Int. Jr. of Mathematical Sciences & Applications
Vol. 8, No. 1, (January-June, 2018)
Copyright: Mind Reader Publications
ISSN No: 2230-9888
www.ijmsa.yolasite.com

Multiple integrals operators and multivariable Aleph-functions

Frédéric Ayant

*Teacher in High School , France
E-mail :fredericayant@gmail.com

ABSTRACT
In thi paper we obtained generalized fractional integrals concerning the product of the multivariable Aleph-functions, general class of polynomials of one and several variables and sequences of functions in the form of four theorems. At the end, we shall two corollaries.

KEYWORDS : Aleph-function of several variables, fractional integral operators, general class of polynomials, sequence of functions.

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20

1 .Introduction and preliminaries.

A K. Sharma and S.C. Sharma [5] have studied four theorems concerning generalized fractional integrals involving the product of two multivariable H-functions and general classes of polynomials of one and several variables. The aim of this paper is to establish four theorems concerning multiple integrals operators of the product of two multivariable Aleph-functions, classes of polynomials of one and several variables and sequence of functions.

For this study, we need the following series formula for the general sequence of functions introduced by Agrawal and Chaubey [1] and was established by Salim [3].

$$R_n^{\alpha,\beta}[x;E,F,g,h;p,q;\gamma;\delta;e^{-\mathfrak{s}x^{\tau}}]=\sum_{w,v,u,t',e,k_1,k_2}\psi(w,v,u,t',e,k_1,k_2)x^Q \tag{1.1}$$

where $\psi(w,v,u,t',e,k_1,k_2)=\dfrac{(-)^{t'+w+k_2}(-v)_u(-t')_e(\alpha)_t l^n}{w!v!u!t'!e!l'_n k_1!k_2!}\dfrac{\mathfrak{s}^{w+k_1}F^{\gamma n-t'}}{(1-\alpha-t')_e}(-\alpha-\gamma n)_e(-\beta-\delta n)_v$

$$g^{v+k_2}h^{\delta n-v-k_2}\,(v-\delta n)_{k_2}E^{t'}\left(\frac{pe+\tau w+\lambda+qu}{l}\right)_n \tag{1.2}$$

and $\displaystyle\sum_{w,v,u,t',e,k_1,k_2}=\sum_{w=0}^{\infty}\sum_{v=0}^{n}\sum_{u=0}^{v}\sum_{t'=0}^{n}\sum_{e=0}^{t'}\sum_{k_1,k_2=0}^{\infty}$

The infinite series in the right hand side of (1.3) is absolutely convergent and $Q=ln+qv+pt'+\tau w+\tau k_1+k_2q$
We shall note $R_n^{\alpha,\beta}[x;E,F,g,h;p,q;\gamma;\delta;e^{-\mathfrak{s}x^{\tau}}]=R_n^{\alpha,\beta}(x)$ (1.3)

The generalized polynomials of multivariables defined by Srivastava [8, p.185, Eq.(7)]], is given in the following manner :

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_\mathfrak{v}}[y_1,\cdots,y_v]=\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_\mathfrak{v}]}\frac{(-N_1)_{\mathfrak{M}_1K_1}}{K_1!}\cdots\frac{(-N_v)_{\mathfrak{M}_\mathfrak{v}K_v}}{K_v!}A[N_1,K_1;\cdots;N_v,K_v]y_1^{K_1}\cdots y_v^{K_v} \tag{1.4}$$

where $\mathfrak{M}_1,\cdots,\mathfrak{M}_\mathfrak{v}$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_v,K_v]$ are arbitrary constants, real or complex.

We shall note

$$a_v=\frac{(-N_1)_{\mathfrak{M}_1K_1}}{K_1!}\cdots\frac{(-N_v)_{\mathfrak{M}_\mathfrak{v}K_v}}{K_v!}A[N_1,K_1;\cdots;N_v,K_v] \tag{1.5}$$

The general class of polynomials $S_N^M(x)$ studied early by Srivastava [7, p.1, Eq.(1)] is defined by

$$S_N^M(x) = \sum_{K=0}^{[N/M]} \frac{(-N)_{MK}}{K!} A_{N,K} x^K \tag{1.6}$$

The Aleph-function of several variables is an extension the multivariable I-function defined by Sharma and Ahmad [6] , itself is a generalisation of G and H-functions of several variables defined by Srivastava et Panda [9,10,11]. The multiple Mellin-Barnes integral occurring in this paper will be referred to as the multivariables Aleph-function of $r-$ variables throughout our present study and will be defined and represented as follows (see Ayant [2]).

We have : $\aleph(z_1,\cdots,z_r) = \aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}} \left(\begin{array}{c|c} \mathrm{z}_1 & [(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}], \\ \cdot & \cdot \\ \cdot & \\ \cdot & \cdots\cdots\cdots\cdots , \\ \mathrm{z}_r & \end{array} \right.$

$$\begin{array}{l} [\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}] : [(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots; \\ \quad\cdot \qquad\qquad\qquad\qquad\qquad \cdot \\ [\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{1,q_i}] : [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots; \end{array}$$

$$\left. \begin{array}{l} [(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ \cdot \\ [(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{array} \right) = \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.7}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.8}$$

and $\theta_k(s_k) = \dfrac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]}$ (1.9)

For more details, see Ayant [2]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|arg z_k| < \frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)} - \tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} - \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)} + \sum_{j=1}^{m_k}\delta_j^{(k)} - \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} > 0 \tag{1.10}$$

with $k=1,\cdots,r, i=1,\cdots,R$, $i^{(k)}=1,\cdots,R^{(k)}$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function. We shall note this function $\aleph_1(z_1,\cdots,z_r)$.

