Int. Jr. of Mathematical Sciences & Applications
Vol. 8, No. 1, (January-June, 2018)
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ISSN No: 2230-9888
www.ijmsa.yolasite.com

A study of finite integrals involving generalized associated Legendre function and special functions

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ABSTRACT
In this paper, we establish the unified infinite integral whose integrand involves the product of (τ,β)-generalized associated Legendre function of first kind ${}^{\tau,\beta}P_k^{m,n}(z)$, general class of multivariable polynomials $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[.]$, $\bar{H}$-function, a multivariable Aleph-function and multivariable I-function. On account of the most general nature of the functions occurring in the integrand of the second integral, our findings provides interesting unifications and extensions of a number of new and known integrals. We shall see several corollaries at the end.

Keywords:Multivariable Aleph-function, infinite integral, multivariable I-function, class of multivariable polynomials, multivariable H-function, generalized associated Legendre function of first kind, $\bar{H}$-function.

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20

1.Introduction and preliminaries.

Recently, Jain and Kumawat [6] have studied an unified finite integral involving the product of (τ,β)-generalized associated Legendre function of first kind ${}^{\tau,\beta}P_k^{m,n}(z)$, multivariable H-function, $\bar{H}$-function and class of polynomials $S_V^U(x)$ polynomials defined by Srivastava [11]. In this paper, we evaluate a general finite integral concerning the product of (τ,β)-generalized associated Legendre function of first kind ${}^{\tau,\beta}P_k^{m,n}(z)$, multivariable Aleph-function, multivariable I-function defined by Prasad [7], $\bar{H}$-function and class of multivariable polynomials with general arguments.

The $\bar{H}$-function occurring in the present paper was introduced by Inayat Hussain [5] and studied by Buschman and Srivastava [2] and others. The following series representationfor the H-function can easily be obtained from a result given by Rathie [8].

$$\bar{H}(z)=\bar{H}_{P,Q}^{M,N}\left(\mathrm{z}\ \middle|\ \begin{array}{l}(\mathrm{e}_j,E_j;\epsilon_j)_{1,N},(E_j,E_j)_{N+1,P}\\ (\mathrm{f}_j,F_j)_{1,M},(f_j,F_j;\mathfrak{F}_j)_{M+1,Q}\end{array}\right)=\frac{1}{2\pi\omega}\int_L\ \Omega_{P,Q}^{M,N}(s)z^s\mathrm{d}s \tag{1.1}$$

for all z different to 0 and

$$\Omega\,{}_{P,Q}^{M,N}(s)=\frac{\prod_{j=1,j\neq g}^{M}\Gamma(f_j-F_js)\prod_{j=1}^{N}\Gamma^{\epsilon_j}(1-e_j+E_js)}{\prod_{j=N+1}^{p}\Gamma(e_j-E_js)\prod_{j=M+1}^{q}\Gamma^{\mathfrak{F}_j}(1-f_j+F_js)} \tag{1.2}$$

The serie representation of $\bar{H}$-function is

$$\bar{H}(z)=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\Omega_{P,Q}^{M,N}(\eta_{G,g})\frac{z^{\eta_{G,g}}(-)^g}{\delta_G g!} \tag{1.3}$$

where

$$\eta_{G,g}=\frac{f_g+G}{F_G} \tag{1.4}$$

The sufficient conditions for the absolute convergence of the defining integral for $\bar{H}$-function given by (1.1) have been given by Gupta, Jain and Agrawal [4]. The behavior of the $\bar{H}_{P,Q}^{M,N}(z)$ function for small values of z is given by Saxena et al. [9, p.112, eq.(2.3)]. It is assumed that $\bar{H}$-function occurring at various places in the present paper satisfy the conditions of existence corresponding appropriately to those mentioned by Gupta, Jain and Agrawal [4].

The generalized polynomials of multivariables defined by Srivastava [12], is given in the following manner :

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[y_1,\cdots,y_v] = \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!}\cdots\frac{(-N_v)_{\mathfrak{M}_v K_v}}{K_v!}A[N_1,K_1;\cdots;N_v,K_v]y_1^{K_1}\cdots y_v^{K_v} \quad (1.5)$$

where $\mathfrak{M}_1,\cdots,\mathfrak{M}_v$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_v,K_v]$ are arbitrary constants, real or complex.

We shall note $a_v = \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!}\cdots\frac{(-N_v)_{\mathfrak{M}_v K_v}}{K_v!}A[N_1,K_1;\cdots;N_v,K_v]$

The Aleph-function of several variables is an extension the multivariable I-function defined by Sharma and Ahmad [10] , itself is an a generalisation of G and H-functions of several variables defined by Srivastava et Panda [13,14]. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function of $r-$ variables throughout our present study and will be defined and represented as follows (see Ayant [1]).

We have : $\aleph(z_1,\cdots,z_r) = \aleph_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}\left(\begin{array}{c|c} \mathrm{z}_1 & [(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}], \\ \cdot & \cdot \\ \cdot & \cdots\cdots\cdots\cdots\cdots\cdots , \\ \cdot & \\ \mathrm{z}_r & \end{array}\right.$

$$\begin{array}{l}[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}] : [(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots; \\ \cdot \\ [\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}] : [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;\end{array}$$

$$\left.\begin{array}{l}[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ \cdot \\ [(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\end{array}\right) = \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\prod_{k=1}^{r}\phi_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \quad (1.6)$$

with $\omega=\sqrt{-1}$ and

$$\phi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \quad (1.7)$$

and $$\phi_k(s_k) = \frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \quad (1.8)$$

For more details, see Ayant [1]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|arg z_k| < \frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)} - \tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)} - \tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} - \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)} + \sum_{j=1}^{m_k}\delta_j^{(k)} - \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} > 0,$$

with $k=1,\cdots,r, i=1,\cdots,R\ , i^{(k)}=1,\cdots,R^{(k)}$ (1.9)

