta+2)x]\frac{\partial\theta}{\partial x}+(\alpha+\beta)\theta \tag{7.5}$$

and the solution (6.2) writes

$$\theta(x,t)=2^{\rho+\sigma}\sum_{\upsilon=0}^{\infty}\sum_{N=0}^{\upsilon}\frac{\upsilon!\Gamma(\alpha+\beta+2\upsilon+2)\Gamma(\alpha+\beta+1)(-\upsilon)_N(\alpha+\beta+\upsilon+1)_N}{N!(\alpha+1)_N\Gamma(\alpha+\upsilon+1)\Gamma(\upsilon+\beta+1)\Gamma(\alpha+\beta+2\upsilon+1)}$$

$$\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{r_1,\cdots,r_v=0}^{\infty}\sum_{G_i=1}^{M_i}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^{s}\phi_i z_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{s}g_i}}{\prod_{j=1}^{s}\delta_{G^{(i)}}^{(i)}\prod_{i=1}^{s}g_i!}\frac{y_1^{K_1}\cdots y_u^{K_u}t_1^{r_1}\cdots t_v^{r_v}}{r_1!\cdots r_v!}A'B'$$

$$exp[-C_2(\upsilon+1)(\alpha+\beta+\upsilon)t]P_\upsilon^{(\alpha,\beta)}(x)\ F(\rho+\alpha,\sigma+\beta;N) \tag{7.6}$$

under the same conditions that (6.2) with $\gamma=0$

8. Aleph-function of two variables

If $r=2$, the multivariable Aleph-function reduces to Aleph-function of two variables defined by Sharma [11], the solution of problem is :

$$\theta(x,t)=2^{\rho+\sigma}\sum_{\upsilon=0}^{\infty}\sum_{N=0}^{\upsilon}\frac{(\alpha-\gamma+\beta+2\upsilon+1)\Gamma(\alpha+\upsilon+1)}{N!(\alpha+1)_N\upsilon!\Gamma(\upsilon+\alpha-\gamma+\beta+1)\Gamma^2(-\upsilon)\Gamma(\upsilon-\gamma+\beta+1)}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{r_1,\cdots,r_v=0}^{\infty}$$

$$\sum_{h_i=1}^{M_i}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{j=1}^{s}\phi_j z_i^{\eta_{h_j,g_j}}(-)^{\sum_{j=1}^{s}g_j}}{\prod_{j=1}^{s}\delta_{h^{(i)}}^{(j)}\prod_{i=1}^{s}g_i!}\frac{y_1^{K_1}\cdots y_u^{K_u}t_1^{r_1}\cdots t_v^{r_v}}{r_1!\cdots r_v!}exp[C_2(\upsilon+1)(\gamma-\alpha-\beta-\upsilon)t]A'B'$$

$$G_{2,2}^{1,2}\left[\begin{array}{c|c}\frac{1}{2}(\text{x-1}) & \begin{array}{c}1+\upsilon,\gamma-\upsilon-\alpha-\beta\\ .\\ 0,\ \text{-}\alpha\end{array}\end{array}\right]F_2(\rho+\alpha,\sigma+\beta;N) \tag{8.1}$$

$$F_2(\rho+\alpha,\sigma+\beta;N) = \aleph^{0,\mathbf{n}+2:V}_{p_i+2,q_i+1,\tau_i;R:W}\left(\begin{array}{c} \mathrm{Z}_1 2^{h_1+k_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Z}_2 2^{h_2+k_2} \end{array} \middle| \begin{array}{c} (\text{-}\sigma-\beta-\sum_{i=1}^u f_iK_i-\sum_{i=1}^v a_ir_i-\sum_{i=1}^s c_i\eta_{G_i,g_i}:h_1,h_2), \\ \cdot \\ \cdot \\ \cdots \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-}\rho-\alpha-N-\sum_{i=1}^u w_iK_i-\sum_{i=1}^v b_ir_i-\sum_{i=1}^s d_i\eta_{G_i,g_i}:k_1,k_2), A \\ \cdot \\ \cdot \\ (\text{-}1\text{-}\sigma-\alpha-\beta-\rho-N-\sum_{i=1}^u(f_i+w_i)K_i-\sum_{i=1}^v(a_i+b_i)r_i-\sum_{i=1}^s(c_i+d_i)\eta_{G_i,g_i}:h_1+k_1,h_2+k_2), B \end{array}\right) \tag{8.2}$$

under the same condition that (6.2) with $r=2$.

9. I_function of two variables defined by Sharma

If $r=2$ and $\tau_i,\tau_{i'},\tau_{i''}\rightarrow 1$, the multivariable Aleph-function reduces to I-function defined by Sharma et al [10]. The solution of the problem is :

$$\theta(x,t) = 2^{\rho+\sigma}\sum_{\upsilon=0}^{\infty}\sum_{N=0}^{\upsilon}\frac{(\alpha-\gamma+\beta+2\upsilon+1)\Gamma(\alpha+\upsilon+1)}{N!(\alpha+1)_N\upsilon!\Gamma(\upsilon+\alpha-\gamma+\beta+1)\Gamma^2(-\upsilon)\Gamma(\upsilon-\gamma+\beta+1)}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{r_1,\cdots,r_v=0}^{\infty}$$

$$\sum_{G_i=1}^{M_i}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^s\phi_i z_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^s g_i}}{\prod_{j=1}^s\delta_{G^{(i)}}^{(i)}\prod_{i=1}^s g_i!}\frac{y_1^{K_1}\cdots y_u^{K_u}t_1^{r_1}\cdots t_v^{r_v}}{r_1!\cdots r_v!}exp[C_2(\upsilon+1)(\gamma-\alpha-\beta-\upsilon)t]A'B'$$