We define the second Aleph-function

$$\aleph(z'_1,\cdots,z'_r)=\aleph^{0,n':m'_1,n'_1,\cdots,m'_r,n'_r}_{p'_i,q'_i,\iota_i;R':p'_{i^{(1)}},q'_{i^{(1)}},\iota_{i^{(1)}};R'^{(1)};\cdots;p'_{i^{(r)}},q'_{i^{(r)}};\iota_{i^{(r)}};R'^{(s)}}\left(\begin{array}{c|c} \mathrm{z}'_1 & [(\mathrm{u}_j;\mu_j^{(1)},\cdots,\mu_j^{(r)})_{1,n'}], \\ \cdot & \cdot \\ \cdot & \\ \cdot & \cdots\cdots\cdots\cdots\cdots\cdots , \\ \mathrm{z}'_r & \end{array}\right.$$

$$\begin{array}{l} [\iota_i(u_{ji};\mu_{ji}^{(1)},\cdots,\mu_{ji}^{(r)})_{n'+1,p'_i}]:[(\mathrm{a}_j^{(1)});\alpha_j^{(1)})_{1,n'_1}],[\iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)};\alpha_{ji^{(1)}}^{(1)})_{n'_1+1,p_i'^{(1)}}];\cdots; \\ \cdot \qquad\qquad\qquad\qquad\qquad\qquad \cdot \\ [\iota_i(v_{ji};v_{ji}^{(1)},\cdots,v_{ji}^{(r)})_{1,q'_i}]:\quad[(\mathrm{b}_j^{(1)});\beta_j^{(1)})_{1,m'_1}],[\iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)};\beta_{ji^{(1)}}^{(1)})_{m'_1+1,q_i'^{(1)}}];\cdots; \end{array}$$

$$\left.\begin{array}{l} [(\mathrm{a}_j^{(r)});\alpha_j^{(r)})_{1,n'_r}],[\iota_{i^{(r)}}(a_{ji^{(r)}}^{(r)};\alpha_{ji^{(r)}}^{(r)})_{n'_r+1,p_i'^{(r)}}] \\ \cdot \\ [(\mathrm{b}_j^{(r)});\beta_j^{(r)})_{1,m'_r}],[\iota_{i^{(r)}}(b_{ji^{(r)}}^{(r)};\beta_{ji^{(r)}}^{(r)})_{m'_r+1,q_i'^{(r)}}] \end{array}\right)=\frac{1}{(2\pi\omega)^r}\int_{L'_1}\cdots\int_{L'_r}\zeta(t_1,\cdots,t_r)\prod_{k=1}^{r}\phi_k(t_k)z_k'^{\,t_k}\,\mathrm{d}t_1\cdots\mathrm{d}t_r \tag{1.11}$$

with $\omega=\sqrt{-1}$

$$\zeta(t_1,\cdots,t_r)=\frac{\prod_{j=1}^{n'}\Gamma(1-u_j+\sum_{k=1}^{r}\mu_j^{(k)}t_k)}{\sum_{i=1}^{R'}[\iota_i\prod_{j=n'+1}^{p'_i}\Gamma(u_{ji}-\sum_{k=1}^{r}\mu_{ji}^{(k)}t_k)\prod_{j=1}^{q'_i}\Gamma(1-v_{ji}+\sum_{k=1}^{r}v_{ji}^{(k)}t_k)]} \tag{1.12}$$

and $$\phi_k(t_k)=\frac{\prod_{j=1}^{m'_k}\Gamma(b_j^{(k)}-\beta_j^{(k)}t_k)\prod_{j=1}^{n'_k}\Gamma(1-a_j^{(k)}+\alpha_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R'^{(k)}}[\iota_{i^{(k)}}\prod_{j=m'_k+1}^{q'_{i^{(k)}}}\Gamma(1-b_{ji^{(k)}}^{(k)}+\beta_{ji^{(k)}}^{(k)}t_k)\prod_{j=n'_k+1}^{p'_{i^{(k)}}}\Gamma(a_{ji^{(k)}}^{(k)}-\alpha_{ji^{(k)}}^{(k)}s_k)]} \tag{1.13}$$

For more details, see Ayant [2]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable H-function given by as :
The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz'_k|<\frac{1}{2}B_i^{(k)}\pi$, where

$$B_i^{(k)}=\sum_{j=1}^{n'}\mu_j^{(k)}-\iota_i\sum_{j=n'+1}^{p'_i}\mu_{ji}^{(k)}+\sum_{j=1}^{n'_k}\alpha_j^{(k)}-\iota_{i^{(k)}}\sum_{j=n'_k+1}^{p'_{i^{(k)}}}\alpha_{ji^{(k)}}^{(k)}+\sum_{j=1}^{m'_k}\beta_j^{(k)}-\iota_{i^{(k)}}\sum_{j=m'_k+1}^{q'_{i^{(k)}}}\beta_{ji^{(k)}}^{(k)}>0 \tag{1.14}$$

with $k=1,\cdots,r,i=1,\cdots,R'$, $i^{(k)}=1,\cdots,R'^{(k)}$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\alpha'_1},\cdots,|z_r|^{\alpha'_r}\,),max(\,|z_1|,\cdots,|z_r|\,)\to 0$

$\aleph(z_1,\cdots,z_r)\;=0(\,|z_1|^{\beta'_1},\cdots,|z_r|^{\beta'_r}\,),min(\,|z_1|,\cdots,|z_r|\,)\to\infty$

where $k=1,\cdots,r:\alpha'_k=min[Re(b_j^{(k)}/\beta_j^{(k)})],j=1,\cdots,m'_k$ and

$\beta'_k=max[Re((a_j^{(k)}-1)/\alpha_j^{(k)})],j=1,\cdots,n'_k$

We shall note $\aleph(z'_1,\cdots,z'_r)=\aleph_2(z'_1,\cdots,z'_r)$

2. Generalized multiple fractional integrals operators

The fractional integral operators defined and represented in the following manner by Sharma [4] are studied in this paper ([1], p. 149-152, Eq. 5.2(1)-5.3(6)) :