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

If all the poles of (1.4) are simples ,then the integral (1.6) can be evaluated with the help of the residue theorem to give

$$\aleph(z_1,\cdots,z_r)=\sum_{G_k=1}^{m_k}\sum_{g_k=0}^{\infty}\phi\frac{\prod_{k=1}^{r}\phi_k z_k^{\eta_{G_k,g_k}}(-)^{\sum_{k=1}^{r}g_k}}{\prod_{k=1}^{r}\delta_{G^{(k)}}^{(k)}\prod_{k=1}^{r}g_k!} \tag{1.10}$$

where

$$\phi=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(i)}S_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}S_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}S_k)]} \tag{1.11}$$

$$\phi_k=\frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}S_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}S_k)}{\sum_{i^{(i)}=1}^{R^{(i)}}[\tau_{i^{(i)}}\prod_{j=m_i+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}S_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}S_k)]} \tag{1.12}$$

$$\sum_{G_k=1}^{m_i}\sum_{g_k=0}^{\infty}=\sum_{G_1,\cdots,G_r=1}^{m_1,\cdots,m_r}\sum_{g_1,\cdots,g_r=0}^{\infty} \tag{1.13}$$

and

$$S_k=\eta_{G_k,g_k}=\frac{d_{g_k}^{(k)}+G_k}{\delta_{g_k}^{(k)}}\text{ for } k=1,\cdots,r \tag{1.14}$$

which is valid under the following conditions : $\epsilon_{M_k}^{(k)}[p_j^{(k)}+p'_k]\neq\epsilon_j^{(k)}[p_{M_k}+g_k]$

The multivariable I-function of s-variables generalizes the multivariable H-function defined by Srivastava and Panda [13,14]. This representation of multiple Mellin-Barnes type integral is:

$$I(z_1,\cdots,z_s)=I_{p'_2,q'_2,p'_3,q'_3;\cdots;p'_s,q'_s:p'^{(1)},q'^{(1)};\cdots;p'^{(s)},q'^{(s)}}^{0,n'_2;0,n'_3;\cdots;0,n'_s:m'^{(1)},n'^{(1)};\cdots;m'^{(s)},n'^{(s)}}\left(\begin{array}{c|l}\mathrm{z}'_1 & (\mathrm{a}'_{2j};\alpha_{2j}'^{(1)},\alpha_{2j}'^{(2)})_{1,p_2};\cdots;\\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}'_s & (\mathrm{b}'_{2j};\beta_{2j}'^{(1)},\beta_{2j}'^{(2)})_{1,q_2};\cdots;\end{array}\right.$$

$$\left.\begin{array}{l}(\mathrm{a}'_{sj};\alpha_{sj}'^{(1)},\cdots,\alpha'_{sj}{}^{(s)})_{1,p'_s}:(a_j'^{(1)},\alpha_j'^{(1)})_{1,p'^{(1)}};\cdots;(a'_j{}^{(s)},\alpha'_j{}^{(s)})_{1,p'^{(s)}}\\ \\ (\mathrm{b}'_{sj};\beta_{sj}'^{(1)},\cdots,\beta'_{sj}{}^{(s)})_{1,q_s}:(b_j'^{(1)},\beta_j'^{(1)})_{1,q'^{(1)}};\cdots;(b'_j{}^{(s)},\beta'_j{}^{(s)})_{1,q'^{(s)}}\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^s}\int_{L'_1}\cdots\int_{L'_s}\phi(t_1,\cdots,t_s)\prod_{i=1}^{s}\phi_i(t_i)z_i'^{t_i}\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.15}$$

where

$$\phi_i(t_i)=\frac{\prod_{j=1}^{m'^{(i)}}\Gamma(b_j'^{(i)}-\beta_j'^{(i)}t_i)\prod_{j=1}^{n'^{(i)}}\Gamma(1-a_j'^{(i)}+\alpha_j'^{(i)}t_i)}{\prod_{j=m'^{(i)}+1}^{q'^{(i)}}\Gamma(1-b_j'^{(i)}+\beta_j'^{(i)}t_i)\prod_{j=n'^{(i)}+1}^{p'^{(i)}}\Gamma(a_j'^{(i)}-\alpha_j'^{(i)}t_i)}\ ,i=1,\cdots,s \tag{1.16}$$

and

$$\phi(t_1,\cdots,t_s)=\frac{\prod_{j=1}^{n'_2}\Gamma(1-a'_{2j}+\sum_{i=1}^{2}\alpha_{2j}'^{(i)}t_i)\prod_{j=1}^{n'_3}\Gamma(1-a'_{3j}+\sum_{i=1}^{3}\alpha_{3j}'^{(i)}t_i)\cdots}{\prod_{j=n'_2+1}^{p_2}\Gamma(a'_{2j}-\sum_{i=1}^{2}\alpha_{2j}'^{(i)}t_i)\prod_{j=n'_3+1}^{p'_3}\Gamma(a'_{3j}-\sum_{i=1}^{3}\alpha_{3j}'^{(i)}t_i)\cdots}$$

$$\frac{\cdots\prod_{j=1}^{n'_s}\Gamma(1-a'_{sj}+\sum_{i=1}^{s}\alpha'^{(i)}_{sj}t_i)}{\cdots\prod_{j=n'_s+1}^{p'_s}\Gamma(a'_{sj}-\sum_{i=1}^{s}\alpha'^{(i)}_{sj}t_i)\prod_{j=1}^{q'_2}\Gamma(1-b'_{2j}-\sum_{i=1}^{2}\beta'^{(i)}_{2j}t_i)}$$

$$\times\frac{1}{\prod_{j=1}^{q'_3}\Gamma(1-b'_{3j}+\sum_{i=1}^{3}\beta'^{(i)}_{3j}t_i)\cdots\prod_{j=1}^{q'_s}\Gamma(1-b'_{sj}-\sum_{i=1}^{s}\beta'^{(i)}_{sj}t_i)} \tag{1.17}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [7]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.8) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz'_i| < \frac{1}{2}\Omega'_i\pi$, where

$$\Omega'_i=\sum_{k=1}^{n'^{(i)}}\alpha'^{(i)}_k-\sum_{k=n'^{(i)}+1}^{p'^{(i)}}\alpha'^{(i)}_k+\sum_{k=1}^{m'^{(i)}}\beta'^{(i)}_k-\sum_{k=m'^{(i)}+1}^{q'^{(i)}}\beta'^{(i)}_k+\left(\sum_{k=1}^{n'_2}\alpha'^{(i)}_{2k}-\sum_{k=n'_2+1}^{p'_2}\alpha'^{(i)}_{2k}\right)+$$

$$+\cdots+\left(\sum_{k=1}^{n'_s}\alpha'^{(i)}_{sk}-\sum_{k=n'_s+1}^{p'_s}\alpha'^{(i)}_{sk}\right)-\left(\sum_{k=1}^{q'_2}\beta'^{(i)}_{2k}+\sum_{k=1}^{q'_3}\beta'^{(i)}_{3k}+\cdots+\sum_{k=1}^{q'_s}\beta'^{(i)}_{sk}\right) \tag{1.18}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$I(z'_1,\cdots,z'_s)=0(\,|z'_1|^{\alpha'_1},\cdots,|z'_s|^{\alpha'_s}\,),max(\,|z'_1|,\cdots,|z'_s|\,)\rightarrow 0$

