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# Multiple integrals operators and multivariable I-functions

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ABSTRACT
In thi paper we obtained generalized fractional integrals concerning the product of the multivariable I-functions, general class of polynomials of one and several variables and generalized Riemann Zeta function in the form of six theorems. At the end, we shall three corollaries.

KEYWORDS : I-function of several variables, fractional integral operators, general class of polynomials, generalized Riemann Zeta function.

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## 1.Introduction and preliminaries.

A. K. Sharma and S.C. Sharma [5], Kant and Koul [2] have studied several theorems concerning generalized fractional integrals involving the product of two multivariable H-functions and general classes of polynomials of one and several variables. The aim of this paper is to establish six theorems concerning simple and multiple integrals operators of the product of two multivariable I-functions, classes of polynomials of one and several variables and generalized Riemann Zeta function.

The Riemann Zeta-function given to by Goyal and Laddha [1] is defined :

$$\phi_\mu(z,\mathbf{s},a,g)=\sum_{g=0}^{\infty}(\mu)_g(a+g)^{-\mathbf{s}}\frac{z^g}{g!} \tag{1.1}$$

where $\mu \geqslant 1, |z|<1, Re(a)>0$ (these conditions will be respected in the paper).

If $\mu=1$, in (1.1) then it reduces to generalized Riemann zeta function.

$$\phi(z,\mathbf{s},a,g)=\sum_{g=0}^{\infty}(a+g)^{-\mathbf{s}}\frac{z^g}{g!} \tag{1.2}$$

where $|z|<1, Re(a)>0$

The generalized polynomials of multivariables defined by Srivastava [7, p.185, Eq.(7)]], is given in the following manner :

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[y_1,\cdots,y_v]=\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\frac{(-N_1)_{\mathfrak{M}_1K_1}}{K_1!}\cdots\frac{(-N_v)_{\mathfrak{M}_vK_v}}{K_v!}A[N_1,K_1;\cdots;N_v,K_v]y_1^{K_1}\cdots y_v^{K_v} \tag{1.3}$$

where $\mathfrak{M}_1,\cdots,\mathfrak{M}_v$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_v,K_v]$ are arbitrary constants, real or complex.

We shall note

$$a_v=\frac{(-N_1)_{\mathfrak{M}_1K_1}}{K_1!}\cdots\frac{(-N_v)_{\mathfrak{M}_vK_v}}{K_v!}A[N_1,K_1;\cdots;N_v,K_v] \tag{1.4}$$

The general class of polynomials $S_N^M(x)$ studied early by Srivastava [6, p.1, Eq.(1)] is defined by

$$S_N^M(x)=\sum_{K=0}^{[N/M]}\frac{(-N)_{MK}}{K!}A_{N,K}x^K \tag{1.5}$$

Another class of polynomials of r variables studied here is defined and represented by Srivastava and Garg ([9], p. 686, Eq. (1.4)) :

$$T_n^{m_1,\cdots,m_r}[x_1,\cdots,x_r]=\sum_{K_1,\cdots,K_r=0}^{M\leqslant n}\left\{(-n)_M A(n,K_1,\cdots,K_r)\prod_{i=1}^{r}\frac{x_i^{K_i}}{K_i!}\right\} \tag{1.6}$$

where $M=\sum_{i=1}^{r} m_i K_i ; K_i, m_i \geqslant 1, i=1,\cdots,r$ and $A(n,K_1,\cdots,K_r)$ are arbitrary constants, real or complex. By suitably specialising the coefficients $A(.)$, the polynomials $\{T_n^{m_1,\cdots,m_r}[x_1,\cdots,x_r]\}$ reduce to certain known polynomials, see ([10], p.459, 455, 382).

The multivariable I-function defined by Prasad [3] generalizes the multivariable H-function studied by Srivastava and Panda [8,11,12]. This function of r-variables is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1,z_2,\cdots,z_r)=I^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;\cdots;p_r,q_r:p^{(1)},q^{(1)};\cdots;p^{(r)},q^{(r)}}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};\cdots; \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};\cdots; \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj})_{1,p_r}:(a_j^{(1)},\alpha_j^{(1)})_{1,p^{(1)}};\cdots;(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}} \\ \\ (\mathrm{b}_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj})_{1,q_r}:(b_j^{(1)},\beta_j^{(1)})_{1,q^{(1)}};\cdots;(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}} \end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\prod_{i=1}^{r}\phi_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.7}$$

where

$$\phi_i(s_i)=\frac{\prod_{j=1}^{m^{(i)}}\Gamma(b_j^{(i)}-\beta_j^{(i)}s_i)\prod_{j=1}^{n^{(i)}}\Gamma(1-a_j^{(i)}+\alpha_j^{(i)}s_i)}{\prod_{j=m^{(i)}+1}^{q^{(i)}}\Gamma(1-b_j^{(i)}+\beta_j^{(i)}s_i)\prod_{j=n^{(i)}+1}^{p^{(i)}}\Gamma(a_j^{(i)}-\alpha_j^{(i)}s_i)}\ ,i=1,\cdots,r \tag{1.8}$$

and

$$\phi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n_2}\Gamma(1-a_{2j}+\sum_{i=1}^{2}\alpha_{2j}^{(i)}s_i)\prod_{j=1}^{n_3}\Gamma(1-a_{3j}+\sum_{i=1}^{3}\alpha_{3j}^{(i)}s_i)\cdots}{\prod_{j=n_2+1}^{p_2}\Gamma(a_{2j}-\sum_{i=1}^{2}\alpha_{2j}^{(i)}s_i)\prod_{j=n_3+1}^{p_3}\Gamma(a_{3j}-\sum_{i=1}^{3}\alpha_{3j}^{(i)}s_i)\cdots}$$

$$\frac{\cdots\prod_{j=1}^{n_r}\Gamma(1-a_{rj}+\sum_{i=1}^{r}\alpha_{rj}^{(i)}s_i)}{\cdots\prod_{j=n_r+1}^{p_r}\Gamma(a_{rj}-\sum_{i=1}^{r}\alpha_{rj}^{(i)}s_i)\prod_{j=1}^{q_2}\Gamma(1-b_{2j}-\sum_{i=1}^{2}\beta_{2j}^{(i)}s_i)}$$

$$\times\frac{1}{\prod_{j=1}^{q_3}\Gamma(1-b_{3j}+\sum_{i=1}^{3}\beta_{3j}^{(i)}s_i)\cdots\prod_{j=1}^{q_r}\Gamma(1-b_{rj}-\sum_{i=1}^{r}\beta_{rj}^{(i)}s_i)} \tag{1.9}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [3]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.7) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_i|<\frac{1}{2}\Omega_i\pi$ , where

$$\Omega_i=\sum_{k=1}^{n^{(i)}}\alpha_k^{(i)}-\sum_{k=n^{(i)}+1}^{p^{(i)}}\alpha_k^{(i)}+\sum_{k=1}^{m^{(i)}}\beta_k^{(i)}-\sum_{k=m^{(i)}+1}^{q^{(i)}}\beta_k^{(i)}+\left(\sum_{k=1}^{n_2}\alpha_{2k}^{(i)}-\sum_{k=n_2+1}^{p_2}\alpha_{2k}^{(i)}\right)+\cdots+$$