Generalized single fractional integral operators :

$$R\left[f(x)\right]=\left[\begin{array}{ccc}\eta,\alpha:k & & \\ . & ; & f(x)\\ \mathrm{a}_1,\cdots,a_r:x & & \end{array}\right]=kx^{-\eta-k\alpha-1}\int_0^x t^{\eta}(x^k-t^k)^{\alpha}\aleph_1(a_1V,\cdots,a_rV)f(t)\mathrm{d}t \tag{2.1}$$

and

$$\mathbb{K}\left[f(x)\right]=\left[\begin{array}{ccc}\eta,\alpha:\mu & & \\ . & ; & f(x)\\ \mathrm{a}_1,\cdots,a_r:x & & \end{array}\right]=\mu x^{\eta}\int_x^{\infty} t^{-\eta-\mu\alpha-1}(t^{\mu}-x^{\mu})^{\alpha}\aleph_1(a_1V',\cdots,a_rV')f(t)\mathrm{d}t \tag{2.2}$$

where $\aleph_1(.)$ is given by (1.7) and

$$V=\left(\frac{t^k}{x^k}\right)^{\rho}\left(1-\frac{t^k}{x^k}\right)^{\sigma} \tag{2.3}$$

$$V'=\left(\frac{x^{\mu}}{t^{\mu}}\right)^{\rho'}\left(1-\frac{x^{\mu}}{t^{\mu}}\right)^{\sigma'} \tag{2.4}$$

The fractional integral operators $R\left[f(x)\right]$ and $\mathbb{K}\left[f(x)\right]$ exit under the following set of conditions :

$1\leqslant p,q<\infty,\dfrac{1}{p}+\dfrac{1}{q}=1;f(x)\in L_p(0,\infty),$

$$Re\,(\eta)+k\rho\sum_{i=1}^{r}\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>max[-p^{-1},-q^{-1}];\;Re\,(\eta)+\mu\rho'\sum_{i=1}^{n}\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>max[-p^{-1},-q^{-1}]$$

$$Re\,(\alpha)+\sigma\sum_{i=1}^{r}\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-q^{-k};\;\;Re\,(\alpha)+\sigma'\sum_{i=1}^{r}\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-q^{-\mu}$$

Generalized multiple fractional integral operators :

$$\mathbf{R}\left[f(x_1,\cdots,x_r)\right]=\left[\begin{array}{ccc}\eta_1\cdots,\eta_r,\alpha_1,\cdots,\alpha_r:k_1,\cdots,k_r & & \\ . & ; & f(x_1,\cdots,x_r)\\ \mathrm{a}_1,\cdots,a_r:x_1,\cdots,x_r & & \end{array}\right]$$

$$=\prod_{i=1}^{r}k_ix^{-\eta_i-k_i\alpha_i-1}\int_0^{x_1}\cdots\int_0^{x_r}\prod_{i=1}^{r}\{t_i^{\eta_i}(x_i^{k_i}-t_i^{k_i})^{\alpha_i}\aleph_1(a_1V_1,\cdots,a_rV_r)f(t_1,\cdots,t_r)\mathrm{d}t_1\cdots\mathrm{d}t_r \tag{2.5}$$

and

$$\mathbf{K}\left[f(x_1,\cdots,x_r)\right]=\left[\begin{array}{ccc}\eta_1\cdots,\eta_r,\alpha_1,\cdots,\alpha_r:\mu_1,\cdots,\mu_r & & \\ . & ; & f(x_1,\cdots,x_r)\\ \mathrm{a}_1,\cdots,a_r:x_1,\cdots,x_r & & \end{array}\right]$$

$$=\prod_{i=1}^{r}\mu_ix^{\eta_i}\int_{x_1}^{\infty}\cdots\int_{x_r}^{\infty}\prod_{i=1}^{r}\{t_i^{-\eta_i-\mu_i\alpha_i-1}(t_i^{\mu_i}-x_i^{\mu_i})^{\alpha_i}\aleph_1(a_1V_1',\cdots,a_rV_r')f(t_1,\cdots,t_r)\mathrm{d}t_1\cdots\mathrm{d}t_r \tag{2.6}$$

where, for $i = 1, \cdots, r$

$$V_i = \left(\frac{t_i^{k_i}}{x_i^{k_i}}\right)^{\rho_i} \left(1 - \frac{t_i^{k_i}}{x_i^{k_i}}\right)^{\sigma_i} \tag{2.7}$$

$$V'_i = \left(\frac{x_i^{\mu_i}}{t_i^{\mu_i}}\right)^{\rho'_i} \left(1 - \frac{x_i^{\mu_i}}{t_i^{\mu_i}}\right)^{\sigma'_i} \tag{2.8}$$

The fractional integrals operators $\mathbf{R}\left[f(x_i)\right]$ and $\mathbf{K}\left[f(x_i)\right]$ exit under the following set of conditions :

$$1 \leqslant p_i, q_i < \infty, \frac{1}{p_i} + \frac{1}{q_i} = 1; f(x_1, \cdots, x_r) \in L_{p_i}(0, \infty), i = 1, \cdots, r$$

$$\sum_{i=1}^{r} \left[Re\,(\eta_i) + k_i \rho_i \min_{1 \leqslant j \leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > max[-p_i^{-1}, -q_i^{-1}]$$

$$\sum_{i=1}^{r} \left[Re\,(\eta_i) + \mu_i \rho'_i \min_{1 \leqslant j \leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > max[-p_i^{-1}, -q_i^{-1}]$$

$$\sum_{i=1}^{r} \left[Re\,(k_i \alpha_i) + k_i \sigma_i \min_{1 \leqslant j \leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > max\left\{-\frac{1}{q_i}\right\};$$

$$\sum_{i=1}^{r} \left[Re\,(\mu_i \alpha_i) + \mu_i \sigma'_i \min_{1 \leqslant j \leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > max\left\{-\frac{1}{q_i}\right\}$$

3. Main Results

In this section, we calculate four images involving the product of multivariable Aleph-function, sequence of functions, classes of polynomials (one and several variables). We establish four theorems in this section.