$I(z'_1,\cdots,z'_s)\,=0(\,|z'_1|^{\beta'_1},\cdots,|z'_s|^{\beta'_s}\,),min(\,|z'_1|,\cdots,|z'_s|\,)\rightarrow\infty$

where $k=1,\cdots,s:\alpha''_k=min[Re(b'^{(k)}_j/\beta'^{(k)}_j)],j=1,\cdots,m'_k$ and

$\beta''_k=max[Re((a'^{(k)}_j-1)/\alpha'^{(k)}_j)],j=1,\cdots,n'_k$

The (τ,β)-generalized associated Legendre function of first kind is defined and represented in the following form [17, p.180, eq.(15-16)].

$${}^{\tau,\beta}P_k^{m,n}(z)=\frac{(z+1)^{\frac{n}{2}}(z-1)^{-\frac{m}{2}}}{\Gamma(1-m)}{}_2F_1^{\tau,\beta}\left[k-\frac{m-n}{2}+1,-k-\frac{m-n}{2};1-m;\frac{1-z}{2}\right] \tag{1.19}$$

where $|1-z|<2;k+\frac{m-n}{2}+1\neq 0,-1,-2,\cdots;k-\frac{m-n}{2}\neq 0,\pm 1,\pm 2,\cdots$

$m\neq 0,-1,-2,\cdots;|arg(1\pm z)|<\pi;\{\tau,\beta\}\subset\mathbb{R},\tau>0,\tau-\beta\leqslant 1$ where ${}_2F_1^{\tau,\beta}$ is known as (τ,β)-generalized hypergeometric function and it is given in the following form [17, p.176, eq.(1)].

$${}_2F_1^{\tau,\beta}(a,b;c;z)={}_2F_1^{\tau,\beta}(z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}{}_2\psi_1\,[(a,1),(c,\tau);(c,\beta);zt^\tau]\mathrm{d}t \tag{1.20}$$

where $[\{\tau,\beta\}\subset\mathbb{R},min(\tau,\beta)>0,\tau-\beta\leqslant 1,Re(c)>Re(b)>0,Re(a)>0]$.

The contour representation of (τ,β)-generalized associated Legendre function of first kind which is used in this paper is in the following form

$$ {}^{\tau,\beta}P_k^{m,n}(z)=\frac{(z+1)^{\frac{n}{2}}(z-1)^{\frac{-m}{2}}}{\Gamma\left(k-\frac{m-n}{2}+1\right)\Gamma\left(-k-\frac{m-n}{2}\right)} $$

$$ \int_L \frac{\Gamma\left(k-\frac{m-n}{2}+1+\zeta\right)\Gamma\left(-k-\frac{m-n}{2}+\tau\zeta\right)\Gamma(-\zeta)}{\Gamma(1-m+\beta\zeta)}\left(\frac{z-1}{2}\right)^{\zeta}\mathrm{d}\zeta \tag{1.21} $$

$$ =\frac{(z+1)^{\frac{n}{2}}(z-1)^{\frac{-m}{2}}}{\Gamma\left(k-\frac{m-n}{2}+1\right)\Gamma\left(-k-\frac{m-n}{2}\right)}H_{2,2}^{1,2}\left[\begin{array}{c}\frac{z-1}{2}\end{array}\left|\begin{array}{c}\left(-k+\frac{m-n}{2};1\right),\left(k+\frac{m-n}{2}+1;\tau\right)\\ (0;1),\ (\mathrm{m};\beta)\end{array}\right.\right] \tag{1.22} $$

2. Required integral

We have the following integral

Lemma

$$ \int_0^c z^{\upsilon-1}(c-z)^{\mu-1}(z+c)^{-\rho}\left(a+bz^q\right)^{-\sigma}\mathrm{d}z=a^{-\rho}c^{-\rho+\mu+\upsilon-1}\frac{\Gamma(\mu)}{\Gamma(\rho)\Gamma(\sigma)} $$

$$ H_{1,1:1,1;1,1}^{0,1:1,1;1,1}\left[\begin{array}{c}\frac{b}{a}\mathrm{c}^q\\ 1\end{array}\left|\begin{array}{c}(1\text{-}\upsilon;q,1):(1-\sigma;1);(1-\rho;1)\\ .\\ (1\text{-}\mu-\upsilon;q,1):(0,1);(0;1)\end{array}\right.\right] \tag{2.1} $$

Provided that $Re(\mu)>0$ and $Re(\upsilon)>0$. Concerning the proof, see Jain and Kumawat [6] for more details.

3. Main integral

Let

$$ X(z;\upsilon,\mu,\rho,\sigma)=z^{\upsilon}(z-1)^{-\mu}(z+1)^{-\rho}(a+bz^q)^{-\sigma} \tag{3.1} $$

We have the following general finite integral .

Theorem

$$ \int_0^c z^{\upsilon-1}(c-z)^{\mu-1}(z+c)^{-\rho}\left(a+bz^q\right)^{-\sigma}\bar{H}\left(xX(z;\upsilon',\mu',\rho',\sigma')\right)S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\left(\begin{array}{c}\mathrm{y}_1X(z;\upsilon_1'',\mu_1'',\rho_1'',\sigma_1'')\\ .\\ .\\ .\\ \mathrm{y}_vX(z;\upsilon_v'',\mu_v'',\rho_v'',\sigma_v'')\end{array}\right) $$

$$ {}^{\tau,\beta}P_k^{m,n}\left(\frac{z}{c}\right)\aleph\left(\begin{array}{c}\mathrm{z}_1'X(z;\upsilon_1',\mu_1',\rho_1',\sigma_1')\\ .\\ .\\ .\\ \mathrm{z}_r'X(z;\upsilon_r',\mu_r',\rho_r',\sigma_r')\end{array}\right)I\left(\begin{array}{c}\mathrm{z}_1X(z;\upsilon_1,\mu_1,\rho_1,\sigma_1)\\ .\\ .\\ .\\ \mathrm{z}_sX(z;\upsilon_s,\mu_s,\rho_s,\sigma_s)\end{array}\right)\mathrm{d}z=\frac{(-)^{-\frac{m}{2}+1}}{\Gamma\left(k-\frac{m-n}{2}+1\right)\Gamma\left(-k-\frac{m-n}{2}\right)} $$