$$\left(\sum_{k=1}^{n_r}\alpha_{rk}^{(i)}-\sum_{k=n_r+1}^{p_r}\alpha_{rk}^{(i)}\right)-\left(\sum_{k=1}^{q_2}\beta_{2k}^{(i)}+\sum_{k=1}^{q_3}\beta_{3k}^{(i)}+\cdots+\sum_{k=1}^{q_r}\beta_{rk}^{(i)}\right) \tag{1.10}$$

where $i=1,\cdots,r$

The complex numbers $z_i$ are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.We may establish the the asymptotic expansion in the following convenient form :

$$I(z_1,\cdots,z_r)=0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,)\,,max(\,|z_1|,\cdots,|z_r|\,)\rightarrow 0$$

$$I(z_1,\cdots,z_r)\;=0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,)\,,min(\,|z_1|,\cdots,|z_r|\,)\rightarrow\infty$$

where $k=1,\cdots,r:\alpha_k'=min[Re(b_j^{(k)}/\beta_j^{(k)})],j=1,\cdots,m^{(k)}$ and

$$\beta_k'=max[Re((a_j^{(k)}-1)/\alpha_j^{(k)})],j=1,\cdots,n^{(k)}$$

We shall note this function $I_1(z_1,\cdots,z_r)$.

We Consider the second multivariable I-function

$$I'(z_1,\cdots,z_r)=\;I^{0,n_2';0,n_3';\cdots;0,n_r':m'^{(1)},n'^{(1)};\cdots;m'^{(r)},n'^{(r)}}_{p_2',q_2',p_3',q_3';\cdots;p_r',q_r':p'^{(1)},q'^{(1)};\cdots;p'^{(r)},q'^{(r)}}\left(\begin{array}{c|c} \mathrm{z}_1' & (\mathrm{a'}_{2j};\alpha_{2j}'^{(1)},\alpha_{2j}'^{(2)})_{1,p_2};\cdots; \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r' & (\mathrm{b'}_{2j};\beta_{2j}'^{(1)},\beta_{2j}'^{(2)})_{1,q_2};\cdots; \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{a}_{rj}';\alpha_{rj}'^{(1)},\cdots,\alpha_{rj}'^{(r)})_{1,p_r'}:(a_j'^{(1)},\alpha_j'^{(1)})_{1,p'^{(1)}};\cdots;(a_j'^{(r)},\alpha_j'^{(r)})_{1,p'^{(r)}} \\ \\ (\mathrm{b}_{rj}';\beta_{rj}'^{(1)},\cdots,\beta_{rj}'^{(r)})_{1,q_r}:(b_j'^{(1)},\beta_j'^{(1)})_{1,q'^{(1)}};\cdots;(b_j'^{(r)},\beta_j'^{(r)})_{1,q'^{(r)}} \end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1'}\cdots\int_{L_r'}\phi(t_1,\cdots,t_r)\prod_{i=1}^{r}\phi_i(t_i)z_i'^{t_i}\mathrm{d}t_1\cdots\mathrm{d}t_r \tag{1.11}$$

where

$$\phi_i(t_i)=\frac{\prod_{j=1}^{m'^{(i)}}\Gamma(b_j'^{(i)}-\beta_j'^{(i)}t_i)\prod_{j=1}^{n'^{(i)}}\Gamma(1-a_j'^{(i)}+\alpha_j'^{(i)}t_i)}{\prod_{j=m'^{(i)}+1}^{q'^{(i)}}\Gamma(1-b_j'^{(i)}+\beta_j'^{(i)}t_i)\prod_{j=n'^{(i)}+1}^{p'^{(i)}}\Gamma(a_j'^{(i)}-\alpha_j'^{(i)}t_i)}\;,i=1,\cdots,r \tag{1.12}$$

and

$$\phi(t_1,\cdots,t_s)=\frac{\prod_{j=1}^{n_2'}\Gamma(1-a_{2j}'+\sum_{i=1}^{2}\alpha_{2j}'^{(i)}t_i)\prod_{j=1}^{n_3'}\Gamma(1-a_{3j}'+\sum_{i=1}^{3}\alpha_{3j}'^{(i)}t_i)\cdots}{\prod_{j=n_2'+1}^{p_2}\Gamma(a_{2j}'-\sum_{i=1}^{2}\alpha_{2j}'^{(i)}t_i)\prod_{j=n_3'+1}^{p_3'}\Gamma(a_{3j}'-\sum_{i=1}^{3}\alpha_{3j}'^{(i)}t_i)\cdots}$$

$$\frac{\cdots\prod_{j=1}^{n_r'}\Gamma(1-a_{rj}'+\sum_{i=1}^{r}\alpha_{rj}'^{(i)}t_i)}{\cdots\prod_{j=n_r'+1}^{p_r'}\Gamma(a_{rj}'-\sum_{i=1}^{r}\alpha_{rj}'^{(i)}t_i)\prod_{j=1}^{q_2'}\Gamma(1-b_{2j}'-\sum_{i=1}^{2}\beta_{2j}'^{(i)}t_i)}$$

$$\times\frac{1}{\prod_{j=1}^{q_3'}\Gamma(1-b_{3j}'+\sum_{i=1}^{3}\beta_{3j}'^{(i)}t_i)\cdots\prod_{j=1}^{q_r'}\Gamma(1-b_{rj}'-\sum_{i=1}^{r}\beta_{rj}'^{(i)}t_i)} \tag{1.13}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [3]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.11) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz'_i| < \frac{1}{2}\Omega'_i\pi$ , where

$$\Omega'_i = \sum_{k=1}^{n'^{(i)}} \alpha'^{(i)}_k - \sum_{k=n'^{(i)}+1}^{p'^{(i)}} \alpha'^{(i)}_k + \sum_{k=1}^{m'^{(i)}} \beta'^{(i)}_k - \sum_{k=m'^{(i)}+1}^{q'^{(i)}} \beta'^{(i)}_k + \left(\sum_{k=1}^{n'_2} \alpha'^{(i)}_{2k} - \sum_{k=n'_2+1}^{p'_2} \alpha'^{(i)}_{2k}\right) +$$

$$+ \cdots + \left(\sum_{k=1}^{n'_r} \alpha'^{(i)}_{rk} - \sum_{k=n'_r+1}^{p'_r} \alpha'^{(i)}_{rk}\right) - \left(\sum_{k=1}^{q'_2} \beta'^{(i)}_{2k} + \sum_{k=1}^{q'_3} \beta'^{(i)}_{3k} + \cdots + \sum_{k=1}^{q'_r} \beta'^{(i)}_{rk}\right) \tag{1.14}$$

The complex numbers $z_i$ are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$$I(z'_1, \cdots, z'_r) = 0( |z'_1|^{\alpha'_1}, \cdots, |z'_r|^{\alpha'_r} ) , max( |z'_1|, \cdots, |z'_r| ) \rightarrow 0$$

$$I(z'_1, \cdots, z'_r) \ = 0( |z'_1|^{\beta'_1}, \cdots, |z'_r|^{\beta'_r} ) , min( |z'_1|, \cdots, |z'_r| ) \rightarrow \infty$$

where $k = 1, \cdots, r : \alpha''_k = min[Re(b'^{(k)}_j / \beta'^{(k)}_j)], j = 1, \cdots, m'^{(k)}$ and

$$\beta''_k = max[Re((a'^{(k)}_j - 1)/\alpha'^{(k)}_j)], j = 1, \cdots, n'^{(k)}$$

We shall note this function $I_2(z'_1, \cdots, z'_r)$.