Theorem 1

If $R\left[f(x)\right]$ be given by (2.1), we have

$$R\left[x^{\lambda} R_n^{(\alpha,\beta)}(ax^{\zeta}) S_N^M(cx^{\gamma}) S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[c_1 x^{\gamma_1}, \cdots, c_v x^{\gamma_v}] \aleph_2(b_1 x^{\beta_1}, \cdots, b_r x^{\beta_r})\right]$$

$$\sum_{w,v,u,t',e,k_1,k_2} \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} \sum_{K=0}^{[N/M]} \frac{(-N)_{MK}}{K!} A_{N,K}\, a_v\, \psi(w,v,u,t',e,k_1,k_2) \prod_{i=1}^{v} c_i^{k_i} x^{\gamma_i K_i} x^{\lambda+\gamma K+Q\zeta} c^K a^Q$$

$$\aleph_{U_{21}:W}^{0,\mathbf{n}+n'+2;V}\left(\begin{array}{c|c} \mathrm{b}_1 x^{\beta_1} & \mathrm{A}_1, \mathbf{A} \\ . & . \\ . & . \\ \mathrm{b}_r x^{\beta_r} & . \\ \mathrm{a}_1 & . \\ . & \mathrm{B}_1, \mathbf{B} \\ . & \\ \mathrm{a}_r & \end{array}\right) \tag{3.1}$$

where

$$U_{21} = 2,1; p'_i, q'_i, \iota_i; R'; p_i, q_i, \tau_i; R \tag{3.2}$$

$$V = m'_1, n'_1; \cdots ; m'_r, n'_r; m_1, n_1; \cdots ; m_r, n_r \tag{3.3}$$

$$W' = p'_{i^{(1)}}, q'_{i^{(1)}}, \iota_{i^{(1)}}; R'^{(1)}; \cdots ; p'_{i^{(r)}}, q'_{i^{(r)}}, \iota_{i^{(r)}}; R'^{(r)}; p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}; \cdots ; p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{3.4}$$

$$A_1 = \left(1 - \frac{\eta+\lambda+\gamma K+\zeta Q+\sum_{i=1}^{v}\gamma_i K_i+1}{k}; \frac{\beta_1}{k}, \cdots, \frac{\beta_r}{k}, \underbrace{\rho, \cdots, \rho}_{r}\right), (-\alpha; \underbrace{0, \cdots, 0}_{r}, \underbrace{\sigma, \cdots, \sigma}_{r}) \tag{3.5}$$

$$\mathbf{A} = \{(u_j; \mu_j^{(1)}, \cdots, \mu_j^{(r)}, \underbrace{0, \cdots, 0}_{r})_{1,n'}\}, \{\iota_i(u_{ji}; \mu_{ji}^{(1)}, \cdots, \mu_{ji}^{(r)}, \underbrace{0, \cdots, 0}_{r})_{n'+1,p'_i}\} :$$

$$\{(a_j; \underbrace{0, \cdots, 0}_{r}, \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji}; \underbrace{0, \cdots, 0}_{r}, \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{n+1,p_i}\}$$

$$\{(a_j^{(1)}; \alpha_j^{(1)})_{1,n'_1}, \iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)}; \alpha_{ji^{(1)}}^{(1)})_{n'_1+1,p'_{i^{(1)}}}\}; \cdots ; \{(a_j^{(r)}; \alpha_j^{(r)})_{1,n'_r}, \iota_{i^{(r)}}(a_{ji^{(r)}}^{(r)}; \alpha_{ji^{(r)}}^{(r)})_{n'_r+1,p'_{i^{(r)}}}\};$$

$$\{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\}, \tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\} ; \cdots ; \{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{3.6}$$

$$B_1 = \left(-\alpha - \frac{\eta+\lambda+\gamma K+\zeta Q+\sum_{i=1}^{v}\gamma_i K_i+1}{k}; \frac{\beta_1}{k}, \cdots, \frac{\beta_r}{k}, \underbrace{\rho+\sigma, \cdots, \rho+\sigma}_{r}\right) \tag{3.7}$$

$$\mathbf{B} = \{\iota_i(v_{ji}; v_{ji}^{(1)}, \cdots, v_{ji}^{(r)}, \underbrace{0, \cdots, 0}_{r})_{m'+1,q'_i}\}, \{\tau_i(b_{ji}; \underbrace{0, \cdots, 0}_{r}, \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{m+1,q_i}\} :$$

$$\{(b_j^{(1)}; \beta_j^{(1)})_{1,m'_1}, \iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)}; \beta_{ji^{(1)}}^{(1)})_{m'_1+1,q'_{i^{(1)}}}\}; \cdots ; \{(b_j^{(r)}; \beta_j^{(r)})_{1,m'_r}, \iota_{i^{(r)}}(\beta_{ji^{(r)}}^{(r)}; \beta_{ji^{(r)}}^{(r)})_{m'_r+1,q'_{i^{(r)}}}\};$$

$$\{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\} ; \cdots ; \{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{3.8}$$

Provided that

$$Re(\eta+\lambda+1) + \sum_{i=1}^{r}\left[\rho k \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) + \beta_i \min_{1\leqslant j\leqslant m'_i} Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > 0 \text{ and}$$

$$Re(\alpha+1) + \sigma\sum_{i=1}^{r} \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0$$

$|arg(a_k V_k)| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.10)

$|arg(b_k x^{\beta_k})| < \frac{1}{2}B_i^{(k)}\pi$, where $B_i^{(k)}$ is defined by (1.14) and the multiple series in the left-hand side of (3.1)converge absolutely.