$$ \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{G_i=1}^{m_i}\sum_{g_i=0}^{\infty}\phi\frac{\prod_{i=1}^s\phi_i z_i'^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^s g_i}}{\prod_{i=1}^s\delta_{G^{(i)}}^{(i)}\prod_{i=1}^s g_i!}\Omega_{P,Q}^{M,N}(\eta_{G,g})\frac{x^{\eta_{G,g}}(-)^g}{\delta_G g!} $$

$$a_v y_1^{K_1}\cdots y_v^{K_v} a^{-(\sigma+\sigma'\eta_{G,g}+\sum_{i=1}^{v}\sigma_i''K_i+\sum_{j=1}^{r}\sigma_j'\eta_{G_j,g_j})}$$

$$c^{\upsilon+\mu-\rho+(\upsilon'+\mu'-\rho')\eta_{G,g}+\sum_{i=1}^{v}(\upsilon_i''+\mu_i''-\rho_i'')K_i+\sum_{j=1}^{r}(\upsilon_j'+\mu_j'-\rho_j')\eta_{G_j,g_j}}$$

$$I_{U:p_s'+4,q_s'+3;Y}^{V;0,n_s'+4;X}\left(\begin{array}{c|c} \mathrm{z}_1 a^{\sigma_1}u^{\upsilon_1+\mu_1-\rho_1} & \mathbb{A}' \\ . & . \\ . & . \\ . & . \\ \mathrm{z}_s a^{\sigma_s}u^{\upsilon_s+\mu_s-\rho_s} & . \\ -\frac{1}{2} & . \\ \frac{b}{a}u^q & \mathbb{B}' \\ 1 & \end{array}\right) \tag{3.1}$$

Provided

$min\{\upsilon',\mu',\rho',\sigma',\upsilon_i'',\mu_i'',\rho_i'',\sigma_i'',\sigma_i'',\upsilon_j',\mu_j',\rho_j',\sigma_j',\upsilon_k,\mu_k,\rho_k,\sigma_k\}>0$; $i=1,\cdots,v;j=1,\cdots,r;k=1,\cdots,s$

$$Re\left(\mu-\frac{m}{2}+\sum_{i=1}^{r}\mu_i'\eta_{G_i,g_i}+\mu'\eta_{G,g}\right)+\sum_{k=1}^{s}\mu_k\min_{1\leqslant K\leqslant m'^{(k)}}Re\left(\frac{b_K'^{(k)}}{\beta_K'^{(k)}}\right)>0$$

$$Re\left(\upsilon+\sum_{i=1}^{r}\upsilon_i'\eta_{G_j,g_j}+\upsilon'\eta_{G,g}\right)+\sum_{k=1}^{s}\upsilon_k\min_{1\leqslant K\leqslant m'^{(k)}}Re\left(\frac{b_K'^{(k)}}{\beta_K'^{(k)}}\right)>0$$

$|argz_k'| \quad <\frac{1}{2}A_i^{(k)}\pi$,where $A_i^{(k)}$ is defined by (1.9). The multiple series in the left-hand side of (3.1) converges absolutely.

$|argz_i|<\frac{1}{2}\Omega_i'\pi$ where Ω_i' is defined by (1.18) and $c>0$.

Where

$$U=p_2',q_2';p_3',q_3';\cdots;p_{s-1}',q_{s-1}' \tag{3.2}$$

$$V=0,n_2';0,n_3';\cdots;0,n_{s-1}' \tag{3.3}$$

$$X=m'^{(1)},n'^{(1)};\cdots;m'^{(s)},n'^{(s)};1,2;1,0;1,0 \tag{3.4}$$

$$Y=p'^{(1)},q'^{(1)};\cdots;p'^{(s)},q'^{(s)};2,2;0,1;0,1 \tag{3.5}$$

$$\mathbb{A}'=(a_{2k}';\alpha_{2k}'^{(1)},\alpha_{2k}'^{(2)})_{1,p_2'};\cdots;(a_{(s-1)k}';\alpha_{(s-1)k}'^{(1)},\alpha_{(s-1)k}'^{(2)},\cdots,\alpha_{(s-1)k}'^{(s-1)})_{1,p_{s-1}'};$$

$$\left(1-\rho+\frac{n}{2}-\rho'\eta_{G,g}-\sum_{i=1}^{v}K_i\rho_i''-\sum_{j=1}^{r}\rho_j'\eta_{G_j,g_j};\rho_1,\cdots,\rho_s,0,0,1\right),$$

$$\left(1-\mu+\frac{m}{2}-\mu'\eta_{G,g}-\sum_{i=1}^{v}K_i\mu_i''-\sum_{j=1}^{r}\mu_j'\eta_{G_j,g_j};\mu_1,\cdots,\mu_s,1,0,0\right),$$

$$\left(1-\upsilon-\upsilon'\eta_{G,g}-\sum_{i=1}^{v}K_i\upsilon_i''-\sum_{j=1}^{r}\upsilon_j'\eta_{G_j,g_j};\upsilon_1,\cdots,\upsilon_s,0,q,1\right),$$

$$\left(1-\sigma-\sigma'\eta_{G,g}-\sum_{i=1}^{v}K_i\sigma_i''-\sum_{j=1}^{r}\sigma_j'\eta_{G_j,g_j};\sigma_1,\cdots,\sigma_s,0,1,0\right),$$

$$(a_{sk}';\alpha_{sk}'^{(1)},\alpha_{sk}'^{(2)},\cdots,\alpha_{sk}'^{(s)},0,0,0)_{1,p_s'}:(a_k'^{(1)},\alpha_k'^{(1)})_{1,p'^{(1)}};\cdots;(a_k'^{(s)},\alpha_k'^{(s)})_{1,p'^{(s)}};\left(-k+\frac{m-n}{2};1\right),$$

$$\left(k+\frac{m-n}{2}+1;\tau\right);-;- \tag{3.6}$$

$$\mathbb{B}'=(b_{2k}';\beta_{2k}'^{(1)},\beta_{2k}'^{(2)})_{1,q_2'};\cdots;(b_{(s-1)k}';\beta_{(s-1)k}'^{(1)},\beta_{(s-1)k}'^{(2)},\cdots,\beta_{(s-1)k}'^{(s-1)})_{1,q_{s-1}'}$$

