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FRIEZE GROUPS AND FRIEZE PATTERNS

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***Abstract**: In analyzing Art mathematics is very useful. In mathematical sense all the art consists of articles that are pleasing to someone's eye because they are symmetrical in nature. Felix Klein (1949-1925) pays attention to the fact that classifying different kind of geometries that preserve some interesting properties such as Euclidean, Spherical and Bolyai-Lobachevsky geometry. Analyzing Art major tool in mathematics is Group Theory and Groups of isomerties . An isometry is a transformation that preserves distances. In this paper we discuss isometries and its groups and seven patterns formed by different isometries and give some examples of different designs formed by these patterns.*

***Keywords**: Isometry, Translation, Rotation, Reflection, Metric Space, Glide reflection, Symmetry.*

1. INTRODUCTION

One can also look at the symmetry of a single motif or ornament. Examples of such motifs (illustrated from patterns used in batiks) are shown below. Such ornaments typically have rotational symmetry and/or reflection symmetry.

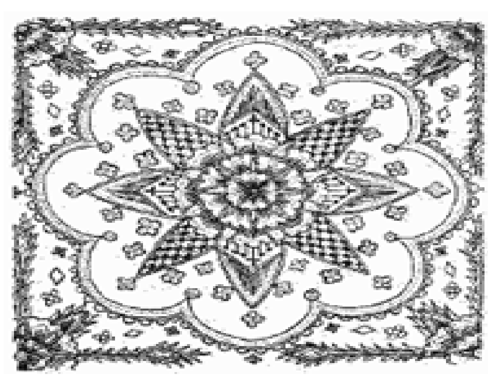


Figure 1

A more complex pattern such as the one below can be built up from simple motifs. Such patterns have translational symmetry in one direction. Designs or patterns of this kind are known as strip, band, or frieze patterns.

Figure 2

Preliminary concepts:

Definition: Let X be a set, a map d: $XxX \to Y$ such that

(1) $d(x, y) \geq 0$ for all $x, y \in X$

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$

(3) $d(x, y) = 0$ iff $x = y$

(4) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$

The map d is called a metric on the set X and the pair (X, d) is called a Metric Space.

Definition: An Isometry of a metric space (X, d) is a bijection map $i : X \to X$ such that for all $x, y \in X$, $d(x, y) = d(ix, iy)$. Isom (X, d) denote the set of all Isometries of a metric space (X, d). It can be stated that isometry is a function from the set X to X which preserve distances, that is if $x = (x_1, y_1)$ and $y = (x_2, y_2) \in R^2$

Then $d(x, y) = \sqrt{(x_i - y_i)^2}$ is a metric on R^2.

Theorem: The set of Isometries of a set X, i.e. Isom (X, d) form a group under the operation composition of maps.

Proof: Consider I: $X \to X$ s.t. $I(x) = x$ is obviously a bijection and for al $x, y \in X$

$d(1x, 1y) = d(x, y)$ as $1(x) = x$ and $1(y) = y$ shows that $I \in$ Isom (X, d) is the identity element.

Composition of maps is obviously associative.

Further let $u \in$ Isom (X, d) then u is a bijection there exist u^{-1} such that $uu^{-1} = u^{-1} u = \mathrm{I}$

For $x, y \in X$, $d\,(xu^{-1}, yu^{-1} = d(xu^{-1}u, yu^{-1}u) = d(xI, yI) = d(x, y)$, shows that $u^{-1} \in$ Isom (X, d) Proves that Isom (X, d) is a group.

The most well-known Isometries of the plane are Translation, Rotation and Reflections.

Definition:

Translation: A Translation is a map t that moves every point to a fixed distance in a fixed direction. Symbolically in R^2 for a fixed point (a, b), define $T_{a,b} : R^2 \to R^2$ by $T_{a,b}\,(x, y) = (x + a, y + b)$ and the vector $a = (a, b)$ is called axis of the translation $T_{a,b}$.