Theorem 2

If $\mathbb{K}\left[f(x)\right]$ be given by (2.2), we have

$$\mathbb{K}\left[x^{-\lambda}R_n^{(\alpha,\beta)}(ax^{\zeta})S_N^M(cx^{\gamma})S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[c_1x^{\gamma_1},\cdots,c_vx^{\gamma_v}]\aleph_2(b_1x^{-\beta_1},\cdots,b_rx^{-\beta_r})\right.$$

$$\sum_{w,v,u,t',e,k_1,k_2}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{K=0}^{[N/M]}\frac{(-N)_{MK}}{K!}A_{N,K}\,a_v\,\psi(w,v,u,t',e,k_1,k_2)\prod_{i=1}^{v}c_i^{k_i}x^{\gamma_iK_i}x^{-\lambda+\gamma K+Q\zeta}c^Ka^Q$$

$$\aleph_{U_{21}:W}^{0,\mathbf{n}+n'+2;V}\left(\begin{array}{c|c} \mathrm{b}_1x^{-\beta_1} & \\ . & \mathrm{A}_2,\mathbf{A} \\ . & . \\ \mathrm{b}_rx^{-\beta_r} & . \\ \mathrm{a}_1 & . \\ . & . \\ . & \mathrm{B}_2,\mathbf{B} \\ \mathrm{a}_r & \end{array}\right) \tag{3.9}$$

where

$$A_2=\left(1-\frac{\eta+\lambda-\gamma K-\zeta Q-\sum_{i=1}^{v}\gamma_iK_i}{\mu};\frac{\beta_1}{\mu},\cdots,\frac{\beta_r}{\mu},\underbrace{\rho',\cdots,\rho'}_{r}\right),(-\alpha;\underbrace{0,\cdots,0}_{r},\underbrace{\sigma',\cdots,\sigma'}_{r}) \tag{3.10}$$

$$B_2=\left(-\alpha-\frac{\eta+\lambda-\gamma K-\zeta Q-\sum_{i=1}^{v}\gamma_iK_i}{\mu};\frac{\beta_1}{\mu},\cdots,\frac{\beta_r}{\mu},\underbrace{\rho'+\sigma',\cdots,\rho'+\sigma'}_{r}\right) \tag{3.11}$$

U_{21},V,W,$\mathbf{A}$ and $\mathbf{B}$ are defined respectively by (3.2), (3.3), (3.4), (3.6) and (3.8).

Provided that

$$Re\left(\eta+\lambda-\gamma[n/m]-\sum_{i=1}^{v}\gamma_i[N_i/M_i]-Q\zeta\right)+\sum_{i=1}^{r}\left[\rho'\mu\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)+\beta_i\min_{1\leqslant j\leqslant m_i'}Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right]>0 \text{ and}$$

$$Re(\alpha+1)+\sigma'\sum_{i=1}^{r}\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$$

$|arg(a_kV_k)|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$is defined by (1.10)

$|arg(b_kx^{-\beta_k})|<\frac{1}{2}B_i^{(k)}\pi$, where $B_i^{(k)}$ is defined by (1.14) and the multiple series in the left-hand side of (3.9) converge absolutely.

Theorem 3

If $\mathbf{R}\left[f(x_1,\cdots,x_r)\right]$ be given by (2.5), we obtain

$$\mathbf{R}\left[\prod_{i=1}^{r}x_i^{\lambda_i}R_n^{(\alpha,\beta)}\left(a\prod_{i=1}^{r}x_i^{\zeta_i}\right)S_N^M\left(c\prod_{i=1}^{r}x_i^{\gamma_i'}\right)S_{N_1,\cdots,N_r}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_r}[c_1x_1^{\gamma_1},\cdots,c_rx_r^{\gamma_r}]\aleph_2(b_1x_1^{\beta_1},\cdots,b_rx_r^{\beta_r})\right]=$$

$$\sum_{w,v,u,t',e,k_1,k_2}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_r=0}^{[N_r/\mathfrak{M}_r]}\sum_{K=0}^{[N/M]}\frac{(-N)_{MK}}{K!}A_{N,K}\,a_r\,\psi(w,v,u,t',e,k_1,k_2)\prod_{i=1}^{r}c_i^{k_i}x_i^{\lambda_i+\gamma_iK_i+\gamma'_iK+Q\zeta_i}c^Ka^Q$$

$$\aleph_{U_{rr}:W'}^{0,\mathbf{n}+n'+r;V'}\left(\begin{array}{c|c}\mathrm{b}_1x^{\beta_1} & \mathrm{A}_3,\mathbf{A}'\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{b}_rx^{\beta_r} & \cdot\\ \mathrm{a}_1 & \cdot\\ \cdot & \cdot\\ \cdot & \mathrm{B}_3,\mathbf{B}\\ \mathrm{a}_r & \end{array}\right)\tag{3.12}$$

where

$$U_{rr}=r,r;p'_i,q'_i,\iota_i;R';p_i,q_i,\tau_i;R\tag{3.13}$$

$$V'=m'_1,n'_1;\cdots;m'_r,n'_r;m_1,n_1+1;\cdots;m_r,n_r+1\tag{3.14}$$

$$W'=p'_{i^{(1)}},q'_{i^{(1)}},\iota_{i^{(1)}};R'^{(1)};\cdots;p'_{i^{(r)}},q'_{i^{(r)}},\iota_{i^{(r)}};R'^{(r)};p_{i^{(1)}}+1,q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}}+1,q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)}\tag{3.15}$$

$$A_3=\left(1-\frac{\eta_1+\lambda_1+\gamma'_1K+\zeta_1Q+\gamma_1K_1+1}{k_1};\frac{\beta_1}{k_1},\underbrace{0,\cdots,0}_{r-1},\rho_1,\underbrace{0,\cdots,0}_{r-1}\right),$$

$$\left(1-\frac{\eta_2+\lambda_2+\gamma'_2K+\zeta_2Q+\gamma_2K_2+1}{k_1};0,\frac{\beta_2}{k_2},\underbrace{0,\cdots,0}_{r-2},0,\rho_2,\underbrace{0,\cdots,0}_{r-2}\right),\cdots,$$