## 2. Generalized multiple fractional integrals operators

The fractional integral operators defined and represented in the following manner by Sharma [4] are studied in this paper ([4], p. 149-152, Eq. 5.2(1)-5.3(6)) :

Generalized single fractional integral operators

$$R\,[f(x)] = \left[ \begin{array}{cc} \eta, \alpha : k & \\ . & ;\ f(x) \\ \mathrm{a}_1, \cdots, a_r : x & \end{array} \right] = kx^{-\eta-k\alpha-1} \int_0^x t^\eta (x^k - t^k)^\alpha I_1(a_1 V, \cdots, a_r V) f(t) \mathrm{d}t \tag{2.1}$$

and

$$\mathbb{K}\,[f(x)] = \left[ \begin{array}{cc} \eta, \alpha : \mu & \\ . & ;\ f(x) \\ \mathrm{a}_1, \cdots, a_r : x & \end{array} \right] = \mu x^\eta \int_x^\infty t^{-\eta-\mu\alpha-1} (t^\mu - x^\mu)^\alpha I_1(a_1 V', \cdots, a_r V') f(t) \mathrm{d}t \tag{2.2}$$

where $I_1(.)$ is given by (1.7) and

$$V = \left(\frac{t^k}{x^k}\right)^\rho \left(1 - \frac{t^k}{x^k}\right)^\sigma \tag{2.3}$$

$$V' = \left(\frac{x^\mu}{t^\mu}\right)^{\rho'} \left(1 - \frac{x^\mu}{t^\mu}\right)^{\sigma'} \tag{2.4}$$

The fractional integral operators $R\,[f(x)]$ and $\mathbb{K}\,[f(x)]$ exit under the following set of conditions :

$$1 \leqslant p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1; f(x) \in L_p(0,\infty),$$

$$Re\,(\eta) + k\rho \sum_{i=1}^{r} \min_{1 \leqslant j \leqslant m^{(i)}} Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right) > max[-p^{-1}, -q^{-1}];$$

$$Re\,(\eta) + \mu\rho' \sum_{i=1}^{n} \min_{1 \leqslant j \leqslant m^{(i)}} Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right) > max[-p^{-1}, -q^{-1}]$$

$$Re\,(\alpha) + \sigma \sum_{i=1}^{r} \min_{1 \leqslant j \leqslant m^{(i)}} Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right) > -q^{-k}\,;\; Re\,(\alpha) + \sigma' \sum_{i=1}^{r} \min_{1 \leqslant j \leqslant m^{(i)}} Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right) > -q^{-\mu}$$

Generalized multiple fractional integral operators :

$$\mathbf{R}\,[f(x_1,\cdots,x_r)] = \left[\begin{array}{cc} \eta_1 \cdots, \eta_r, \alpha_1, \cdots, \alpha_r : k_1, \cdots, k_r & \\ . & ;\, f(x_1,\cdots,x_r) \\ \mathrm{a}_1, \cdots, a_r : x_1, \cdots, x_r & \end{array}\right]$$

$$= \prod_{i=1}^{r} k_i x^{-\eta_i - k_i\alpha_i - 1} \int_0^{x_1} \cdots \int_0^{x_r} \prod_{i=1}^{r} \{t_i^{\eta_i}(x_i^{k_i} - t_i^{k_i})^{\alpha_i} I_1(a_1V_1, \cdots, a_rV_r) f(t_1,\cdots,t_r)\mathrm{d}t_1\cdots\mathrm{d}t_r \tag{2.5}$$

and

$$\mathbf{K}\,[f(x_1,\cdots,x_r)] = \left[\begin{array}{cc} \eta_1 \cdots, \eta_r, \alpha_1, \cdots, \alpha_r : \mu_1, \cdots, \mu_r & \\ . & ;\, f(x_1,\cdots,x_r) \\ \mathrm{a}_1, \cdots, a_r : x_1, \cdots, x_r & \end{array}\right]$$

$$= \prod_{i=1}^{r} \mu_i x^{\eta_i} \int_{x_1}^{\infty} \cdots \int_{x_r}^{\infty} \prod_{i=1}^{r} \{t_i^{-\eta_i - \mu_i\alpha_i - 1}(t_i^{\mu_i} - x_i^{\mu_i})^{\alpha_i} I_1(a_1V_1', \cdots, a_rV_r') f(t_1,\cdots,t_r)\mathrm{d}t_1\cdots\mathrm{d}t_r \tag{2.6}$$

where, for $i = 1, \cdots, r$

$$V_i = \left(\frac{t_i^{k_i}}{x_i^{k_i}}\right)^{\rho_i} \left(1 - \frac{t_i^{k_i}}{x_i^{k_i}}\right)^{\sigma_i} \tag{2.7}$$

$$V_i' = \left(\frac{x_i^{\mu_i}}{t_i^{\mu_i}}\right)^{\rho_i'} \left(1 - \frac{x_i^{\mu_i}}{t_i^{\mu_i}}\right)^{\sigma_i'} \tag{2.8}$$

The fractional integrals operators $\mathbf{R}\,[f(x_i)]$ and $\mathbf{K}\,[f(x_i)]$ exit under the following set of conditions :

$$1 \leqslant p_i, q_i < \infty, \frac{1}{p_i} + \frac{1}{q_i} = 1; f(x_1,\cdots,x_r) \in L_{p_i}(0,\infty), i = 1, \cdots, r$$

$$\sum_{i=1}^{r}\left[Re\,(\eta_i) + k_i\rho_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > max[-p_i^{-1}, -q_i^{-1}]$$

$$\sum_{i=1}^{r}\left[Re\,(\eta_i) + \mu_i\rho_i' \min_{1 \leqslant j \leqslant m^{(i)}} Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > max[-p_i^{-1}, -q_i^{-1}]$$

$$\sum_{i=1}^{r}\left[Re\,(k_i\alpha_i) + k_i\sigma_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > max\left\{-\frac{1}{q_i}\right\};$$

$$\sum_{i=1}^{r}\left[Re\left(\mu_i\alpha_i\right)+\mu_i\sigma'_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right]>max\left\{-\frac{1}{q_i}\right\}$$

## 3. Main Results

In this section, we calculate six images involving the product of multivariable I-function defined by Prasad [3], generalized Riemann Zeta function and classes of polynomials (one and several variables). We establish six theorems in this section.