$$\left(1-\mu-\upsilon+\frac{m}{2}-\mu'\eta_{G,g}-\sum_{i=1}^{v}K_i(\mu_i''+\upsilon_i'')-\sum_{j=1}^{r}(\mu_j'+\upsilon_j')\eta_{G_j,g_j};\mu_1+\upsilon_1,\cdots,\mu_s+\upsilon_s,1,q,1\right),$$

$$\left(1-\rho+\frac{n}{2}-\rho'\eta_{G,g}-\sum_{i=1}^{v}K_i\rho_i''-\sum_{j=1}^{r}\rho_j'\eta_{G_j,g_j};\rho_1,\cdots,\rho_s,0,0,0\right),$$

$$\left(1-\sigma-\sigma'\eta_{G,g}-\sum_{i=1}^{v}K_i\sigma_i''-\sum_{j=1}^{r}\sigma_j'\eta_{G_j,g_j};\sigma_1,\cdots,\sigma_s,0,0,0\right),(b_{sk}';\beta_{sk}'^{(1)},\beta_{sk}'^{(2)},\cdots,\beta_{sk}'^{(s)},0,0,0)_{1,q_s'}:$$

$$(b_k'^{(1)},\beta_k'^{(1)})_{1,q'^{(1)}};\cdots;(b_k'^{(s)},\beta_k'^{(s)})_{1,q'^{(s)}};(0;1),(m;\beta);(0;1);(0;1)$$

Proof

To evaluate the main integral (3.1), first we replace the class of multivariable polynomials $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_\mathfrak{v}}[.]$,the multivariable aleph-function, the function $\bar{H}_{P,Q}^{M,N}(z)$ and the (τ,β)-generalized associated Legendre function ${}^{\tau,\beta}P_k^{m,n}(z)$ occurring in the left-hand side of the main integral in term of series and integral contour form with the help of equations (1.5), (1.10), (1,3) and (1.21) respectively. Now we express the multivariable I-function in Mellin-Barnes integrals contour with the help of (1.15). Nest, we change the order of the $(t_1,\cdots,t_{s+1})$-integrals and z-integral, (which is justified under the conditions stated) and we obtain the following result (say L.H.S.) :

$$\text{L.H.S} = \frac{(-)^{-\frac{m}{2}+1}}{\Gamma\left(k-\frac{m-n}{2}+1\right)\Gamma\left(-k-\frac{m-n}{2}\right)}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_\mathfrak{v}]}\sum_{G=1}^{M}\sum_{g=0}^{\infty}$$

$$\sum_{G_i=1}^{m_i}\sum_{g_i=0}^{\infty}\phi\frac{\prod_{i=1}^{s}\phi_i z_i'^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{s}g_i}}{\prod_{i=1}^{s}\delta_{G^{(i)}}^{(i)}\prod_{i=1}^{s}g_i!}\Omega_{P,Q}^{M,N}(\eta_{G,g})\frac{x^{\eta_{G,g}}(-)^g}{\delta_G g!}\ a_v y_1^{K_1}\cdots y_v^{K_v}$$

$$=\frac{1}{(2\pi\omega)^s}\int_{L_1'}\cdots\int_{L_{s+1}'}\phi(t_1,\cdots,t_s)\prod_{i=1}^{s}\phi_i(t_i)z_k^{t_k}\frac{\Gamma\left(k-\frac{m-n}{2}+1+t_{s+1}\right)\Gamma\left(-k-\frac{m-n}{2}+\tau t_{s+1}\right)\Gamma(-t_{s+1})}{\Gamma\left(1-m+\beta t_{s+1}\right)}$$

$$\left(-\frac{1}{2c}\right)^{t_{s+1}}\left[\int_0^c (c-z)^{\mu+\mu''\eta_{G,g}+\sum_{i=1}^{v}K_i\mu_i''+\sum_{j=1}^{r}\mu_j'\eta_{G_j,g_j}+\sum_{k=1}^{s}\mu_k t_k-\frac{m}{2}+t_{s+1}-1}\right.$$

$$z^{\upsilon+\upsilon''\eta_{G,g}+\sum_{i=1}^{v}K_i\upsilon_i''+\sum_{j=1}^{r}\upsilon_j'\eta_{G_j,g_j}+\sum_{k=1}^{s}\upsilon_k t_k-1}(z+c)^{-\left(\rho+\rho''\eta_{G,g}+\sum_{i=1}^{v}K_i\rho_i''+\sum_{j=1}^{r}\rho_j'\eta_{G_j,g_j}+\sum_{k=1}^{s}\rho_k t_k-\frac{n}{2}\right)}$$

$$\left.(a+bz^q)^{-\left(\sigma+\sigma''\eta_{G,g}+\sum_{i=1}^{v}K_i\sigma_i''+\sum_{j=1}^{r}\sigma_j'\eta_{G_j,g_j}+\sum_{k=1}^{s}\sigma_k t_k\right)}\,\mathrm{d}z\right]\mathrm{d}t_1\cdots\mathrm{d}t_{s+1} \tag{3.7}$$

Now we evaluate the z-integral occuring in (3.7) with the help of the lemma (2.1), finally reinterpreting the result thus obtained in terms of I-function of $(s+3)$-variables. We obtain the right side of (3.1) after algebric manipulations.

4. Corollaries

The integral (3.1) is a general character. In this section, we studies several particular cases. First time, the class of multivariable polynomials reduces to class of polynomials of one variable defined by Srivastava [11], we get

Corollary 1

$$\int_0^c z^{\upsilon-1}(c-z)^{\mu-1}(z+c)^{-\rho}\,(a+bz^q)^{-\sigma}\,\bar{H}\,(xX(z;\upsilon',\mu',\rho',\sigma'))S_N^{\mathfrak{M}}\left(\ \mathrm{y}_1X(z;\upsilon_1'',\mu_1'',\rho_1'',\sigma_1'')\ \right)$$

$$\,^{\tau,\beta}P_k^{m,n}\left(\frac{z}{c}\right)\aleph\begin{pmatrix}\mathrm{z}_1'X(z;\upsilon_1',\mu_1',\rho_1',\sigma_1')\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r'X(z;\upsilon_r',\mu_r',\rho_r',\sigma_r')\end{pmatrix}I\begin{pmatrix}\mathrm{z}_1X(z;\upsilon_1,\mu_1,\rho_1,\sigma_1)\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_sX(z;\upsilon_s,\mu_s,\rho_s,\sigma_s)\end{pmatrix}\mathrm{d}z=\frac{(-)^{-\frac{m}{2}+1}}{\Gamma\left(k-\frac{m-n}{2}+1\right)\Gamma\left(-k-\frac{m-n}{2}\right)}$$