$$\left(1-\frac{\eta_r+\lambda_r+\gamma'_rK+\zeta_rQ+\gamma_rK_r+1}{k_r};\underbrace{0,\cdots,0}_{r-1},\frac{\beta_r}{k_r},\underbrace{0,\cdots,0}_{r-1},\rho_r\right)\tag{3.16}$$

$$\mathbf{A}=\{(u_j;\mu_j^{(1)},\cdots,\mu_j^{(r)},\underbrace{0,\cdots,0}_{r})_{1,n'}\},\{\iota_i(u_{ji};\mu_{ji}^{(1)},\cdots,\mu_{ji}^{(r)},\underbrace{0,\cdots,0}_{r})_{n'+1,p'_i}\}:$$

$$\{(a_j;\underbrace{0,\cdots,0}_{r},\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,n}\},\{\tau_i(a_{ji};\underbrace{0,\cdots,0}_{r},\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{n+1,p_i}\}$$

$$\{(a_j^{(1)};\alpha_j^{(1)})_{1,n'_1},\iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)};\alpha_{ji^{(1)}}^{(1)})_{n'_1+1,p'_{i^{(1)}}}\};\cdots;\{(a_j^{(r)};\alpha_j^{(r)})_{1,n'_r},\iota_{i^{(r)}}(a_{ji^{(r)}}^{(r)};\alpha_{ji^{(r)}}^{(r)})_{n'_r+1,p'_{i^{(r)}}}\};$$

$$(-\alpha_1;\sigma_1),\{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\},\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\};\cdots;(-\alpha_r;\sigma_r),\{(c_j^{(r)};\gamma_j^{(r)})_{1,n_r}\},\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\}\tag{3.17}$$

$$B_3=\left(-\alpha_1-\frac{\eta_1+\lambda_1+\gamma'_1K+\zeta_1Q+\gamma_1K_1+1}{k_1};\frac{\beta_1}{k_1},\underbrace{0,\cdots,0}_{r-1},\rho_1+\sigma_1,\underbrace{0,\cdots,0}_{r-1}\right),$$

$$\left(-\alpha_2-\frac{\eta_2+\lambda_2+\gamma'_2K+\zeta_2Q+\gamma_2K_2+1}{k_1};0,\frac{\beta_2}{k_2},\underbrace{0,\cdots,0}_{r-2},0,\rho_2+\sigma_2,\underbrace{0,\cdots,0}_{r-2}\right),\cdots,$$

$$\left(-\alpha_r-\frac{\eta_r+\lambda_r+\gamma'_rK+\zeta_rQ+\gamma_rK_r+1}{k_r};\underbrace{0,\cdots,0}_{r-1},\frac{\beta_r}{k_r},\underbrace{0,\cdots,0}_{r-1},\rho_r+\sigma_r\right) \tag{3.18}$$

$$\mathbf{B}=\{\iota_i(v_{ji};v_{ji}^{(1)},\cdots,v_{ji}^{(r)},\underbrace{0,\cdots,0}_{r})_{m'+1,q'_i}\},\{\tau_i(b_{ji};\underbrace{0,\cdots,0}_{r},\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}\}:$$

$$\{(b_j^{(1)};\beta_j^{(1)})_{1,m'_1},\iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)};\beta_{ji^{(1)}}^{(1)})_{m'_1+1,q'_{i^{(1)}}}\};\cdots;\{(b_j^{(r)};\beta_j^{(r)})_{1,m'_r},\iota_{i^{(r)}}(\beta_{ji^{(r)}}^{(r)};\beta_{ji^{(r)}}^{(r)})_{m'_r+1,q'_{i^{(r)}}}\};$$

$$\{(d_j^{(1)};\delta_j^{(1)})_{1,m_1}\},\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}\;;\cdots;\;\{(d_j^{(r)};\delta_j^{(r)})_{1,m_r}\},\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{3.19}$$

Provided that

$$Re(\eta_i+\lambda_i+1)+\left[\rho_ik_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)+\beta_i\min_{1\leqslant j\leqslant m'_i}Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right]>0$$

and $Re(\alpha_i+1)+\sigma_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$ for $i=1,\cdots,r$

$|arg(a_kV_k)|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$is defined by (1.10)

$|arg(b_kx_k^{\beta_k})|<\frac{1}{2}B_i^{(k)}\pi$, where $B_i^{(k)}$ is defined by (1.14) and the multiple series in the left-hand side of (3.12) converge absolutely.

Theorem 4

If $\mathbf{K}\left[f(x_1,\cdots,x_r)\right]$ be given by (2.6), we obtain

$$\mathbf{K}\left[\prod_{i=1}^r x_i^{\lambda_i}R_n^{(\alpha,\beta)}\left(a\prod_{i=1}^r x_i^{\zeta_i}\right)S_N^M\left(c\prod_{i=1}^r x_i^{\gamma'_i}\right)S_{N_1,\cdots,N_r}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_r}[c_1x_1^{\gamma_1},\cdots,c_rx_r^{\gamma_r}]\aleph_2(b_1x_1^{-\beta_1},\cdots,b_rx_r^{-\beta_r})\right]=$$

$$\sum_{w,v,u,t',e,k_1,k_2}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_r=0}^{[N_r/\mathfrak{M}_r]}\sum_{K=0}^{[N/M]}\frac{(-N)_{MK}}{K!}A_{N,K}\,a_r\,\psi(w,v,u,t',e,k_1,k_2)\prod_{i=1}^r c_i^{k_i}x_i^{-\lambda_i+\gamma_iK_i+\gamma'_iK+Q\zeta_i}a^Qc^K$$

$$\aleph_{U_{rr}:W'}^{0,\mathbf{n}+n'+r;V'}\left(\begin{array}{c|c}\mathrm{b}_1x^{-\beta_1} & \mathrm{A}_4,\mathbf{A}'\\ . & .\\ . & .\\ \mathrm{b}_rx^{-\beta_r} & .\\ \mathrm{a}_1 & .\\ . & \mathrm{B}_4,\mathbf{B}\\ . & \\ \mathrm{a}_r & \end{array}\right) \tag{3.20}$$