Now $T(R^2) = \{T_{a,b}/\, a,b \in R\}$ is a group under the operation $T_{a,b}.T_{c,d} = T_{a+c,b+d}$ where $T_{0,0}$ is the Identity and $T_{-a,-b}$ is the Inverse of T. It is also Abelian group.

Rotation: The Rotation r of the plane is a map that moves every point through a fixed angle about a fixed point, called center. If center be taken as origin then using polar coordinate system $r(\mathrm{l}, \theta) \to (l, \theta + a)$ where (l, θ) are the polar co-ordinates of an arbitrary point in R^2 and a be the fixed angle for which the Rotation takes place. We denote such a rotation by $r = r(0, \alpha)$. Rotation with same center O can be composed according to $r(0, \alpha)$. $r(0, \beta) = r(0, \beta)$ where $r(0, \alpha + \beta)$:$(\rho, \theta) \to (\rho\theta, + \alpha + \beta)$ where α, β are constant angles.

Reflection: A reflection across a line L is that function that leaves every point of L fixed and takes any point q not on L to the point q' so that L the perpendicular bisector of the line segment joining q and q'. A reflection across a plane in three dimensions is defined analogously. The line L is called the axis of reflection. In x, y coordinate plane whereas $(x, y) \to (y, x)$ is the reflection across the line $y = x$. Some authors call an axis of reflective symmetry L a mirror because L acts like a two sided mirror. Reflections are called opposite isometric because they reverse orientation. For example the reflected image of a clockwise spiral.

Glide Reflection: Glide reflection is the product of a translation and a reflection across the line containing the translation vector. For example there is an isometry of the reflection on X-axis followed by a unit translation parallel to it as $(x, y) \rightarrow (x + 1, -y)$, shown in the figure 3.

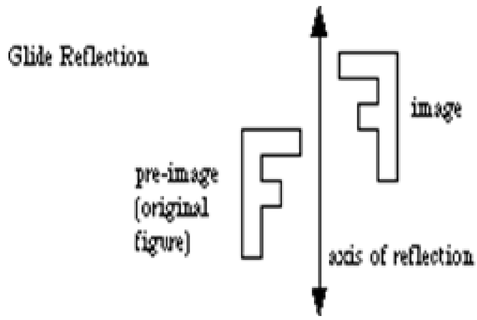


Figure 3

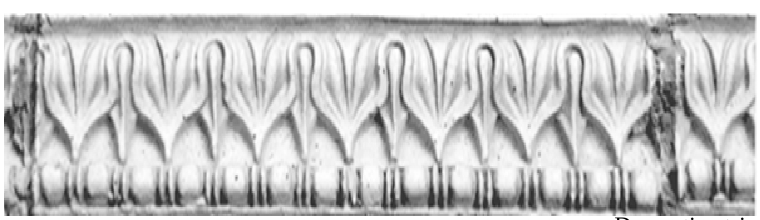
The vector gives the direction and length of the translation and is contained in the axis of reflection. A glide reflection is also an opposite isometry. Successive foot prints in wet sand are related by a glide reflection.

Here it is mentioned that translations, rotations and reflections are Isometries of R^2 and satisfy their described properties regarding Orientation, fixed points and compositions. Having defined the notion of Translation, Rotation and Reflection now it is important to mention here that every isometry of the plane be written as a composition of all the three.

Dihedral group: For each $n \geq 3$, a group G is called Dihedral group of order n if it is generated by two elements a, b such that $o(a) = n$; $o(b) = 2$ and $ba = a^{-1}b$

It can be shown that for each $n \geq 3$, Dn is a group of order $2n$ and is unique up to isomorphism. It can be thought as a group of rotations and reflections of a regular n-gons with n vertices where a is the rotation through angle$2\pi/n$ and b is the reflection.