$$\sum_{K=0}^{[\mathfrak{N}/\mathfrak{M}]}\ \sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{G_i=1}^{m_i}\sum_{g_i=0}^{\infty}\phi\frac{\prod_{i=1}^{s}\phi_i z_i'^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{s}g_i}}{\prod_{i=1}^{s}\delta_{G^{(i)}}^{(i)}\prod_{i=1}^{s}g_i!}\Omega_{P,Q}^{M,N}(\eta_{G,g})\frac{x^{\eta_{G,g}}(-)^g}{\delta_G g!}$$

$$\frac{(-\mathfrak{N})_{\mathfrak{M}K}A_{\mathfrak{N},K}}{K!}y_1^K a^{-(\sigma+\sigma'\eta_{G,g}+\sigma''K+\sum_{j=1}^{r}\sigma_j'\eta_{G_j,g_j})}c^{\upsilon+\mu-\rho+(\upsilon'+\mu'-\rho')\eta_{G,g}+(\upsilon''+\mu''-\rho'')K_i+\sum_{j=1}^{r}(\upsilon_j'+\mu_j'-\rho_j')\eta_{G_j,g_j}}$$

$$I_{U:p_s'+4,q_s'+3;Y}^{V;0,n_s'+4;X}\left(\begin{array}{c|c}\mathrm{z}_1a^{\sigma_1}u^{\upsilon_1+\mu_1-\rho_1} & \mathbb{A}'\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_sa^{\sigma_s}u^{\upsilon_s+\mu_s-\rho_s} & \cdot\\ -\frac{1}{2} & \\ \frac{b}{a}u^q & \mathbb{B}'\\ 1 & \end{array}\right) \tag{4.1}$$

Provided

$$min\{\upsilon',\mu',\rho',\sigma',\upsilon'',\mu'',\rho'',\sigma_i'',\sigma_i'',\upsilon_j',\mu_j',\rho_j',\sigma_j',\upsilon_k,\mu_k,\rho_k,\sigma_k\}>0\ ;\ j=1,\cdots,r;k=1,\cdots,s$$

the other conditions are the same that (3.1), where

$$\mathbb{A}'=(a_{2k}';\alpha_{2k}'^{(1)},\alpha_{2k}'^{(2)})_{1,p_2'};\cdots;(a_{(s-1)k}';\alpha_{(s-1)k}'^{(1)},\alpha_{(s-1)k}'^{(2)},\cdots,\alpha_{(s-1)k}'^{(s-1)})_{1,p_{s-1}'};$$

$$\left(1-\rho+\frac{n}{2}-\rho'\eta_{G,g}-K\rho''-\sum_{j=1}^{r}\rho'_j\eta_{G_j,g_j};\rho_1,\cdots,\rho_s,0,0,1\right),\left(1-\mu+\frac{m}{2}-\mu'\eta_{G,g}-K\mu''-\sum_{j=1}^{r}\mu'_j\eta_{G_j,g_j};\mu_1,\cdots,\mu_s,1,0,0\right),$$

$$\left(1-\sigma-\sigma'\eta_{G,g}-K\sigma''-\sum_{j=1}^{r}\sigma'_i\eta_{G_j,g_j};\sigma_1,\cdots,\sigma_s,0,1,0\right),\left(1-\upsilon-\upsilon'\eta_{G,g}-K\upsilon''-\sum_{j=1}^{r}\upsilon'_j\eta_{G_j,g_j};\upsilon_1,\cdots,\upsilon_s,0,q,1\right),$$

$$(a'_{sk};\alpha'^{(1)}_{sk},\alpha'^{(2)}_{sk},\cdots,\alpha'^{(s)}_{sk},0,0,0)_{1,p'_s}:(a'^{(1)}_k,\alpha'^{(1)}_k)_{1,p'^{(1)}};\cdots;(a'^{(s)}_k,\alpha'^{(s)}_k)_{1,p'^{(s)}};\left(-k+\frac{m-n}{2};1\right),$$

$$\left(k+\frac{m-n}{2}+1;\tau\right);-;- \tag{4.2}$$

$$\mathbb{B}'=(b'_{2k};\beta'^{(1)}_{2k},\beta'^{(2)}_{2k})_{1,q'_2};\cdots;(b'_{(s-1)k};\beta'^{(1)}_{(s-1)k},\beta'^{(2)}_{(s-1)k},\cdots,\beta'^{(s-1)}_{(s-1)k})_{1,q'_{s-1}}$$

$$\left(1-\mu-\upsilon+\frac{m}{2}-\mu'\eta_{G,g}-K(\mu''+\upsilon'')-\sum_{j=1}^{r}(\mu'_j+\upsilon'_j)\eta_{G_j,g_j};\mu_1+\upsilon_1,\cdots,\mu_s+\upsilon_s,1,q,1\right),$$

$$\left(1-\rho+\frac{n}{2}-\rho'\eta_{G,g}-K\rho''-\sum_{j=1}^{r}\rho'_j\eta_{G_j,g_j};\rho_1,\cdots,\rho_s,0,0,0\right),\left(1-\sigma-\sigma'\eta_{G,g}-K\sigma''-\sum_{j=1}^{r}\sigma'_j\eta_{G_j,g_j};\sigma_1,\cdots,\sigma_s,0,0,0\right),$$

$$(b'_{sk};\beta'^{(1)}_{sk},\beta'^{(2)}_{sk},\cdots,\beta'^{(s)}_{sk},0,0,0)_{1,q'_s}:(b'^{(1)}_k,\beta'^{(1)}_k)_{1,q'^{(1)}};\cdots;(b'^{(s)}_k,\beta'^{(s)}_k)_{1,q'^{(s)}};(0;1),(m;\beta);(0;1);(0;1) \tag{4.3}$$

Consider the above corollary, if the multivariable Aleph-function and the multivariable I-function reduce respectively to Aleph-function of one variable defined by Sudland [15] and Fox H-function [3] of one variable, we get