Where

$$A_4 = \left(1 - \frac{\eta_1 + \lambda_1 - \gamma_1' K - \zeta_1 Q - \gamma_1 K_1}{\mu_1}; \frac{\beta_1}{\mu_1}, \underbrace{0, \cdots, 0}_{r-1}, \rho_1', \underbrace{0, \cdots, 0}_{r-1}\right),$$

$$\left(1 - \frac{\eta_2 + \lambda_2 - \gamma_2' K - \zeta_2 Q - \gamma_2 K_2}{\mu_2}; 0, \frac{\beta_2}{\mu_2}, \underbrace{0, \cdots, 0}_{r-2}, 0, \rho_2', \underbrace{0, \cdots, 0}_{r-2}\right), \cdots,$$

$$\left(1 - \frac{\eta_r + \lambda_r - \gamma_r' K - \zeta_r Q - \gamma_r K_r}{\mu_r}; \underbrace{0, \cdots, 0}_{r-1}, \frac{\beta_r}{\mu_r}, \underbrace{0, \cdots, 0}_{r-1}, \rho_r'\right) \tag{3.21}$$

$$B_4 = \left(-\alpha_1 - \frac{\eta_1 + \lambda_1 - \gamma_1' K - \zeta_1 Q - \gamma_1 K_1}{\mu_1}; \frac{\beta_1}{\mu_1}, \underbrace{0, \cdots, 0}_{r-1}, \rho_1' + \sigma_1', \underbrace{0, \cdots, 0}_{r-1}\right),$$

$$\left(-\alpha_2 - \frac{\eta_2 + \lambda_2 - \gamma_2' K - \zeta_2 Q - \gamma_2 K_2}{\mu_2}; 0, \frac{\beta_2}{\mu_2}, \underbrace{0, \cdots, 0}_{r-2}, 0, \rho_2' + \sigma_2', \underbrace{0, \cdots, 0}_{r-2}\right), \cdots,$$

$$\left(-\alpha_r - \frac{\eta_r + \lambda_r - \gamma_r' K - \zeta_r Q - \gamma_r K_r}{\mu_r}; \underbrace{0, \cdots, 0}_{r-1}, \frac{\beta_r}{\mu_r}, \underbrace{0, \cdots, 0}_{r-1}, \rho_r' + \sigma_r'\right) \tag{3.22}$$

U_{rr}, V', W' and $\mathbf{A}'$ are defined respectively by (3.13), (3.14), (3.15) and (3.17).

Provided that

$$Re\,(\eta_i + \lambda_i - \gamma_i'[n/m] - \gamma_r[N_r/M_r] - Q\zeta_i) + \left[\rho_i'\mu_i \min_{1 \leqslant j \leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) + \beta_i \min_{1 \leqslant j \leqslant m_i'} Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > 0$$

and $Re(\alpha_i + 1) + \sigma_i' \min_{1 \leqslant j \leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0$ for $i = 1, \cdots, r$

$|arg(a_k V_k)| < \frac{1}{2} A_i^{(k)} \pi$, where $A_i^{(k)}$ is defined by (1.10)

$|arg(b_k x_k^{-\beta_k})| < \frac{1}{2} B_i^{(k)} \pi$, where $B_i^{(k)}$ is defined by (1.14) and the multiple series in the left-hand side of (3.20) converge absolutely.

Proof

To establish the theorem 1, we first use the definition of the operator R from (2.1), then express general classes of polynomials of one and several variables and the sequence of functions $R_n^{(\alpha,\beta)}(.)$ in series with the help of (1.6), (1.4) and (1.1) respectively and multivariable Aleph-function involved therein in terms of Mellin-Barnes type integral contour, integrate term by term and then evaluate the resulting Beta integral to get the desired result. The theorems 2 to 4 are similarly proved.

4. Corollaries

In this section, we consider the theorem 1 and 4. If the classes of polynomials of one and several variables vanish, we get the following results.

Corollary 1

If $R\left[f(x)\right]$ be given by (2.1), we have

$$R\left[x^{\lambda}R_n^{(\alpha,\beta)}(ax^{\zeta})\aleph_2(b_1x^{\beta_1},\cdots,b_rx^{\beta_r})\right]=\sum_{w,v,u,t',e,k_1,k_2}\psi(w,v,u,t',e,k_1,k_2)a^Q$$

$$x^{\lambda+Q\zeta}\aleph^{0,\mathbf{n}+n'+2;V}_{U_{21}:W}\left(\begin{array}{c|c} \mathrm{b}_1x^{\beta_1} & \\ . & \mathrm{A}_1,\mathbf{A} \\ . & . \\ \mathrm{b}_rx^{\beta_r} & . \\ \mathrm{a}_1 & . \\ . & . \\ . & \mathrm{B}_1,\mathbf{B} \\ \mathrm{a}_r & \end{array}\right) \tag{4.1}$$

where

$$A_1=\left(1-\frac{\eta+\lambda+\zeta Q+1}{k};\frac{\beta_1}{k},\cdots,\frac{\beta_r}{k},\underbrace{\rho,\cdots,\rho}_{r}\right),(-\alpha;\underbrace{0,\cdots,0}_{r},\underbrace{\sigma,\cdots,\sigma}_{r}) \tag{4.2}$$

$$B_1=\left(-\alpha-\frac{\eta+\lambda+\zeta Q+1}{k};\frac{\beta_1}{k},\cdots,\frac{\beta_r}{k},\underbrace{\rho+\sigma,\cdots,\rho+\sigma}_{r}\right) \tag{4.3}$$

U_{21},V,W,$\mathbf{A}$ and $\mathbf{B}$ are defined respectively by (3.2), (3.3), (3.4), (3.6) and (3.8), under the same conditions that (3.1).