The Frieze Groups: Dictionary meaning of the term Frieze is "A broad Horizontal Band of Sculpted or Painted Decoration". Especially on a wall nears the ceiling. Frieze groups are collection of infinite symmetry groups that arises from periodic designs in a plane. Figs (4) & (5).



Decorative trim on a building

Figure 4

Figure 5

These are of the two types:

1. **Discrete Frieze Groups:** That is plane symmetry groups of patterns whose subgroups of translation are isometric to *Z*, the set of Integers. These kinds of designs are used for decorative strips and for patterns of jewelry as shown in the figure 6.

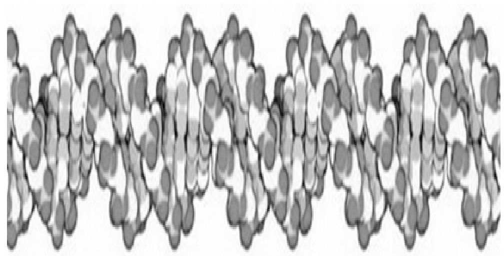


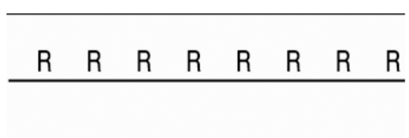
Figure 6

Similar examples in mathematics are $y = \sin x$, $y = \cos x$, $y = \tan x$, $y = |\sin x|$ or

$$y = |\cos x|$$

2. **Discrete Symmetry Groups:** are those plane patterns whose subgroups $Z \oplus Z$ of translations are isomorphic to Z?Z. There are seven patterns in the Frieze Groups We here discuss them as under.

Pattern I: Also known as (F_1) **Hop**: Consists of translation only. If *x* denote a translation to right of one unit we may write the symmetry group of patterns *I* as F_1 as $F_1 = \{x^n / n \in Z\}$ as shown in the figures 7.



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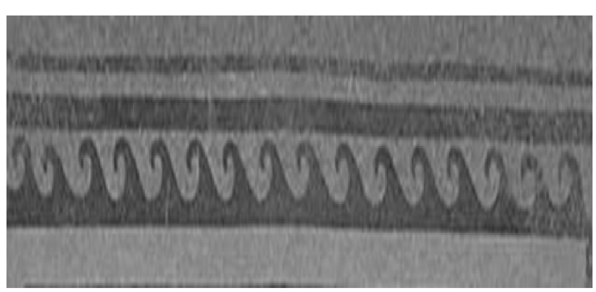
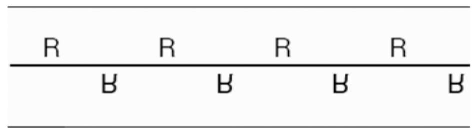


Figure 7

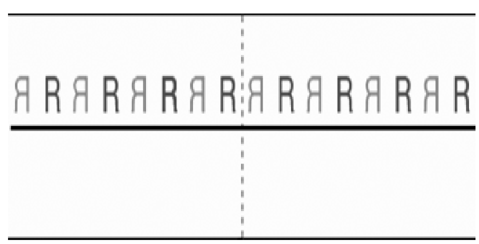
Pattern II: Also known as (F_2) **STEP**: Consists of translation and Glide reflection symmetries. If x is a glide reflection then we may write symmetry group of Patterns F_2 as $F_2 = \{x^n / n \in Z\}$ as shown in the figure 8.



Or

Figure 8

Pattern III: Also known as F_3 **SIDLE**: Consists of translation x and vertical reflection y called axis of symmetry and there are infinitely many axis of symmetry. The entire group is written as $F_3 = \{x^n\, y^m / n \in Z, m = 0 \text{ or } 1\}$, each pair of elements xy and y have order 2 and they generate F_3. Thus by a theorem: A group generated by a pair of elements of order 2 is dihedral. The group is shown by the figure 9.



or

It forms a design as under

Figure 9

Pattern IV: Also known as (F_4) **SPINNING HOP**: This symmetry generated by translation x and rotation y of 180° about a point p midway between consecutive *R's* (called Half Turn). As like F_3 it is also infinitely dihedral (Another rotation point lies between a top and bottom R. As in pattern III, the distance between consecutive points of rotational symmetry is half of the length of the smallest translation vector) and is denoted by $F_4 = \{x^n y^m / n \in Z, m = 0 \text{ or } 1\}$ shown by the figure (10).