**Theorem 1**

If $R\left[f(x)\right]$ be given by (2.1), we have

$$R\left[x^{\lambda}\phi_{\mu}(ex^{\zeta},\mathbf{s},a,g)S_N^M(cx^{\gamma})S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[c_1x^{\gamma_1},\cdots,c_vx^{\gamma_v}]I_2(b_1x^{\beta_1},\cdots,b_rx^{\beta_r})\right]=$$

$$\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{K=0}^{[N/M]}\frac{(-N)_{MK}}{K!}A_{N,K}\,(\mu)_g(a+g)^{-\mathbf{s}}\frac{e^g}{g!}\,a_v\prod_{i=1}^{v}c_i^{K_i}x^{\gamma_iK_i}x^{\lambda+\gamma K+g\zeta}c^K$$

$$I_{U:p_r+p'_r+2,q_r+q'_r+1;Y}^{V;0,n_r+n'_r+2;X}\left(\begin{array}{c|c}\mathrm{b}_1x^{\beta_1} & \\ . & \mathbb{A}\ ;\ \mathrm{A}_1,\mathbf{A}\\ . & .\\ \mathrm{b}_rx^{\beta_r} & .\\ \mathrm{a}_1 & .\\ . & .\\ . & \mathbb{B}\ ;\ \mathrm{B}_1,\mathbf{B}\\ \mathrm{a}_r & \end{array}\right) \qquad (3.1)$$

where

$$U=p'_2,q'_2;p'_3,q'_3;\cdots;p'_{r-1},q'_{r-1}\ ;p_2,q_2;p_3,q_3;\cdots;p_{r-1},q_{r-1} \qquad (3.2)$$

$$V=0,n'_2;0,n'_3;\cdots;0,n'_{r-1}\ \ ;0,n_2;0,n_3;\cdots;0,n_{r-1} \qquad (3.3)$$

$$X=m'^{(1)},n'^{(1)};\cdots;m'^{(r)},n'^{(r)};m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)} \qquad (3.4)$$

$$Y=p'^{(1)},q'^{(1)};\cdots;p'^{(r)},q'^{(r)};p^{(1)},q^{(1)};\cdots;p^{(r)},q^{(r)} \qquad (3.5)$$

$$\mathbb{A}=(a'_{2k};\alpha'^{(1)}_{2k},\alpha'^{(2)}_{2k})_{1,p'_2};\cdots;(a'_{(r-1)k};\alpha'^{(1)}_{(r-1)k},\alpha'^{(2)}_{(r-1)k},\cdots,\alpha'^{(r-1)}_{(r-1)k})_{1,p'_{r-1}};$$

$$(a_{2k};\alpha^{(1)}_{2k},\alpha^{(2)}_{2k})_{1,p_2};\cdots;(a_{(r-1)k};\alpha^{(1)}_{(r-1)k},\alpha^{(2)}_{(r-1)k},\cdots,\alpha^{(r-1)}_{(r-1)k})_{1,p_{r-1}} \qquad (3.6)$$

$$A_1=\left(1-\frac{\eta+\lambda+\gamma K+\zeta g+\sum_{i=1}^{v}\gamma_iK_i+1}{k};\frac{\beta_1}{k},\cdots,\frac{\beta_r}{k},\underbrace{\rho,\cdots,\rho}_{r}\right),(-\alpha;\underbrace{0,\cdots,0}_{r},\underbrace{\sigma,\cdots,\sigma}_{r}) \qquad (3.7)$$

$$\mathbf{A}=(a'_{rk};\alpha'^{(1)}_{rk},\alpha'^{(2)}_{rk},\cdots,\alpha'^{(r)}_{rk},\underbrace{0,\cdots,0}_{r})_{1,p'_r},(a_{rk};\underbrace{0,\cdots,0}_{r},\alpha^{(1)}_{rk},\alpha^{(2)}_{rk},\cdots,\alpha^{(r)}_{rk})_{1,p_r}:$$

$$(a_k'^{(1)},\alpha_k'^{(1)})_{1,p'^{(1)}};\cdots;(a_k'^{(r)},\alpha_k'^{(r)})_{1,p'^{(r)}};(a_k^{(1)},\alpha_k^{(1)})_{1,p^{(1)}};\cdots;(a_k^{(r)},\alpha_k^{(r)})_{1,p^{(r)}}; \tag{3.8}$$

$$\mathbb{B}=(b'_{2k};\beta_{2k}'^{(1)},\beta_{2k}'^{(2)})_{1,q'_2};\cdots;(b'_{(r-1)k};\beta_{(r-1)k}'^{(1)},\beta_{(r-1)k}'^{(2)},\cdots,\beta_{(r-1)k}'^{(r-1)})_{1,q'_{r-1}};$$

$$(b_{2k};\beta_{2k}^{(1)},\beta_{2k}^{(2)})_{1,q_2};\cdots;(b_{(r-1)k};\beta_{(r-1)k}^{(1)},\beta_{(r-1)k}^{(2)},\cdots,\beta_{(r-1)k}^{(r-1)})_{1,q_{r-1}} \tag{3.9}$$

$$B_1=\left(-\alpha-\frac{\eta+\lambda+\gamma K+\zeta g+\sum_{i=1}^{v}\gamma_i K_i+1}{k};\frac{\beta_1}{k},\cdots,\frac{\beta_r}{k},\underbrace{\rho+\sigma,\cdots,\rho+\sigma}_{r}\right) \tag{3.10}$$

$$\mathbf{B}=(b'_{rk};\beta_{rk}'^{(1)},\beta_{rk}'^{(2)},\cdots,\beta_{rk}'^{(r)},\underbrace{0,\cdots,0}_{r})_{1,q'_r};(b_{rk};\underbrace{0,\cdots,0}_{r},\beta_{rk}^{(1)},\beta_{rk}^{(2)},\cdots,\beta_{rk}^{(r)})_{1,q_r},$$

$$(b_k'^{(1)},\beta_k'^{(1)})_{1,q'^{(1)}};\cdots;(b_k'^{(r)},\beta_k'^{(r)})_{1,q'^{(r)}};(b_k^{(1)},\beta_k^{(1)})_{1,q^{(1)}};\cdots;(b_k^{(r)},\beta_k^{(r)})_{1,q^{(r)}} \tag{3.11}$$

Provided that

$$Re(\eta+\lambda+1)+\sum_{i=1}^{r}\left[\rho k\min_{1\leqslant j\leqslant m^{(i)}}Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)+\beta_i\min_{1\leqslant j\leqslant m'^{(i)}}Re\left(\frac{b_j'^{(i)}}{\beta_j'^{(i)}}\right)\right]>0 \text{ and}$$

$$Re(\alpha+1)+\sigma\sum_{i=1}^{r}\min_{1\leqslant j\leqslant m^{(i)}}Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)>0$$

$|arg(a_kV_k)| < \frac{1}{2}\Omega_i\pi$, where $\Omega_i$is defined by (1.10)

$|arg(b_kx^{\beta_k})| < \frac{1}{2}\Omega'_i\pi$, where $\Omega'_i$ is defined by (1.14) and the multiple series in the left-hand side of (3.1) converge absolutely.