Corollary 2

If $\mathbf{K}\left[f(x_1,\cdots,x_r)\right]$ be given by (2.5), we obtain

$$\mathbf{K}\left[\prod_{i=1}^{r}x_i^{\lambda_i}R_n^{(\alpha,\beta)}\left(a\prod_{i=1}^{r}x_i^{\zeta_i}\right)\aleph_2(b_1x_1^{-\beta_1},\cdots,b_rx_r^{-\beta_r})\right]=\sum_{w,v,u,t',e,k_1,k_2}\psi(w,v,u,t',e,k_1,k_2)a^Q$$

$$\prod_{i=1}^{v}x_i^{-\lambda_i+Q\zeta_i}\aleph^{0,\mathbf{n}+n'+r;V'}_{U_{rr}:W'}\left(\begin{array}{c|c} \mathrm{b}_1x^{-\beta_1} & \\ . & \mathrm{A}_4,\mathbf{A'} \\ . & . \\ \mathrm{b}_rx^{-\beta_r} & . \\ \mathrm{a}_1 & . \\ . & . \\ . & \mathrm{B}_4,\mathbf{B} \\ \mathrm{a}_r & \end{array}\right) \tag{4.4}$$

where

$$A_4=\left(1-\frac{\eta_1+\lambda_1-\zeta_1Q}{\mu_1};\frac{\beta_1}{\mu_1},\underbrace{0,\cdots,0}_{r-1},\rho'_1,\underbrace{0,\cdots,0}_{r-1}\right),$$

$$\left(1-\frac{\eta_2+\lambda_2-\zeta_2 Q}{\mu_2};0,\frac{\beta_2}{\mu_2},\underbrace{0,\cdots,0}_{r-2},0,\rho'_2,\underbrace{0,\cdots,0}_{r-2}\right),\cdots,$$

$$\left(1-\frac{\eta_r+\lambda_r-\zeta_r Q}{\mu_r};\underbrace{0,\cdots,0}_{r-1},\frac{\beta_r}{\mu_r},\underbrace{0,\cdots,0}_{r-1},\rho'_r\right)$$

$$B_3=\left(-\alpha_1-\frac{\eta_1+\lambda_1+\zeta_1 Q}{\mu_1};\frac{\beta_1}{\mu_1},\underbrace{0,\cdots,0}_{r-1},\rho'_1+\sigma'_1,\underbrace{0,\cdots,0}_{r-1}\right),$$

$$\left(-\alpha_2-\frac{\eta_2+\lambda_2-\zeta_2 Q}{\mu_2};0,\frac{\beta_2}{\mu_2},\underbrace{0,\cdots,0}_{r-2},0,\rho'_2+\sigma'_2,\underbrace{0,\cdots,0}_{r-2}\right),\cdots,$$

$$\left(-\alpha_r-\frac{\eta_r+\lambda_r-\zeta_r Q}{\mu_r};\underbrace{0,\cdots,0}_{r-1},\frac{\beta_r}{\mu_r},\underbrace{0,\cdots,0}_{r-1},\rho'_r+\sigma'_r\right) \tag{4.5}$$

U_{rr},V',W' and $\mathbf{A}'$ are defined respectively by (3.13), (3.14), (3.15) and (3.17).

Provided that

$$Re\,(\eta_i+\lambda_i-Q\zeta_i)+\left[\rho'_i\mu_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)+\beta_i\min_{1\leqslant j\leqslant m'_i}Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right]>0$$

and $Re(\alpha_i+1)+\sigma'_i\min\limits_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$ for $i=1,\cdots,r$

The others conditions are the same that (3.22).

5. Conclusion

In this paper, we have obtained four theorems involving the generalized fractional integrals operators (simple and multiple). The images have been developed in terms of the product of the two multivariable Aleph-functions and general classes of polynomials and sequence of functions in a compact and elegant form with the help of fractional integral operators. The formulae established in this paper are very general nature. Thus, the results established in this research work would serve as a key formulae from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables, simple and multiple integrals operators can be obtained.

References

[1] B.D. Agarwal and J.P. Chaubey, Operational derivation of generating relations for generalized polynomials. Indian J.Pure Appl. Math. 11 (1980), 1155-1157.

[2] F. Y. Ayant, On general multiple Eulerian integrals involving the multivariable Aleph-function, a general class of polynomials and generalized incomplete hypergeometric function, Int Jr. of Mathematical Sciences & Applications, 6(2), (2016), 1011-1030.

[3] Tariq O. Salim, Tariq, A series formula of a generalized class of polynomials associated with Laplace Transform and fractional integral operators. J. Rajasthan Acad. Phy. Sci. 1, No. 3 (2002), 167-176.

[4] S.C. Sharma, Integral transforms and their application, Ph.D. Thesis, University of Rajasthan (1987), 149-152.

[5] A.K. Sharma and S.C. Sharma, On fractional integral operator, Ganita Sandesh, 15(2) (2001), 67-76.

[6] C.K. Sharma and S.S. Ahmad, On the multivariable I-function. Acta ciencia Indica Math , 1994 vol 20,no2, 113-116.

[7] H.M. Srivastava, A contour integral involving Fox's H-function. Indian. J. Math, (14) (1972), 1-6.

[8] H.M. Srivastava, A multilinear generating function for the Konhauser set of biorthogonal polynomials suggested by Laguerre polynomial, Pacific. J. Math. 177(1985), 183-191.

[9] H.M. srivastava, K.C. Gupta and S.P. Goyal, The H-function of one and two variables with applications, South Asian Publisher, New Delhi, 1982.

[10] H.M.Srivastava and R. Panda, Some expansion theorems and generating relations for the H-function of several complex variables. Comment. Math. Univ. St. Paul. 24 (1975),119-137.

[11] H.M.Srivastava and R.Panda, Some expansion theorems and generating relations for the H-function of several complex variables II. Comment. Math. Univ. St. Paul. 25 (1976)., 167-197