$$G^{1,2}_{2,2}\left[\ \tfrac{1}{2}(\text{x-1})\ \middle|\ \begin{array}{c} 1{+}\upsilon,\gamma-\upsilon-\alpha-\beta \\ \cdot \\ 0,\text{-}\alpha \end{array}\right]\ F'_2(\rho+\alpha,\sigma+\beta;N) \tag{9.1}$$

$$F'_2(\rho+\alpha,\sigma+\beta;N) = I^{0,\mathbf{n}+2:V}_{p_i+2,q_i+1;R:W}\left(\begin{array}{c} \mathrm{Z}_1 2^{h_1+k_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Z}_2 2^{h_2+k_2} \end{array} \middle| \begin{array}{c} (\text{-}\sigma-\beta-\sum_{i=1}^u f_iK_i-\sum_{i=1}^v a_ir_i-\sum_{i=1}^s c_i\eta_{G_i,g_i}:h_1,h_2), \\ \cdot \\ \cdot \\ \cdots \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-}\rho-\alpha-N-\sum_{i=1}^u w_iK_i-\sum_{i=1}^v b_ir_i-\sum_{i=1}^s d_i\eta_{G_i,g_i}:k_1,k_2), A \\ \cdot \\ \cdot \\ (\text{-}1\text{-}\sigma-\alpha-\beta-\rho-N-\sum_{i=1}^u(f_i+w_i)K_i-\sum_{i=1}^v(a_i+b_i)r_i-\sum_{i=1}^s(c_i+d_i)\eta_{G_i,g_i}:h_1+k_1,h_2+k_2), B \end{array}\right) \tag{9.2}$$

under the same condition that (6.2) with $r=2$ and $\tau_i,\tau_{i'},\tau_{i''}\rightarrow 1$.

10. I-function of two variables defined by Rathie

If $s=2$, the multivariable I-function defined by Prathima et al [7] reduces to I-function defined by Rathie et al [8]. The solution of problem is :

$$\theta(x,t) = 2^{\rho+\sigma}\sum_{\upsilon=0}^{\infty}\sum_{N=0}^{\upsilon}\frac{\Gamma(2\upsilon+2)(-\upsilon)_N(\upsilon+1)_N}{N!(\alpha+1)_N\Gamma(\upsilon-\alpha+1)\Gamma(-\upsilon)\Gamma(2\upsilon+1)}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{r_1,\cdots,r_v=0}^{\infty}$$

$$\sum_{G_i=1}^{M_i}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{j=1}^2\phi_j z_i^{\eta_{G_j,g_j}}(-)^{\sum_{j=1}^2 g_j}}{\prod_{j=1}^2\delta_{G^{(i)}}^{(j)}\prod_{i=1}^2 g_i!}\frac{y_1^{K_1}\cdots y_u^{K_u}t_1^{r_1}\cdots t_v^{r_v}}{r_1!\cdots r_v!}exp[-C_2(\upsilon+1)\upsilon t]A'B'$$

$$G_{2,2}^{1,2}\left[\begin{array}{c|c}\frac{1}{2}(\text{x-1}) & \begin{array}{c}1+\upsilon,-\upsilon\\ .\\ 0,\text{-}\alpha\end{array}\end{array}\right]F_2''(\rho+\alpha,\sigma+\beta;N) \tag{10.1}$$

where

$$F_2''(\rho+\alpha,\sigma+\beta;N) = \aleph_{p_i+2,q_i+1,\tau_i;R:W}^{0,\mathbf{n}+2:V}\left(\begin{array}{c|c}\mathrm{Z}_1 2^{h_1+k_1} & (\text{-}\sigma-\beta-\sum_{i=1}^u f_iK_i-\sum_{i=1}^v a_ir_i-\sum_{i=1}^2 c_i\eta_{G_i,g_i}:h_1,\cdots,h_r),\\ . & .\\ . & .\\ . & \cdots\\ \mathrm{Z}_r 2^{h_r+k_r} & \end{array}\right.$$

$$\left.\begin{array}{c}(\text{-}\rho-\alpha-N-\sum_{i=1}^u w_iK_i-\sum_{i=1}^v b_ir_i-\sum_{i=1}^2 d_i\eta_{G_i,g_i}:k_1,\cdots,k_r),A\\ .\\ .\\ (\text{-1-}\sigma-\sigma-\alpha-\rho-N-\sum_{i=1}^u(f_i+w_i)K_i-\sum_{i=1}^v(a_i+b_i)r_i-\sum_{i=1}^2(c_i+d_i)\eta_{G_i,g_i}:h_1+k_1,\cdots,h_r+k_r),B\end{array}\right) \tag{10.2}$$

under the same condition that (6.2) with $s=2$

Remark

Singh et al [12] have obtained the solution of the same problem wi th the modified multivariable H-function defined by Prasad et al [6].

10. Conclusion

Similarly, specializing the coefficient $A[N_1,K_1;\cdots;N_u,K_u]$ and parameters of multivariable I-function defined by Prathima et al [2], the multivariable A-function defined by Gautam et al [3], the Srivastava-Daoust function and the Meijer's G-function, we can obtain large number of results involving various special functions of one and several variables useful in Mathematics analysis, Applied Mathematics, Physics and Mechanics, for example, the application to temperature distribution in a moving medium.

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