**Theorem 2**

If $\mathbb{K}\left[f(x)\right]$ be given by (2.2), we have

$$\mathbb{K}\left[x^{\lambda}\phi_{\mu}(ex^{\zeta},\mathbf{s},a,g)S_N^M(cx^{\gamma})S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[c_1x^{\gamma_1},\cdots,c_vx^{\gamma_v}]I_2(b_1x^{-\beta_1},\cdots,b_rx^{-\beta_r})\right]=$$

$$\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{K=0}^{[N/M]}\frac{(-N)_{MK}}{K!}A_{N,K}\,(\mu)_g(a+g)^{-\mathbf{s}}\frac{e^g}{g!}\,a_v\prod_{i=1}^{v}c_i^{K_i}x^{\gamma_iK_i}x^{-\lambda+\gamma K+g\zeta}c^K$$

$$I_{U:p_r+p'_r+2,q_r+q'_r+1;Y}^{V;0,n_r+n'_r+2;X}\left(\begin{array}{c|c}\mathrm{b}_1x^{-\beta_1} & \mathbb{A}\ ;\ \mathrm{A}_2,\mathbf{A}\\ . & .\\ . & .\\ \mathrm{b}_rx^{-\beta_r} & .\\ \mathrm{a}_1 & .\\ . & \mathbb{B}\ ;\ \mathrm{B}_2,\mathbf{B}\\ . & \\ \mathrm{a}_r & \end{array}\right) \tag{3.12}$$

where

$$A_2 = \left(1 - \frac{\eta + \lambda - \gamma K - \zeta g - \sum_{i=1}^{v} \gamma_i K_i}{\mu}; \frac{\beta_1}{\mu}, \cdots, \frac{\beta_r}{\mu}, \underbrace{\rho', \cdots, \rho'}_{r}\right), (-\alpha; \underbrace{0, \cdots, 0}_{r}, \underbrace{\sigma', \cdots, \sigma'}_{r}) \tag{3.13}$$

$$B_2 = \left(-\alpha - \frac{\eta + \lambda - \gamma K - \zeta g - \sum_{i=1}^{v} \gamma_i K_i}{\mu}; \frac{\beta_1}{\mu}, \cdots, \frac{\beta_r}{\mu}, \underbrace{\rho' + \sigma', \cdots, \rho' + \sigma'}_{r}\right) \tag{3.14}$$

The others quantities are defined in theorem 1.

Provided that

$Re\left(\eta + \lambda - \gamma[n/m] - \sum_{i=1}^{v} \gamma_i [N_i/M_i] - g\zeta\right) + \sum_{i=1}^{r}\left[\rho'\mu \min_{1 \leqslant j \leqslant m^{(i)}} Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right) + \beta_i \min_{1 \leqslant j \leqslant m'^{(i)}} Re\left(\frac{b_j'^{(i)}}{\beta_j'^{(i)}}\right)\right] > 0$ and

$Re(\alpha + 1) + \sigma' \sum_{i=1}^{r} \min_{1 \leqslant j \leqslant m^{(i)}} Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right) > 0$

$|arg(a_k V_k)| < \frac{1}{2}\Omega_i \pi$, where $\Omega_i$is defined by (1.10)

$|arg(b_k x^{-\beta_k})| < \frac{1}{2}\Omega_i' \pi$, where $\Omega_i'$ is defined by (1.14) and the multiple series in the left-hand side of (3.12) converge absolutely.

**Theorem 3**

If $\mathbf{R}\left[f(x_1, \cdots, x_r)\right]$ be given by (2.5), we obtain

$$\mathbf{R}\left[\prod_{i=1}^{r} x_i^{\lambda_i} \phi_\mu\left(e \prod_{i=1}^{r} x_i^{\zeta_i}, \mathbf{s}, a, g\right) S_N^M\left(c \prod_{i=1}^{r} x_i^{\gamma_i'}\right) S_{N_1, \cdots, N_r}^{\mathfrak{M}_1, \cdots, \mathfrak{M}_r}[c_1 x_1^{\gamma_1}, \cdots, c_r x_r^{\gamma_r}] I_2(b_1 x_1^{\beta_1}, \cdots, b_r x_r^{\beta_r})\right] =$$

$$\sum_{g=0}^{\infty} \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_r=0}^{[N_r/\mathfrak{M}_r]} \sum_{K=0}^{[N/M]} \frac{(-N)_{MK}}{K!} A_{N,K}\, a_r\, (\mu)_g (a+g)^{-\mathbf{s}} \frac{e^g}{g!} \prod_{i=1}^{r} c_i^{K_i} x_i^{\lambda_i + \gamma_i K_i + \gamma_i' K + g\zeta_i} c^K$$

$$I_{U:p_r+p_r'+r, q_r+q_r'+r; Y'}^{V;0, n_r+n_r'+r; X'}\left(\begin{array}{c|c} \mathrm{b}_1 x^{\beta_1} & \\ . & \mathbb{A}\ ;\ \mathrm{A}_3, \mathbf{A}' \\ . & . \\ \mathrm{b}_r x^{\beta_r} & . \\ \mathrm{a}_1 & . \\ . & . \\ . & \mathbb{B}\ ;\ \mathrm{B}_3, \mathbf{B} \\ \mathrm{a}_r & \end{array}\right) \tag{3.15}$$

where

$$X' = m'^{(1)}, n'^{(1)}; \cdots; m'^{(r)}, n'^{(r)}; m^{(1)}, n^{(1)} + 1; \cdots; m^{(r)}, n^{(r)} + 1 \tag{3.16}$$

$$Y' = p'^{(1)}, q'^{(1)}; \cdots; p'^{(r)}, q'^{(r)}; p^{(1)} + 1, q^{(1)}; \cdots; p^{(r)} + 1, q^{(r)} \tag{3.17}$$

$$A_3=\left(1-\frac{\eta_1+\lambda_1+\gamma_1'K+\zeta_1g+\gamma_1K_1+1}{k_1};\frac{\beta_1}{k_1},\underbrace{0,\cdots,0}_{r-1},\rho_1,\underbrace{0,\cdots,0}_{r-1}\right),$$

$$\left(1-\frac{\eta_2+\lambda_2+\gamma_2'K+\zeta_2g+\gamma_2K_2+1}{k_1};0,\frac{\beta_2}{k_2},\underbrace{0,\cdots,0}_{r-2},0,\rho_2,\underbrace{0,\cdots,0}_{r-2}\right),\cdots,$$

$$\left(1-\frac{\eta_r+\lambda_r+\gamma_r'K+\zeta_rg+\gamma_rK_r+1}{k_r};\underbrace{0,\cdots,0}_{r-1},\frac{\beta_r}{k_r},\underbrace{0,\cdots,0}_{r-1},\rho_r\right) \tag{3.18}$$

$$\mathbf{A}'=(a'_{rk};\alpha_{rk}'^{(1)},\alpha_{rk}'^{(2)},\cdots,\alpha_{rk}'^{(r)},\underbrace{0,\cdots,0}_{r})_{1,p'_r},(a_{rk};\underbrace{0,\cdots,0}_{r},\alpha_{rk}^{(1)},\alpha_{rk}^{(2)},\cdots,\alpha_{rk}^{(r)})_{1,p_r}:$$

$$(a_k'^{(1)},\alpha_k'^{(1)})_{1,p'^{(1)}};\cdots;(a_k'^{(r)},\alpha_k'^{(r)})_{1,p'^{(r)}};(-\alpha_1;\sigma_1),(a_k^{(1)},\alpha_k^{(1)})_{1,p^{(1)}};\cdots;(-\alpha_r;\sigma_r),(a_k^{(r)},\alpha_k^{(r)})_{1,p^{(r)}} \tag{3.19}$$

$$B_3=\left(-\alpha_1-\frac{\eta_1+\lambda_1+\gamma_1'K+\zeta_1g+\gamma_1K_1+1}{k_1};\frac{\beta_1}{k_1},\underbrace{0,\cdots,0}_{r-1},\rho_1+\sigma_1,\underbrace{0,\cdots,0}_{r-1}\right),$$

$$\left(-\alpha_2-\frac{\eta_2+\lambda_2+\gamma_2'K+\zeta_2g+\gamma_2K_2+1}{k_1};0,\frac{\beta_2}{k_2},\underbrace{0,\cdots,0}_{r-2},0,\rho_2+\sigma_2,\underbrace{0,\cdots,0}_{r-2}\right),\cdots,$$

$$\left(-\alpha_r-\frac{\eta_r+\lambda_r+\gamma_r'K+\zeta_rg+\gamma_rK_r+1}{k_r};\underbrace{0,\cdots,0}_{r-1},\frac{\beta_r}{k_r},\underbrace{0,\cdots,0}_{r-1},\rho_r+\sigma_r\right) \tag{3.20}$$

The others quantities are the same that (3.1).