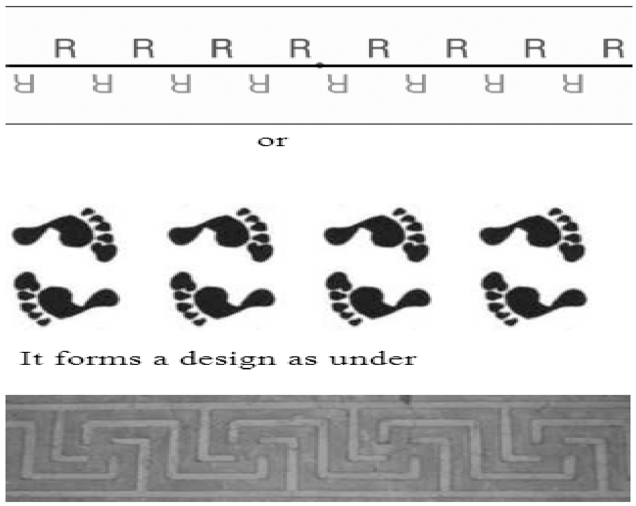


Figure 10

Pattern V: Also known as (F_5) SPINNING SIDLE: It is another dihedral group generated by a glide reflection x and a rotation y of 180° about a point p. Pattern V has vertical reflection symmetry xy and rotation points are midway between the vertical reflection axes. The group is represented as $F_S = \{x^n y^m / n \in Z, m = 0$ or $1\}$ shown by the figure (11).

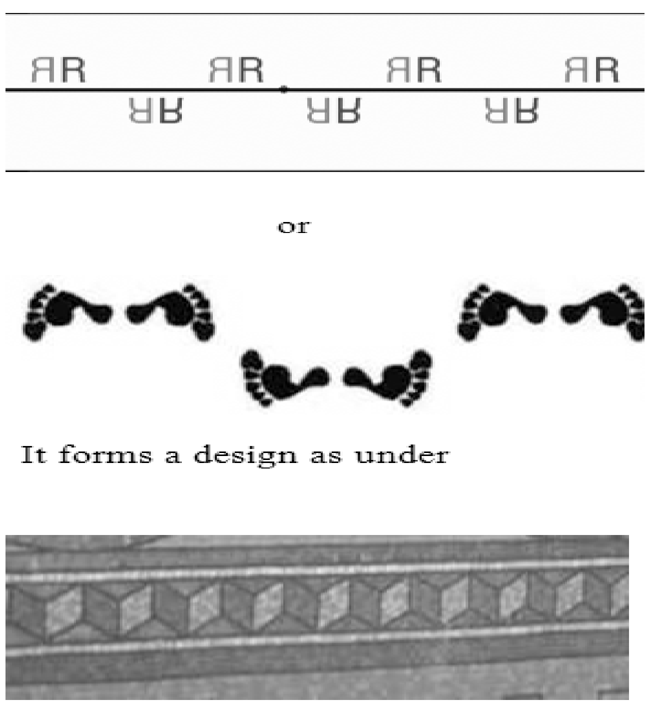
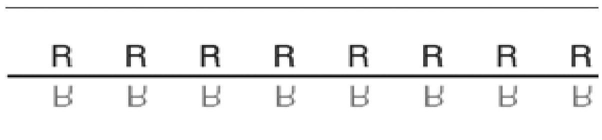