Corollary 2

$$\int_0^c z^{\upsilon-1}(c-z)^{\mu-1}(z+c)^{-\rho}\left(a+bz^q\right)^{-\sigma}\bar{H}\left(xX(z;\upsilon',\mu',\rho',\sigma')\right)S_N^{\mathfrak{M}}\left(\mathrm{y}_1X(z;\upsilon''_1,\mu''_1,\rho''_1,\sigma''_1)\right)$$

$${}^{\tau,\beta}P_k^{m,n}\left(\frac{z}{c}\right)\aleph\left(\mathrm{z}'_1X(z;\upsilon'_1,\mu'_1,\rho'_1,\sigma'_1)\right)H^{m'^{(1)},n'^{(1)}}_{p'^{(1)}q'^{(1)}}\left(\mathrm{z}_1X(z;\upsilon_1,\mu_1,\rho_1,\sigma_1)\right)\mathrm{d}z=$$

$$\frac{(-)^{-\frac{m}{2}+1}}{\Gamma\left(k-\frac{m-n}{2}+1\right)\Gamma\left(-k-\frac{m-n}{2}\right)}\sum_{K=0}^{[\mathfrak{N}/\mathfrak{M}]}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{G_1=1}^{m_1}\sum_{g_1=0}^{\infty}\frac{\phi_1 z_1'^{\eta_{G_1,g_1}}(-)^{g_1}}{\delta^{(1)}_{G^{(1)}}g_1!}\frac{(-\mathfrak{N})_{\mathfrak{M}K}A_{\mathfrak{N},K}}{K!}y_1^K$$

$$a^{-(\sigma+\sigma'\eta_{G,g}+\sigma''K+\sigma'_1\eta_{G_1,g_1})}c^{\upsilon+\mu-\rho+(\upsilon'+\mu'-\rho')\eta_{G,g}+(\upsilon''+\mu''-\rho'')K_i+(\upsilon'_1+\mu'_1-\rho'_1)\eta_{G_1,g_1}}$$

$$H^{m'^{(1)},n'^{(1)}+4;1,2;1,0;1,0}_{p'^{(1)}+4,q'^{(1)}+3;2,2;0,1;0,1}\left(\begin{array}{c|c}\mathrm{z}_1a^{\sigma_1}u^{\upsilon_1+\mu_1-\rho_1} & \mathbb{A}'\\ -\frac{1}{2} & \cdot\\ \frac{b}{a}u^q & \mathbb{B}'\\ 1 & \end{array}\right) \tag{4.4}$$

Provided

$min\{\upsilon',\mu',\rho',\sigma',\upsilon'',\mu'',\rho'',\sigma_1'',\sigma_1'',\upsilon_1',\mu_1',\rho_1',\sigma_1',\upsilon_1,\mu_1,\rho_1,\sigma_1\} > 0$

$$Re\left(\mu-\frac{m}{2}+\mu_1'\eta_{G_1,g_1}+\mu'\eta_{G,g}\right)+\mu_1\min_{1\leqslant K\leqslant m'^{(1)}}Re\left(\frac{b_K'^{(1)}}{\beta_K'^{(1)}}\right)>0$$

$$Re\left(\upsilon+\upsilon_1'\eta_{G_1,g_1}+\upsilon'\eta_{G,g}\right)+\upsilon_1\min_{1\leqslant K\leqslant m'^{(1)}}Re\left(\frac{b_K'^{(1)}}{\beta_K'^{(1)}}\right)>0$$

$|argz_1'| < \frac{1}{2}A_i^{(1)}\pi$ where $A_i^{(k)}$ is defined by (1.9).

$|argz_1| < \frac{1}{2}\Omega_1'\pi$ where Ω_i' is defined by (1.18) and $c > 0$ and

$$\mathbb{A}'=\left(1-\rho+\frac{n}{2}-\rho'\eta_{G,g}-K\rho''-\rho_1'\eta_{G_1,g_1};\rho_1,0,0,1\right),\left(1-\mu+\frac{m}{2}-\mu'\eta_{G,g}-K\mu''-\mu_1'\eta_{G_1,g_1};\mu_1,1,0,0\right),$$

$$(1-\upsilon-\upsilon'\eta_{G,g}-K\upsilon''-\upsilon_1'\eta_{G_1,g_1};\upsilon_1,0,q,1),(1-\sigma-\sigma'\eta_{G,g}-K\sigma''-\sigma_1'\eta_{G_1,g_1};\sigma_1,0,1,0),$$

$$(a_k'^{(1)},\alpha_k'^{(1)})_{1,p'^{(1)}};\left(-k+\frac{m-n}{2};1\right),\left(k+\frac{m-n}{2}+1;\tau\right);-;- \qquad (4.5)$$

$$\mathbb{B}'=\left(1-\mu-\upsilon+\frac{m}{2}-\mu'\eta_{G,g}-K(\mu''+\upsilon'')-(\mu_1'+\upsilon_1')\eta_{G_1,g_1};\mu_1+\upsilon_1,1,q,1\right),$$

$$\left(1-\rho+\frac{n}{2}-\rho'\eta_{G,g}-K\rho''-\rho_1'\eta_{G_1,g_1};\rho_1,0,0,0\right),(1-\sigma-\sigma'\eta_{G,g}-K\sigma''-\sigma_1'\eta_{G_1,g_1};\sigma_1,0,0,0),$$

$$(b_k'^{(1)},\beta_k'^{(1)})_{1,q'^{(1)}};(0;1),(m;\beta);(0;1);(0;1) \qquad (4.6)$$

Consider the above corollary, if the class of polynomials $S_N^{\mathfrak{M}}[.]$ and the generalized associated Legendre function ${}^{\tau,\beta}P_k^{m,n}(z)$ reduce respectively to 1 and $P_k^{(m,n)}(z)$ function [16, p. 241, eq. (5.2)], we get the following result.

Corollary 3

$$\int_0^c z^{\upsilon-1}(c-z)^{\mu-1}(z+c)^{-\rho}(a+bz^q)^{-\sigma}H(xX(z;\upsilon',\mu',\rho',\sigma'))P_k^{m,n}\left(\frac{z}{c}\right)$$

$$\aleph(\ z_1'X(z;\upsilon_1',\mu_1',\rho_1',\sigma_1')\ )H^{m'^{(1)},n'^{(1)}}_{p'^{(1)}q'^{(1)}}(\ z_1X(z;\upsilon_1,\mu_1,\rho_1,\sigma_1)\ )\mathrm{d}z=$$

$$\frac{(-)^{-\frac{m}{2}+1}}{\Gamma\left(k-\frac{m-n}{2}+1\right)\Gamma\left(-k-\frac{m-n}{2}\right)}\sum_{K=0}^{[\mathfrak{N}/\mathfrak{M}]}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{G_1=1}^{m_1}\sum_{g_1=0}^{\infty}\frac{\phi_1 z_1'^{\eta_{G_1,g_1}}(-)^{g_1}}{\delta_{G^{(1)}}^{(1)}g_1!}\frac{(-\mathfrak{N})_{\mathfrak{M}K}A_{\mathfrak{N},K}}{K!}y_1^K$$