Provided that

$$Re(\eta_i+\lambda_i+1)+\left[\rho_ik_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)+\beta_i\min_{1\leqslant j\leqslant m'^{(i)}}Re\left(\frac{b_j'^{(i)}}{\beta_j'^{(i)}}\right)\right]>0$$

and $Re(\alpha_i+1)+\sigma_i\min\limits_{1\leqslant j\leqslant m^{(i)}}Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)>0$ for $i=1,\cdots,r$

$|arg(a_kV_k)|<\frac{1}{2}\Omega_i\pi$, where $\Omega_i$is defined by (1.10)

$|arg(b_kx_k^{\beta_k})|<\frac{1}{2}\Omega_i'\pi$, where $\Omega_i'$ is defined by (1.14) and the multiple series in the left-hand side of (3.15) converge absolutely.

**Theorem 4**

If $\mathbf{K}\left[f(x_1,\cdots,x_r)\right]$ be given by (2.6), we obtain

$$\mathbf{K}\left[\prod_{i=1}^r x_i^{\lambda_i}\phi_\mu\left(e\prod_{i=1}^r x_i^{\zeta_i},\mathbf{s},a,g\right)S_N^M\left(c\prod_{i=1}^r x_i^{\gamma_i'}\right)S_{N_1,\cdots,N_r}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_r}[c_1x_1^{\gamma_1},\cdots,c_rx_r^{\gamma_r}]I_2(b_1x_1^{-\beta_1},\cdots,b_rx_r^{-\beta_r})\right]=$$

$$\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_r=0}^{[N_r/\mathfrak{M}_r]}\sum_{K=0}^{[N/M]}\frac{(-N)_{MK}}{K!}A_{N,K}\,a_r\,(\mu)_g(a+g)^{-\mathbf{s}}\frac{e^g}{g!}\prod_{i=1}^{r}c_i^{K_i}x_i^{-\lambda_i+\gamma_iK_i+\gamma_i'K+g\zeta_i}c^K$$

$$I_{U:p_r+p_r'+r,q_r+q_r'+r;Y'}^{V;0,n_r+n_r'+r;X'}\left(\begin{array}{c|c} \mathrm{b}_1x^{-\beta_1} & \\ \cdot & \mathbb{A}\ ;\ \mathrm{A}_4,\mathbf{A}' \\ \cdot & \cdot \\ \mathrm{b}_rx^{-\beta_r} & \cdot \\ \mathrm{a}_1 & \cdot \\ \cdot & \cdot \\ \cdot & \mathbb{B}\ ;\ \mathrm{B}_4,\mathbf{B} \\ \mathrm{a}_r & \end{array}\right) \tag{3.21}$$

where

$$A_4=\left(1-\frac{\eta_1+\lambda_1-\gamma_1'K-\zeta_1g-\gamma_1K_1}{\mu_1};\frac{\beta_1}{\mu_1},\underbrace{0,\cdots,0}_{r-1},\rho_1',\underbrace{0,\cdots,0}_{r-1}\right),$$

$$\left(1-\frac{\eta_2+\lambda_2-\gamma_2'K-\zeta_2g-\gamma_2K_2}{\mu_2};0,\frac{\beta_2}{\mu_2},\underbrace{0,\cdots,0}_{r-2},0,\rho_2',\underbrace{0,\cdots,0}_{r-2}\right),\cdots,$$

$$\left(1-\frac{\eta_r+\lambda_r-\gamma_r'K-\zeta_rg-\gamma_rK_r}{\mu_r};\underbrace{0,\cdots,0}_{r-1},\frac{\beta_r}{\mu_r},\underbrace{0,\cdots,0}_{r-1},\rho_r'\right) \tag{3.22}$$

$$B_4=\left(-\alpha_1-\frac{\eta_1+\lambda_1-\gamma_1'K-\zeta_1g-\gamma_1K_1}{\mu_1};\frac{\beta_1}{\mu_1},\underbrace{0,\cdots,0}_{r-1},\rho_1'+\sigma_1',\underbrace{0,\cdots,0}_{r-1}\right),$$

$$\left(-\alpha_2-\frac{\eta_2+\lambda_2-\gamma_2'K-\zeta_2g-\gamma_2K_2}{\mu_2};0,\frac{\beta_2}{\mu_2},\underbrace{0,\cdots,0}_{r-2},0,\rho_2'+\sigma_2',\underbrace{0,\cdots,0}_{r-2}\right),\cdots,$$

$$\left(-\alpha_r-\frac{\eta_r+\lambda_r-\gamma_r'K-\zeta_rg-\gamma_rK_r}{\mu_r};\underbrace{0,\cdots,0}_{r-1},\frac{\beta_r}{\mu_r},\underbrace{0,\cdots,0}_{r-1},\rho_r'+\sigma_r'\right) \tag{3.23}$$

The other quantities are the same that (3.15).

Provided that

$$Re\,(\eta_i+\lambda_i-\gamma_i'[n/m]-\gamma_i[N_i/M_i]-g\zeta_i)+\ \left[\rho_i'\mu_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)+\beta_i\min_{1\leqslant j\leqslant m'^{(i)}}Re\left(\frac{b_j'^{(i)}}{\beta_j'^{(i)}}\right)\right]>0$$

and $Re(\alpha_i+1)+\sigma_i'\min_{1\leqslant j\leqslant m^{(i)}}Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)>0$ for $i=1,\cdots,r$

$|arg(a_kV_k)|<\frac{1}{2}\Omega_i\pi$, where $\Omega_i$is defined by (1.10)

$|arg(b_kx_k^{-\beta_k})|<\frac{1}{2}\Omega_i'\pi$, where $\Omega_i'$ is defined by (1.14) and the multiple series in the left-hand side of (3.21) converge absolutely.