Figure 11

Pattern VI: Also known as (F_6) JUMP: Generated by a translation x and a horizontal reflection y. The elements are represented as $F_6 = \{x^n\, y^m / n \in Z, m = 0 \text{ or } 1\}$. In this pattern x and y commute And F_6 is not Infinite dihedral. Infect F_6 isomorphic to $Z \oplus Z$. Pattern VI is left invariant under a glide reflection also. But in this case the glide reflection is called Trivial since it is the product of x and y. (Conversely a glide reflection is non-trivial if its translation component and reflection component are not elements of the symmetry group). Shown by the figure 12.



or

It forms a design as under

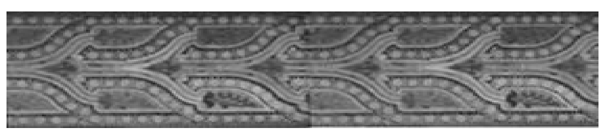


Figure 12

Pattern VII: Also known as (F_7) SPINNING JUMP: is generated by a translation x, a horizontal reflection y and a vertical reflection z. It is isomorphic to the direct product of the infinite dihedral group Z_2 and the product of y and z is a 180° rotation. The group is represented as $F_7 = \{x^n/y^m\, z^k/n \in Z, m = 0 \text{ or } 1, k = 0 \text{ or } 1\}$ and is shown by the figure 13.

or

In form a design under

Figure 13

Preposition: The set of translations T is normal subgroup in a Frieze group G.

Proof: Let $\tau \in T$ so that $\tau(z) \to z+1$ and any $h \in G$. If T is normal in G if h^{-1} Th $\leq$ for all $h \in G$. Let $\tau^m \in T$ for $m \in Z$, chose an arbitrary h of the form $h(z) \to az + \beta$ then $h^{-1}(z) = \frac{z-\beta}{\alpha}$. Then $(h^{-1}o\tau^m oh)(z) = h^{-1}o((\tau^m oh)(z))$. $h^{-1}(\tau^m(h)(z)) - h^{-1}((az+\beta)+m) - \frac{((az+\beta)+m-\beta}{\alpha} - z + \frac{m}{\alpha}$

Also chose an arbitrary h of the form $h = a\bar{z} + \bar{\beta}$

where $$h^{-1} - \frac{\bar{E}-\bar{\beta}}{\alpha} - \frac{\bar{E}-\beta}{\alpha}$$

then $$(h^{-1}o\tau^m oh)(z) - h^{-1}o((\tau^m oh)(z)) - h^{-1}(\tau^m(h)(\beta)) - h^{-1}((a\bar{z}+\beta)+m)$$

$$\frac{((\alpha\bar{z}+\bar{\beta})+\bar{m})-\beta}{\alpha}\ \frac{((az+\beta)+m)-\beta}{\beta} - z + \frac{m}{\alpha} \subseteq G.$$

Thus for every $h \in G$ we shows that $h^{-1}oToh \subseteq T$. Shows that $T \ \Delta\ G$. That is T is normal subgroup of G.

REFERENCES:

[1] Coxeter, H. S. M. (1969). Introduction to Geometry. New York: John Wiley & Sons. pp. 47-49. ISBN 0-471-50458-0.

[2] Cederberg, Judith N. (2001). A Course in Modern Geometries, 2nd ed. New York: Springer-Verlag. pp. 117-118, 165-171. ISBN 0-387-98972-2.

[3] Fisher, G.L.; Mellor, B. (2007), "Three-dimensional finite point groups and the symmetry of beaded beads", Journal for Mathematics and the Art .

[4] Gallian. J.A.(1999). Contemporary Abstract Algebra. Narosa Publishing House , New- Delhi.

[5] http://euler.slu.edu/escher/index.php/Frieze_Patterns.

[6] http://nrich.maths.org/1341.

[7] http://mathcentral.uregina.ca/RR/database/RR.09.01/mcdonald1/

[8] Stolarski, M.(2011) Frieze Groups in R^2.

[9] http://www.math.uchicago.edu/~may/VIGRE/VIGRE2010/REUPapers/Stolarski.pdf