$$a^{-(\sigma+\sigma'\eta_{G,g}+\sigma''K+\sigma_1'\eta_{G_1,g_1})}c^{\upsilon+\mu-\rho+(\upsilon'+\mu'-\rho')\eta_{G,g}+(\upsilon''+\mu''-\rho'')K_i+(\upsilon_1'+\mu_1'-\rho_1')\eta_{G_1,g_1}}$$

$$H_{p'^{(1)}+4,q'^{(1)}+3;2,2;0,1;0,1}^{m'^{(1)},n'^{(1)}+4;1,2;1,0;1,0}\left(\begin{array}{c|c} \mathrm{z}_1 a^{\sigma_1} u^{\upsilon_1+\mu_1-\rho_1} & \mathbb{A}' \\ -\frac{1}{2} & \cdot \\ \frac{b}{a}u^q & \mathbb{B}' \\ 1 & \end{array}\right) \tag{4.7}$$

Provided

$min\{\upsilon',\mu',\rho',\sigma',\upsilon'',\mu'',\rho'',\sigma''_1,\sigma''_1,\upsilon'_1,\mu'_1,\rho'_1,\sigma'_1,\upsilon_1,\mu_1,\rho_1,\sigma_1\} > 0$, the other conditions are the same that (4.4), where

$$\mathbb{A}' = \left(1-\rho+\frac{n}{2}-\rho'\eta_{G,g}-\rho'_1\eta_{G_1,g_1};\rho_1,0,0,1\right),\left(1-\mu+\frac{m}{2}-\mu'\eta_{G,g}-\mu'_1\eta_{G_1,g_1};\mu_1,1,0,0\right),$$

$$(1-\upsilon-\upsilon'\eta_{G,g}-\upsilon'_1\eta_{G_1,g_1};\upsilon_1,0,q,1),(1-\sigma-\sigma'\eta_{G,g}-\sigma'_1\eta_{G_1,g_1};\sigma_1,0,1,0),$$

$$(a_k'^{(1)},\alpha_k'^{(1)})_{1,p'^{(1)}};\left(-k+\frac{m-n}{2};1\right),\left(k+\frac{m-n}{2}+1;1\right);-;- \tag{4.8}$$

$$\mathbb{B}' = \left(1-\mu-\upsilon+\frac{m}{2}-\mu'\eta_{G,g}-(\mu'_1+\upsilon'_1)\eta_{G_1,g_1};\mu_1+\upsilon_1,1,q,1\right),$$

$$\left(1-\rho+\frac{n}{2}-\rho'\eta_{G,g}-\rho'_1\eta_{G_1,g_1};\rho_1,0,0,0\right),(1-\sigma-\sigma'\eta_{G,g}-\sigma'_1\eta_{G_1,g_1};\sigma_1,0,0,0),$$

$$(b_k'^{(1)},\beta_k'^{(1)})_{1,q'^{(1)}};(0;1),(m;1);(0;1);(0;1) \tag{4.9}$$

5. Conclusion

In this paper, we evaluate an unified finite integral involving the product of (τ,β)-generalized associated Legendre function of first kind ${}^{\tau,\beta}P_k^{m,n}(z)$, general class of multivariable polynomials $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[y_1,\cdots,y_v]$, H-function, multivariable aleph-function and multivariable I-function. The formula established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

References

[1] F.Y. Ayant, An integral associated with the Aleph-functions of several variables. International Journal of Mathematics Trends and Technology (IJMTT), Vol 31 (3) (2016), 142-154.

[2] R.G. Buschman and H.M. Srivastava, The -Function associated with a certain class of Feynman integrals, J. Phys. A: Math. Gen., 23 (1990), 4707-4710.

[3] C. Fox, The G and H-functions as symmetrical Fourier Kernels, Trans. Amer. Math. Soc. 98 (1961)., 395-429.

[4] K.C. Gupta, R. Jain and R. Agrawal, On existence conditions for generalized Mellin-Barnes type integral, Nat. Acad. Sci. Lett., 30(5 & 6) (2007), 169-172.

[5] A. Inayat-Hussain, New properties of hypergeometric series derivable from Feynman integrals: II. A generalization of the H-Function, J. Phys. A: Math. Gen., 20 (1987), 4119-4128.

[6] R. Jain and P. Kumawat, On unified finite integral involving ${}^{\tau,\beta}P_k^{m,n}(z)$ function, multivariable H-function and $S_V^U(x)$ polynomial, International Journal of Computational Science and Mathematics, 4(3) (2012), 233-242.

[7] Y.N. Prasad, Multivariable I-function, Vijnana Parishad Anusandhan Patrika 29 (1986), 231-237.

[8] A.K. Rathie, A new generalization of generalized hypergeometric functions, Le Matematiche Fasc. II, 52 (1997), 297-310.

[9] R.K. Saxena, S.L. Kalla and R. Saxena, Multivariate analogue of Generalized Mittag- Leffler function, Integral Transform and special functions I First (2011), 1-6.

[10] C.K. Sharma and S.S. Ahmad, On the multivariable I-function. Acta ciencia Indica Math, 20(2) (1994), 113-116.

[11] H.M. Srivastava, A contour integral involving Fox's H-function. Indian. J. Math, (14) (1972)., 1-6.

[12] H.M. Srivastava, A multilinear generating function for the Konhauser set of biorthogonal polynomials suggested by Laguerre polynomial, Pacific. J. Math. 177(1985), 183-191.

[13] H.M.Srivastava and R. Panda, Some expansion theorems and generating relations for the H-function of several complex variables. Comment. Math. Univ. St. Paul. 24 (1975),119-137.

[14] H.M.Srivastava and R.Panda, Some expansion theorems and generating relations for the H-function of several complex variables II. Comment. Math. Univ. St. Paul. 25 (1976), 167-197.

[15] N. Südland, B. Baumann and T.F. Nonnenmacher, Open problem, who knows about the Aleph-functions? Fract. Calc. Appl. Anal, 1(4) (1998), 401-402.

[16] Nina A.Virchenko, On some Generalizations of the Functions of Hypergeometric Type, Fractional Calculus $ Applied Analysis, 2(3) (1999), 233-244.

[17] Nina A.Virchenko and Olena V. Rumiantseva, On The Generalized Associated Legendre Functions, Fractional Calculus and Applied Analysis, 11(2) (2008), 175-184.