**Theorem 5**

If $\mathbf{R}\left[f(x_1,\cdots,x_r)\right]$ be given by (2.5), we obtain

$$\mathbf{R}\left[\prod_{i=1}^{r}x_i^{\lambda_i}\phi_\mu\left(e\prod_{i=1}^{r}x_i^{\zeta_i},\mathbf{s},a,g\right)S_N^M\left(c\prod_{i=1}^{r}x_i^{\gamma'_i}\right)T_n^{m_1,\cdots,m_r}[c_1x_1^{\gamma_1},\cdots,c_rx_r^{\gamma_r}]I_2(b_1x_1^{\beta_1},\cdots,b_rx_r^{\beta_r})\right]=$$

$$\sum_{g=0}^{\infty}\sum_{K_1,\cdots,K_r=0}^{M'\leqslant n}\sum_{K=0}^{[N/M]}\frac{(-N)_{MK}}{K!}A_{N,K}\,a_r\,(\mu)_g(a+g)^{-\mathbf{s}}\frac{e^g}{g!}(-n)_{M'}A(n,K_1,\cdots,K_r)\prod_{i=1}^{r}\frac{c_i^{K_i}}{K_i!}$$

$$\prod_{i=1}^{r}x_i^{\lambda_i+\gamma_ih_i+\gamma'_iK+g\zeta_i}c^KI_{U:p_r+p'_r+r,q_r+q'_r+r;Y'}^{V;0,n_r+n'_r+r;X'}\left(\begin{array}{c|c}\mathrm{b}_1x^{\beta_1} & \mathbb{A}\ ;\ \mathrm{A}_3,\mathbf{A'}\\ . & .\\ . & .\\ \mathrm{b}_rx^{\beta_r} & .\\ \mathrm{a}_1 & .\\ . & .\\ . & \mathbb{B}\ ;\ \mathrm{B}_3,\mathbf{B}\\ \mathrm{a}_r & \end{array}\right)\tag{3.24}$$

with the same notations and conditions that theorem 3 and $M'=\sum_{i=1}^{r}m_iK_i$

**Theorem 6**

If $\mathbf{K}\left[f(x_1,\cdots,x_r)\right]$ be given by (2.6), we obtain

$$\mathbf{K}\left[\prod_{i=1}^{r}x_i^{\lambda_i}\phi_\mu\left(e\prod_{i=1}^{r}x_i^{\zeta_i},\mathbf{s},a,g\right)S_N^M\left(c\prod_{i=1}^{r}x_i^{\gamma'_i}\right)T_n^{m_1,\cdots,m_r}[c_1x_1^{\gamma_1},\cdots,c_rx_r^{\gamma_r}]I_2(b_1x_1^{-\beta_1},\cdots,b_rx_r^{-\beta_r})\right]=$$

$$\sum_{g=0}^{\infty}\sum_{K_1,\cdots,K_r=0}^{M'\leqslant n}\sum_{K=0}^{[N/M]}\frac{(-N)_{MK}}{K!}A_{N,K}\,a_r\,(\mu)_g(a+g)^{-\mathbf{s}}\frac{e^g}{g!}\ (-n)_{M'}A(n,K_1,\cdots,K_r)\prod_{i=1}^{r}\frac{c_i^{K_i}}{K_i!}$$

$$\prod_{i=1}^{r}x_i^{-\lambda_i+\gamma_ih_i+\gamma'_iK+g\zeta_i}\ c^k\ I_{U:p_r+p'_r+r,q_r+q'_r+r;Y'}^{V;0,n_r+n'_r+r;X'}\left(\begin{array}{c|c}\mathrm{b}_1x^{-\beta_1} & \mathbb{A}\ ;\ \mathrm{A}_4,\mathbf{A'}\\ . & .\\ . & .\\ \mathrm{b}_rx^{-\beta_r} & .\\ \mathrm{a}_1 & .\\ . & .\\ . & \mathbb{B}\ ;\ \mathrm{B}_4,\mathbf{B}\\ \mathrm{a}_r & \end{array}\right)\tag{3.25}$$

under the same notations and conditions that theorem 4 with $M'=\sum_{i=1}^{r}m_iK_i$ and

$$Re\,(\eta_i+\lambda_i-\gamma'_i[n/m]-\gamma_in-g\zeta_i)+\ \left[\rho'_i\mu_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)+\beta_i\min_{1\leqslant j\leqslant m'^{(i)}}Re\left(\frac{b_j'^{(i)}}{\beta_j'^{(i)}}\right)\right]>0$$

Proof

To establish the theorem 1, we first use the definition of the operator $R$ from (2.1), then express general classes of polynomials of one and several variables and the generalized Riemann Zeta function $\phi_\mu(z,\mathbf{s},a,g)$ in series with the help of (1.5), (1.3) and (1.1) respectively and the multivariable I-function involved therein in terms of Mellin-Barnes type integral contour, integrate term by term and then evaluate the resulting Beta integral to get the desired result. The theorems 2 to 6 are similarly proved.

## 4. Corollaries

In this section, we consider the theorem 1, 3 and 6. Concerning the two first corollaries, the classes of polynomials of one and several variables vanish, we obtain.

**Corollary 1**

If $R\left[f(x)\right]$ be given by (2.1), we have

$$R\left[x^{\lambda}\phi_{\mu}(ex^{\zeta},\mathbf{s},a,g)I_2(b_1x^{\beta_1},\cdots,b_rx^{\beta_r})\right]=\sum_{g=0}^{\infty}\ (\mu)_g(a+g)^{-\mathbf{s}}\frac{e^g}{g!}x^{\lambda+g\zeta}$$

$$I^{V;0,n_r+n_r'+2;X}_{U:p_r+p_r'+2,q_r+q_r'+1;Y}\left(\begin{array}{c|c} \mathrm{b}_1x^{\beta_1} & \mathbb{A}\ ;\ \mathrm{A}_1',\mathbf{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{b}_rx^{\beta_r} & \cdot \\ \mathrm{a}_1 & \cdot \\ \cdot & \cdot \\ \cdot & \mathbb{B}\ ;\ \mathrm{B}_1',\mathbf{B} \\ \mathrm{a}_r & \end{array}\right) \tag{4.1}$$

where

$$A_1'=\left(1-\frac{\eta+\lambda+\zeta g+1}{k};\frac{\beta_1}{k},\cdots,\frac{\beta_r}{k},\underbrace{\rho,\cdots,\rho}_{r}\right),(-\alpha;\underbrace{0,\cdots,0}_{r},\underbrace{\sigma,\cdots,\sigma}_{r}) \tag{4.2}$$

and

$$B_1'=\left(-\alpha-\frac{\eta+\lambda+\zeta g+1}{k};\frac{\beta_1}{k},\cdots,\frac{\beta_r}{k},\underbrace{\rho+\sigma,\cdots,\rho+\sigma}_{r}\right) \tag{4.3}$$

The other quantities and conditions are the same that theorem 1.

**Corollary 2**

If $\mathbf{R}\left[f(x_1,\cdots,x_r)\right]$ be given by (2.5), we obtain

$$\mathbf{R}\left[\prod_{i=1}^{r}x_i^{\lambda_i}\phi_{\mu}\left(e\prod_{i=1}^{r}x_i^{\zeta_i},\mathbf{s},a,g\right)I_2(b_1x_1^{\beta_1},\cdots,b_rx_r^{\beta_r})\right]=\sum_{g=0}^{\infty}\ (\mu)_g(a+g)^{-\mathbf{s}}\frac{e^g}{g!}\prod_{i=1}^{r}x_i^{\lambda_i+g\zeta_i}$$

$$I^{V;0,n_r+n_r'+r;X'}_{U:p_r+p_r'+r,q_r+q_r'+r;Y'}\left(\begin{array}{c|c} \mathrm{b}_1x^{\beta_1} & \mathbb{A}\ ;\ \mathrm{A}_3',\mathbf{A}' \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{b}_rx^{\beta_r} & \cdot \\ \mathrm{a}_1 & \cdot \\ \cdot & \cdot \\ \cdot & \mathbb{B}\ ;\ \mathrm{B}_3',\mathbf{B} \\ \mathrm{a}_r & \end{array}\right) \tag{4.4}$$

where

$$A_3 = \left(1 - \frac{\eta_1+\lambda_1+\zeta_1 g+1}{k_1}; \frac{\beta_1}{k_1}, \underbrace{0,\cdots,0}_{r-1}, \rho_1, \underbrace{0,\cdots,0}_{r-1}\right),$$

$$\left(1 - \frac{\eta_2+\lambda_2+\zeta_2 g+1}{k_1}; 0, \frac{\beta_2}{k_2}, \underbrace{0,\cdots,0}_{r-2}, 0, \rho_2, \underbrace{0,\cdots,0}_{r-2}\right), \cdots,$$

$$\left(1 - \frac{\eta_r+\lambda_r+\zeta_r g+1}{k_r}; \underbrace{0,\cdots,0}_{r-1}, \frac{\beta_r}{k_r}, \underbrace{0,\cdots,0}_{r-1}, \rho_r\right) \tag{4.5}$$

and

$$B_3' = \left(-\alpha_1 - \frac{\eta_1+\lambda_1+\zeta_1 g+1}{k_1}; \frac{\beta_1}{k_1}, \underbrace{0,\cdots,0}_{r-1}, \rho_1+\sigma_1, \underbrace{0,\cdots,0}_{r-1}\right),$$

$$\left(-\alpha_2 - \frac{\eta_2+\lambda_2+\zeta_2 g+1}{k_1}; 0, \frac{\beta_2}{k_2}, \underbrace{0,\cdots,0}_{r-2}, 0, \rho_2+\sigma_2, \underbrace{0,\cdots,0}_{r-2}\right), \cdots,$$

$$\left(-\alpha_r - \frac{\eta_r+\lambda_r+\zeta_r g+1}{k_r}; \underbrace{0,\cdots,0}_{r-1}, \frac{\beta_r}{k_r}, \underbrace{0,\cdots,0}_{r-1}, \rho_r+\sigma_r\right) \tag{4.6}$$

The other notations and conditions are the same that theorem 3.

Now, we consider the theorem 6 with $\mu = 1$, the polynomials $S_n^M(.)$ vanishes, $m_1 = \cdots = m_r = 1$

and $A(n, K_1, \cdots, K_r) = \frac{\prod_{i=1}^r (\beta_i')_{K_i}}{(\alpha)_{M'}}$ where $M' = \sum_{i=1}^r K_i$, it yields

**Corollary 3**

If $\mathbf{K}\left[f(x_1,\cdots,x_r)\right]$ be given by (2.6), we obtain

$$\mathbf{K}\left[\prod_{i=1}^r x_i^{\lambda_i}\phi\left(e\prod_{i=1}^r x_i^{\zeta_i}, \mathbf{s}, a, g\right) F_D^{(r)}[-n, \beta_1', \cdots, \beta_r'; \delta; c_1 x_1^{\gamma_1}, \cdots, c_r x_r^{\gamma_r}] I_2(b_1 x_1^{-\beta_1}, \cdots, b_r x_r^{-\beta_r})\right] =$$

$$\sum_{g=0}^{\infty} \sum_{K_1,\cdots,K_r=0}^{M'\leqslant n} (a+g)^{-\mathbf{s}} \frac{e^g}{g!} \frac{(-n)_{M'}}{(\delta)_{M'}} \prod_{i=1}^r \left[\frac{c_i^{K_i} x_i^{-\lambda_i+\gamma_i K_i+g\zeta_i}}{K_i!} (\beta_i')_{K_i}\right]$$

$$I_{U:p_r+p_r'+r,q_r+q_r'+r;Y'}^{V;0,n_r+n_r'+r;X'} \left(\begin{array}{c|c} \mathrm{b}_1 x^{-\beta_1} & \mathbb{A} \; ; \; \mathrm{A}_4', \mathbf{A}' \\ . & . \\ . & . \\ \mathrm{b}_r x^{-\beta_r} & . \\ \mathrm{a}_1 & . \\ . & \\ . & \mathbb{B} \; ; \; \mathrm{B}_4', \mathbf{B} \\ \mathrm{a}_r & \end{array}\right) \tag{4.7}$$

where

$$A'_4 = \left(1 - \frac{\eta_1 + \lambda_1 - \zeta_1 g - \gamma_1 K_1}{\mu_1}; \frac{\beta_1}{\mu_1}, \underbrace{0, \cdots, 0}_{r-1}, \rho'_1, \underbrace{0, \cdots, 0}_{r-1}\right),$$

$$\left(1 - \frac{\eta_2 + \lambda_2 - \zeta_2 g - \gamma_2 K_2}{\mu_2}; 0, \frac{\beta_2}{\mu_2}, \underbrace{0, \cdots, 0}_{r-2}, 0, \rho'_2, \underbrace{0, \cdots, 0}_{r-2}\right), \cdots,$$

$$\left(1 - \frac{\eta_r + \lambda_r - \zeta_r g - \gamma_r K_r}{\mu_r}; \underbrace{0, \cdots, 0}_{r-1}, \frac{\beta_r}{\mu_r}, \underbrace{0, \cdots, 0}_{r-1}, \rho'_r\right) \tag{4.8}$$

$$B_4 = \left(-\alpha_1 - \frac{\eta_1 + \lambda_1 - \zeta_1 g - \gamma_1 K_1}{\mu_1}; \frac{\beta_1}{\mu_1}, \underbrace{0, \cdots, 0}_{r-1}, \rho'_1 + \sigma'_1, \underbrace{0, \cdots, 0}_{r-1}\right),$$

$$\left(-\alpha_2 - \frac{\eta_2 + \lambda_2 - \zeta_2 g - \gamma_2 K_2}{\mu_2}; 0, \frac{\beta_2}{\mu_2}, \underbrace{0, \cdots, 0}_{r-2}, 0, \rho'_2 + \sigma'_2, \underbrace{0, \cdots, 0}_{r-2}\right), \cdots,$$

$$\left(-\alpha_r - \frac{\eta_r + \lambda_r - \zeta_r g - \gamma_r K_r}{\mu_r}; \underbrace{0, \cdots, 0}_{r-1}, \frac{\beta_r}{\mu_r}, \underbrace{0, \cdots, 0}_{r-1}, \rho'_r + \sigma'_r\right) \tag{4.9}$$

under the same notations and conditions that theorem 4 with

$$Re\left(\eta_i + \lambda_i - \gamma_i n - g\zeta_i\right) + \left[\rho'_i \mu_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right) + \beta_i \min_{1 \leqslant j \leqslant m'^{(i)}} Re\left(\frac{b_j'^{(i)}}{\beta_j'^{(i)}}\right)\right] > 0$$

## 5. Conclusion

In this paper, we have obtained six theorems involving the generalized fractional integrals operators (simple and multiple). The images have been developed in terms of the product of the two multivariable I-functions and general classes of polynomials and generalized Riemann Zeta function in a compact and elegant form with the help of fractional integral operators. The formulae established in this paper are very general nature. Thus, the results established in this research work would serve as a key formulae from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables, simple and multiple integrals operators can be obtained.

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